On the ionospheric application of Poynting’s theorem

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[1] It has been proposed that the geomagnetic field-aligned component of the perturbation Poynting vector above the ionosphere, as obtained from the cross product of the electric and magnetic perturbation fields observed on a spacecraft, may be used to estimate the field line-integrated electromagnetic energy dissipation in the ionosphere below. This paper clarifies conditions under which this approximation may be either valid or invalid. It is shown that the downward field-aligned component of the perturbation Poynting vector can underestimate the electromagnetic energy dissipation in regions of high ionospheric Pedersen conductance, and it can significantly overestimate the dissipation in regions of low conductance. Local values of upward perturbation Poynting vector do not necessarily correspond to net ionospheric generation of electromagnetic energy along that geomagnetic field line. An Equipotential Boundary Poynting Flux (EBPF) theorem is presented for quasi-static electromagnetic fields as follows: when a volume of the ionosphere is bounded on the sides by an equipotential surface and on the bottom by the base of the conducting ionosphere, then the area integral of the downward normal component of the perturbation Poynting vector over the top of that volume equals the energy dissipation within the volume. This equality does not apply to volumes with arbitrary side boundaries. However, the EBPF theorem can be applied separately to different components of the electric potential, such as the large- and small-scale components. Since contours of the small-scale component of potential tend to close over relatively localized regions, the associated small-scale structures of downward perturbation Poynting vector tend to be dissipated locally.


1. Introduction

[2] An important energy source for the high-latitude ionosphere and thermosphere is electromagnetic energy transfer from the magnetosphere. This transfer is associated with electric current flow along geomagnetic field lines above the ionosphere and horizontally within the ionosphere, and with strong electric fields. Both the electric field and the magnetic perturbation field associated with the current can be measured above the ionosphere by spacecraft, while radars can determine two-dimensional horizontal distributions of the electric field. Kelley et al. [1991] pointed out how such measurements can be used to determine the downward component of the Poynting vector, or Poynting flux, and to estimate the height-integrated energy dissipation in the ionosphere/thermosphere by application of Poynting’s Theorem. The Poynting vector $\mathbf{S}$ was originally derived by Poynting [1884] in a form equivalent to

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}$$

(for a non-magnetic medium), where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and $\mu_0$ is the permeability of free space. As discussed by Kelley et al. [1991], Thayer and Semeter [2004] and others, it is useful for many ionospheric purposes to use the perturbation Poynting vector

$$\mathbf{S}_p = \frac{\mathbf{E} \times \delta \mathbf{B}}{\mu_0},$$

where $\delta \mathbf{B}$ is the perturbation of the geomagnetic field with respect to the main field $\mathbf{B}_0$. Kelley et al. [1991], Gary et al. [1994, 1995], Deng et al. [1995], Huang and Burke [2004], Golovchanskaya and Maltsev [2004], Olsson et al. [2004], and Janhunen et al. [2005] have analyzed the geomagnetic field-aligned component of $\mathbf{S}_p$, or $\mathbf{S}_{p||}$, using spacecraft data. Waters et al. [2004] combined satellite magnetometer data with SuperDARN radar measurements of plasma velocities to compute $\mathbf{S}_{p||}$. Weimer [2005] calculated $\mathbf{S}_{p||}$ from his empirical models of the electric and magnetic perturbation fields above the ionosphere.

[3] The purpose of this paper is to clarify the relation between the perturbation Poynting flux above the ionosphere and the electromagnetic energy dissipation in the ionosphere and thermosphere below. Under certain circumstances, $\mathbf{S}_{p||}$ at a given point above the ionosphere equals the geomagnetic field line-integrated electromagnetic...
energy dissipation rate along that field line, as elaborated in the next two sections. This equality is often assumed to apply under very general circumstances in the ionosphere. However, examples are shown in section 3 where this equality does not hold. Instead, it is shown in the next section that the relation between $S_p$ and the dissipation applies only in an integrated sense for volumes with specific boundary conditions.

2. Application of Poynting’s Theorem for Quasi-Static Fields

[4] The geomagnetic field is dominated by the Earth’s main field $B_0$, defined in terms of a magnetic potential $V_0$ as

$$B_0 = -\nabla V_0. \quad (3)$$

The perturbation on this, $\delta B$, is usually less than a few percent the magnitude of $B_0$. Both $B_0$ and $\delta B$ are divergence-free. This paper is concerned with quasi-static electric and magnetic fields, those varying on time scales longer than about 100 s. At high latitudes, $\delta B$ may vary by up to a few hundred nanoteslas or more in this time, during strong disturbances and over a limited region. If a change of 100 nT occurs over a region of scale size $L \sim 1000$ km in a time of $\tau \sim 1000$ s, then from Faraday’s Law

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\frac{\partial \delta B}{\partial t}, \quad (4)$$

the induction electric field is of the order of $E_{\text{induction}} \sim (10^{-7} \text{ T})(10^6 \text{ m})/(10^3 \text{ s}) = 10^{-4} \text{ V/m}$, which is small in comparison with typical high-latitude ionospheric electric fields. For short periods of time in localized regions, the induction electric field can occasionally be non-negligible [e.g., Vanhamäki et al., 2007], but such isolated events are ignored here. For many purposes, then, the electric field can be considered electrostatic

$$E = -\nabla \Phi, \quad (5)$$

where $\Phi$ is the electrostatic potential. To an even better approximation, the electric current density $J$ can be expressed in terms of the curl of $\delta B$

$$J = \nabla \times \delta B/\mu_0, \quad (6)$$

where $\mu_0$ is the permeability of free space. Poynting’s Theorem for these quasi-static fields can be derived by calculating the divergence of $S_p$ with the use of the equations above

$$\nabla \cdot S_p = \nabla \cdot E = \nabla \cdot (\nabla \Phi) = -J \cdot E. \quad (7)$$

$S_p$, $S$, and $J\Phi$ represent very different vector fields, even though they all have the same divergence when $E$ is electrostatic. (Note, however, that these quantities generally all have a different divergence if $E$ is not electrostatic.) For example, $S_p$ is generally non-zero below the ionosphere, because $E$ and $\delta B$ are non-zero there, while $J\Phi$ does vanish below the ionosphere, if we assume that $J$ vanishes there. As pointed out by Kelley et al. [1991], $S_p$ is more useful than $S$ for ionospheric applications, because the former removes the very large contribution from $E \times B_0/\mu_0$, which is non-divergent and therefore contributes nothing to the energy conversion. Although $J\Phi$ has sometimes been used to trace electromagnetic energy flow, it is less useful than $S_p$ when dealing with observations, as it requires knowledge of the electric potential, which cannot be directly measured by satellite, and also requires $J$, which is difficult to measure directly in space. Furthermore, $J\Phi$ is not unique, as the zero level of $\Phi$ is arbitrary. Adding a constant to $\Phi$ will change $J\Phi$ everywhere that $J$ is non-zero.

[5] The electromagnetic energy dissipation or conversion rate $J \cdot E$ represents the sum of Joule heating and generation of bulk kinetic energy of the ionosphere/thermosphere medium

$$J \cdot E = J \cdot (E + v \times B) + (J \times B) \cdot v, \quad (8)$$

where $v$ is the bulk velocity of the medium, defined as the mass-weighted average velocity of the neutrals, ions, and electrons (nearly the same as the neutral velocity, since neutrals dominate the total mass density). Both $E$ and $v$ depend on the chosen reference frame, which is assumed here to be fixed with the rotating Earth. The first term on the right-hand side of (8) is Joule heating. The second term is the scalar product of the velocity and the force per unit volume on the medium, $J \times B$, and represents the generation of kinetic energy. The second term can be negative, if the force opposes the velocity. If this term is both negative and greater in magnitude than the Joule heating, then $J \cdot E$ is negative, and the ionospheric wind dynamo acts as a net source of electromagnetic energy. Lu et al. [1995] showed that, integrated over the high-latitude ionosphere, Joule heating is usually much greater than the net generation of kinetic energy. For a particular storm simulation, the latter averaged 6% of the total energy dissipation.

[6] Applying Gauss’ Theorem, (7) can be expressed in integral form as

$$-\int S_p \cdot da = \int J \cdot E dV, \quad (9)$$

where the area of the integral on the left-hand side encloses the volume of the integral on the right-hand side; $da$ is an outward-directed element of area; and $dV$ is an element of volume. As explained by Kelley et al. [1991], if the volume is chosen to enclose the high-latitude ionosphere in such a way that the top of the volume is above the ionosphere and has a non-zero normal component of $S_p$, while the sides and bottom of the volume have zero tangential electric field (so that the normal component of $S_p$ is zero), then all of the downward perturbation Poynting flux passing through the top surface is converted into heat and mechanical energy within the volume. Kelley et al. [1991] chose the sides of the volume to be at subauroral latitudes, where the electric field is small, and the bottom of the volume to be at the Earth’s surface, which is essentially an equipotential for the quasi-static conditions under consideration, so that the horizontal component of $E$ vanishes there.

[7] The model of Kelley et al. [1991] can also be applied to other surfaces that are equipotential everywhere except the top surface. For example, there can exist V-shaped potential structures associated with field-aligned particle
acceleration above the ionosphere [e.g., Mozer et al., 1980; Thayer and Semeter, 2004]. The normal component of \( \mathbf{S}_p \) over any equipotential surface is automatically zero, and so all of the energy flux represented by the downward component of \( \mathbf{S}_p \) over the top surface must be dissipated within the volume bounded by the equipotential surface enclosing the sides and bottom of such a structure.

Kelley et al. [1991] argued that the model could be applied to any high-latitude magnetic flux tube of arbitrarily small cross-section. Weimer [2005] also explained that there are certain circumstances where the local downward component of \( \mathbf{S}_p \) above the ionosphere is dissipated entirely along that magnetic field line in the ionosphere below. This is the case if the ionospheric current below is simply proportional to \( \delta \mathbf{B} \) at the satellite altitude, as it would be if the field-aligned current were purely radial and the horizontal ionospheric current had a zero vertical component of curl. However, these conditions are not generally met.

Let us consider the situation illustrated by Figure 1, in which a volume is bounded on the sides by the equipotential surface of potential \( \Phi_b \), on the bottom by the bottom of the ionosphere, and on the top by an arbitrary surface above the ionosphere. Highly conducting magnetic field lines can be considered to enforce vertical equipotentiality along the sides, although this does not have to be the case, as illustrated by the example discussed earlier of a V-shaped auroral potential structure with a field-aligned potential drop. Equipotentiality along the edges of the top and bottom surfaces is enforced by placing the edges along an equipotential line of potential \( \Phi_b \). That is, this is not an arbitrary boundary, but rather represents a streamline of ionospheric \( \mathbf{J} = \mathbf{E} \times \delta \mathbf{B}/\mu_0 \). Since \( \delta \mathbf{B} \) is created by currents both nearby and distant, which may have complex distributions, \( \delta \mathbf{B} \) may have any orientation, and need not be perpendicular either to the equipotential surface or to the main field \( \mathbf{B}_0 \).

Unlike the model of Kelley et al. [1991], the bottom surface need not be equipotential, but it must lie at an altitude below the ionosphere where current essentially vanishes. In general, neither \( \mathbf{E} \) nor \( \delta \mathbf{B} \) vanishes at the bottom, and the normal component of \( \mathbf{S}_p \) is not zero over this bottom surface, but its integral is zero. To see this, use Stokes’ Theorem and other mathematical manipulations to derive the following, where the area integrals over \( A \) represent either the top or bottom of the volume in Figure 1, while the line integrals over \( L \) represent the edge of this cross section (with elemental length \( dl \)):

\[
\int_A \mathbf{S}_p \cdot d\mathbf{a} = \int_A \frac{\mathbf{E} \times \delta \mathbf{B}}{\mu_0} \cdot d\mathbf{a} = \int_A (\Phi - \Phi_b) \mathbf{J} \cdot d\mathbf{a} = 0.
\]

The integrated perturbation Poynting flux over the cross-sectional surface can therefore be expressed in terms of the integral of the normal component of current density, multiplied by the difference of electric potential from the potential at the edge. Since the current vanishes at the bottom, the integrated perturbation Poynting flux over the bottom also vanishes, even though \( \mathbf{S}_p \) itself does not necessarily vanish there. Therefore, we have proved the following theorem for quasi-static electromagnetic fields in the ionosphere:

Equi-Potential Boundary Poynting Flux (EBPF) Theorem: The area-integrated downward normal component of the perturbation Poynting vector \( \mathbf{S}_p = \mathbf{E} \times \delta \mathbf{B}/\mu_0 \), for a surface above the ionosphere that is bounded on the sides by an intersecting equipotential surface, equals the integrated electromagnetic energy dissipation in the volume bounded by those surfaces and by the base of the ionosphere.

Equivalent theorems apply to the full Poynting vector \( \mathbf{S} \) and to the quantity \( \mathbf{J}(\Phi - \Phi_b) \), in place of \( \mathbf{S}_p \).

If we consider, for example, the equipotential contours around the typical dawn-side potential maximum, then, moving outward from the maximum, the integrated downward perturbation Poynting flux over the ionosphere over the area bounded by each successive contour equals the total electromagnetic energy dissipated in the ionosphere below. We can readily deduce that the integrated perturbation Poynting flux in a band bounded by two successive equipotential contours must also equal the electromagnetic energy dissipated below in that collection of flux tubes passing through the band. However, we cannot extend this reasoning to single flux tube or to any arbitrary collection of flux tubes. The side boundaries of the volume must be equipotential in order for the EBPF Theorem to apply.

The conditions for validity of the EBPF Theorem are that \( \mathbf{E} \) is electrostatic, and that \( \mathbf{J} \) vanishes below the ionosphere. The theorem cannot be applied to situations where either of these conditions is violated, such as the transfer of electromagnetic energy into the solid Earth through induction currents, or current flow between the lower atmosphere and the ionosphere above thunderstorms.

One consequence of the fact that the perturbation Poynting flux for an arbitrary collection of flux tubes need not equal the volume integral of \( \mathbf{J} \cdot \mathbf{E} \) for those same flux tubes, unless the boundary is an equipotential, is that
observations of upward $S_{||}$ [e.g., Kelley et al., 1991; Gary et al., 1994, 1995] may not necessarily coincide with regions where the ionospheric wind dynamo is a net generator of electromagnetic energy (negative $J \cdot E$). An example is presented in the next section. Nevertheless, if there is a net upward perturbation Poynting flux over a surface above the ionosphere bounded by an equipotential, then there must indeed be a net ionospheric/thermospheric source of electromagnetic energy.

3. Simple Examples

[16] To understand better how the geometry of the electric current affects $S_p$, let us consider some idealized examples. All of these examples treat the ionosphere as a horizontal conducting plate in the $r - \theta$ plane (also the $x - y$ plane), into which current flows from above in the $\pm z$ direction. The vertical current will be considered to represent geomagnetic field-aligned current (FAC). In Example 1, the plate has only a Pedersen conductance (field line-integrated conductivity) $\Sigma_p$, with the uniform value $\Sigma_{p0}$, and the FAC flows along a cylinder centered on the positive $z$ axis, with a sheet current density $J_z$ given by

$$K_z = -2K_0 \cos(\theta) \delta(r - r_0), \quad z > 0, \quad (11)$$

where $\delta$ is the Kronecker delta function, and $\cos \theta = x/r$. The solution for the electric potential $\Phi$ in this case, shown by the contours in Figure 2a, is

$$\Phi = \Phi_0 \left( \frac{r}{r_0} \right) \cos \theta, \quad r < r_0,$$

$$= \Phi_0 \left( \frac{r_0}{r} \right) \cos \theta, \quad r > r_0, \quad (12)$$

$$\Phi_0 = \frac{K_0 r_0}{\Sigma_{p0}}. \quad (13)$$

$F$ and $E$ are constant in height, and $E$ is horizontal. The magnitude of $E$ depends only on $r$. It is equal to $E_0(= K_0/\Sigma_{p0})$ in the inner region ($r < r_0$), and to $E_0(r_0/r)^2$ outside. In the inner region $E$ points everywhere in the $-x$ direction. The radial component of $E$ reverses direction at $r = r_0$, without any discontinuity of magnitude, while the azimuthal component of $E$ is continuous across this boundary. The height-integrated horizontal current density in the plate, $K$, is $\Sigma_pE$, and is shown in Figure 3a. Half of the total FAC closes through the inner region ($r < r_0$) of the plate, while half closes outside this region. $\delta B$ above the plate is horizontal, and equals $-\mu_0 k \times K$, where $k$ is a unit vector in the $z$ direction. Loops of $\delta B$ coincide with equipotential contours. $\delta B$ is proportional in magnitude to $E$ everywhere above the plate, and the two are orthogonal. $\delta B$ vanishes below the plate for a current system like this composed of vertical FAC connected to curl-free current in the plate [see Fukushima, 1976]. The pattern of $\delta B$ above the plate is shown in Figure 3e. Above the plate $S_p$ is everywhere downward, constant in $z$, and equal to the rate of Joule heating in the plate directly below, $\Sigma_pE^2$. The downward perturbation Poynting flux $S_{||}$ above the plate and the Joule heating rate in the plate are functions only of $r$, and are shown in Figure 3i. $S_{||}$ and the Joule heating are uniform in the inner region, with a value of $\Sigma_pE_{||}^2$, and they fall off as $\Sigma_pE_{||}^2(r_0/r)^2$ outside. In this first example, $S_{||}$ at any point above the plate indeed gives the electromagnetic energy dissipation rate directly below that point, as discussed by Kelley et al. [1991], Weimer [2005], and others.

[17] Example 1 has some similarities with conditions over the polar ionosphere, if $K_z$ is considered to represent the Region 1 FAC and $2\Phi_0$ the cross-polar-cap potential drop. However, it differs significantly from the polar ionosphere in its neglect of Hall currents and Region 2 FAC. Alternatively, if applied to a smaller scale, this example may be considered a representation of a localized pair of field-aligned currents and their associated electric fields.

[18] For Example 2, let us add a uniform Hall conductance to the plate, of the same magnitude as $\Sigma_p$. For a
uniform Hall conductance, the associated height-integrated Hall current density $K_H$ is divergence-free, and so $E$ is unchanged from Example 1, as shown in Figure 2a. The main geomagnetic field is taken to be in the downward ($-z$) direction, so that Hall current flows 90° clockwise to $E$, as in Figure 3b. This current is additional to the Pedersen current of Figure 3a. The component of $\delta B$ associated with $K_H$, which we call $\delta B_{H}$, is three-dimensional, with vertical as well as horizontal components that exist both above and below the plate. Its horizontal component immediately above the plate, shown in Figure 3f, equals $-\mu_0 k \times \mathbf{K}_H / 2$. This component of $\delta B$ is additional to that of Figure 3e. Immediately below the plate the horizontal component of $\delta B_{H}$ has the same magnitude but is reversed in direction. Note that the magnitude of the horizontal component of $\delta B_{H}$ immediately above the plate is only half as large in relation to its associated current as is the $\delta B$ associated with the Pedersen current of Example 1, because half of the magnetic field change across the sheet current appears below the plate in this case, whereas the field below the plate is zero in Example 1. Along the z axis the $x$ component of $\delta B_{H}$ varies as

$$\delta B_{Hx} = \frac{\pm \mu_0 K_0}{2} \left( 1 - \frac{z}{\sqrt{r_0^2 + z^2}} \right), \quad (14)$$

where the $+$ and $-$ signs are for positive and negative $z$, respectively. There also exists a vertical component of $\delta B_{H}$, which is largest upward (out of the plane of Figure 3b) at the plate in the vicinity of the counterclockwise current focus, and largest downward (into the plane of Figure 3b) at the plate in the vicinity of the clockwise current focus.

[19] Immediately above and below the plate, $\delta B_{H}$ is either parallel or antiparallel to $E$, and so there is no additional $S_p||$ at the plate. However, the pattern of $\delta B_{H}$ changes as we move away from the plate, so that its contribution to $S_p||$ is no longer strictly zero everywhere, although it is zero along the $z$ axis. Nonetheless, the Hall-current contribution to $S_p||$ tends to be small in comparison to $|E||\delta B_{H}|/\mu_0$, because the angle between $E$ and the horizontal component of $\delta B_{H}$ tends
Downward perturbation Poynting flux along the z axis \((r = 0)\). The Hall currents in Example 2, which are additional to the FAC and Pedersen currents of Example 1, contribute nothing to \(S_{p\parallel}(r = 0)\), so that \(S_{p\parallel}(r = 0)\) is the same for Examples 1 and 2.

to be small. In contrast, the vertical component of \(\delta B_H\) interacts with \(E\) to produce a sizable horizontal component of \(S_\phi\) that largely circulates counterclockwise and clockwise, respectively, around the locations of minimum and maximum electric potential, although this horizontal component of \(S_\phi\) associated with the Hall current also has some divergence or convergence, that exactly balances the convergence or divergence of the vertical component of \(S_\phi\) associated with the Hall current. In three dimensions, the component of \(S_\phi\) associated with the Hall current is completely divergence-free, so it does not contribute to the transfer of electromagnetic energy. It decreases in magnitude with increasing distance from the plate, in relation to the decrease of \(\delta B_H\).

[20] For Example 3, let us return to a plate with only Pedersen conductance, but with \(\Sigma_p\) reduced to \(0.02\Sigma_{p0}\) in the inner region \((r < r_0)\), and increased to \(1.98\Sigma_{p0}\) outside this region. The vertical current density is unchanged, and it is readily found that \(\Phi\) and \(E\) remain the same as in Figure 2a. The height-integrated current density in the plate is shown in Figure 3c. In contrast to Example 1, for which half the FAC closes across the inner region, and half closes outside, in Example 3 only 1% of the FAC closes across the inner region, while 99% closes outside. Consequently, the Joule dissipation in the plate is only 2% as large as in Example 1 within the inner region, but is 198% as large outside this region. The horizontally integrated Joule dissipation is the same in the two examples.

[21] Example 3 has some similarities with conditions over the polar ionosphere in winter, where the conductivity in the polar cap is small in comparison with that in the surrounding auroral region, although we must note, as for Example 1, that the absence of Hall currents and Region 2 FAC significantly affects the results. Furthermore, the conductance contrast in the real ionosphere is likely to be less than in Example 3. However, the idealized conductance distribution used in this example allows us to illustrate semiquantitatively how spatially varying conductances affect \(\delta B\) and \(S_{p\parallel}\).

[22] To calculate \(\delta B\), we can note that the current is just the sum of that in Example 1, plus an additional plate current of the form of \(K_H\) in Example 2, rotated 90° clockwise, with a strength 98% as large. The resulting horizontal component of \(\delta B\) immediately above the plate is shown in Figure 3g. With respect to Example 1, the horizontal component of \(\delta B\) in Example 3 is \((1 - 0.5 \times 0.98) = 0.51\) as large immediately above the plate in the inner region, and \((1 + 0.5 \times 0.98) = 1.49\) as large outside this region. Immediately below the plate the horizontal component of \(\delta B\) drops slightly in the inner region, by a factor 0.49/0.51, from its value immediately above, while outside the inner region the direction reverses and the amplitude is smaller by a factor 0.49/1.49. At very large \(z\) distance from the plate, \(\delta B\) becomes the same as in Example 1. Both at the plate and far above it \(\delta B\) is orthogonal to \(E\), which simplifies the calculation of \(S_{p\parallel}\) at these locations.

[23] The changes of \(S_{p\parallel}\) with respect to Example 1 reflect the differences of \(\delta B\). Figure 3k shows that the Joule dissipation, \(\Sigma_p E^2\), is much smaller than \(S_{p\parallel}\) in the inner region, by a factor 0.02/0.51 with respect to \(S_{p\parallel}\) just above the plate, decreasing to a factor 0.02 with respect to \(S_{p\parallel}\) at very high altitude, at which point \(S_{p\parallel}\) becomes the same as in Example 1 (i.e., the same as in Figure 3i). Outside \(r_0\), \(\Sigma_p E^2\) is larger than \(S_{p\parallel}\), by a factor 1.98/1.49 immediately above the plate, increasing to a factor 1.98 with respect to \(S_{p\parallel}\) at very high altitude. The variation of \(S_{p\parallel}\) with \(z\) is shown in Figure 4. Even though the plate current and Joule dissipation in the inner region are small, neither \(\delta B\) nor \(S_{p\parallel}\) is small at any altitude above the plate. The “surplus” downward \(S_{p\parallel}\) above the plate in the inner region represents Poynting flux that traverses the plate and curves around below the plate to come back to the plate beyond \(r = r_0\). This Poynting flux from below compensates the deficit of Poynting flux from above in the outer region, such that the difference of \(S_{p\parallel}\) across the plate matches the Joule heating rate. The horizontal spreading of Poynting flux starts well above the plate, accounting for decreasing \(S_{p\parallel}\) with decreasing height in the inner region, seen in Figure 4, and for increasing \(S_{p\parallel}\) with decreasing height above the plate in the outer region. Since \(S_{p\parallel}\) is divergence-free above and below the plate, the vertical variations of \(S_{p\parallel}\) must be accompanied by a horizontal component of \(S_\phi\), that diverges or converges in such a way to offset the convergence or divergence of the vertical component. This horizontal component of \(S_\phi\) is associated with the vertical component of \(\delta B\).

[24] Example 4 is the opposite of Example 3, with \(\Sigma_p\) set to \(1.98\Sigma_{p0}\) in the inner region and to \(0.02\Sigma_{p0}\) in the outer region. The results are shown in Figures 3d, 3h, 3l, and 4. At very high altitude, \(S_{p\parallel}\) is again the same as for Example 1. Example 4 may be crudely representative of a localized region of enhanced conductance that is connected to upward and downward FAC on opposite edges.

[25] Examples 1–4 all have the same distribution of \(E\), and all result in downward (positive) \(S_{p\parallel}\) everywhere above the plate, even if this downward \(S_{p\parallel}\) does not always match
the Joule heating below. To see how it is possible to have upward (negative) $S_{\parallel}$ over some region above the plate, even though energy dissipation in the plate is everywhere positive, let us consider Example 5, in which $E$ is altered. We start from Example 3, which has strongly reduced $\Sigma_p$ in the inner region. To this we add a small uniform downward current density over the entire inner region, of magnitude 0.064$K_0/r_0$, together with a balancing azimuthally uniform upward sheet current in the cylinder of magnitude 0.032$K_0$. This small additional current modifies $E$ only in the inner region, with the change of electric potential, $\Phi'$, given by

$$\Phi' = 0.8\Phi_0\left[1 - \left(\frac{r}{r_0}\right)^2\right], \quad r < r_0. \quad (15)$$

This changes the total electric potential and electric field as shown in Figure 2b. The pattern of the potential contours over the inner region bears some similarity to northern polar-cap potentials when the interplanetary magnetic field has a large negative $B_z$ component [e.g., Heelis, 1984]. Although the additional FAC is relatively weak, it is associated with a large change of $E$ in the inner region, owing to the small conductance there. On the right side of this inner region, shifted in Figure 2b, $E_x$ is reversed with respect to its value in Examples 1–4. On the other hand, the change in $\delta B$ associated with this small additional current is only 6.3% or less the magnitude of $\delta B$ for Example 3. The small change in $\delta B$ is confined to the inner region and is non-zero only above the plate, where it is constant in height. The plate current and the total $\delta B$ immediately above the plate are similar to Example 3 (Figures 3c and 3g). Since $\delta B$, remains positive over the entire inner region, the reversal of $E_x$ over part of the inner region is associated with a reversal of the direction of $S_{\parallel}$, which becomes upward over a region approximately the same as the shaded region in Figure 2b. Upward $S_{\parallel}$ is found at all values of $z$, not only close to the plate; in fact, it has a larger upward magnitude at high altitude, where $\delta B$ is larger than it is close to the plate. This occurs even though $J \cdot E$ remains positive everywhere in the plate, with no source of electromagnetic energy in the plate. In the region of reversed $E_x$, the altered current slightly decreases the horizontal component of $\delta B$ above the plate and slightly increases it below, such that $S_{\parallel}$ below the plate is upward and slightly larger in magnitude than $S_{\parallel}$ above the plate, and the change of $S_{\parallel}$ across the plate tends to correspond to the Joule dissipation in the plate, as it must. The integrated downward perturbation Poynting flux above the plate is slightly larger than in Example 3, because of the additional small current flow in the inner region. Note that the electric-potential contours in the region of upward $S_{\parallel}$ extend outside this region, and also enclose an area of downward $S_{\parallel}$. The EBPF Theorem tells us that these contours must enclose more downward than upward perturbation Poynting flux, since the dissipation within them is positive.

4. Discussion

The examples of the previous section illustrate how it is possible for measurements of $S_{\parallel}$ above the ionosphere not to represent the local energy dissipation along individual geomagnetic field lines in the ionosphere below, even though the integrated downward perturbation Poynting flux over a surface bounded by an equipotential does represent the volume-integrated energy dissipation below. These examples emphasize effects associated with the three-dimensionality of the currents. In contrast, a strictly two-dimensional current and electric field configuration, in which the electric field is non-zero only in the direction perpendicular to $B$ in the cross-sectional plane of the currents, would result in $S_{\parallel}$ being proportional to $B$ along field lines above the ionosphere, and equal to the flux tube-integrated energy dissipation below, as has often been assumed to hold more generally.

From these examples we can draw some general inferences. Firstly, $S_{\parallel}$ above the ionosphere tends to be more spread out horizontally than the distribution of the dissipation in the ionosphere. $S_{\parallel}$ tends to underestimate the energy dissipation in high-conductance regions, and to overestimate the dissipation in low-conductance regions. Secondly, the amount of horizontal spreading of $S_{\parallel}$ is related to the spatial scale of the current system, which is $r_0$ in the examples. Smaller-scale current structures have more concentrated perturbation Poynting flux that is associated with more-localized energy dissipation. Thirdly, the horizontal spreading of $S_{\parallel}$ must be accompanied by a divergent/convergent horizontal component of $S_{\parallel}$ that is associated with a vertical component of $\delta B$. Vertical components of $\delta B$ have indeed been measured above the ionosphere [e.g., Langel, 1974], as well as at the ground, although these tend to be dominated by the effects of auroral Hall currents, which do not contribute to the energy transfer. A more detailed analysis of the data will be necessary to determine the component of vertical magnetic perturbation associated with the horizontal spreading of $S_{\parallel}$. Fourthly, we must be cautious interpreting observations of a localized region of upward $S_{\parallel}$. These cannot automatically be used to identify a net ionospheric-dynamo source of electromagnetic energy (negative $J \cdot E$).

The fact that the perturbation Poynting flux above the ionosphere, $S_{\parallel}$, cannot be used to determine precisely the electromagnetic energy dissipation on individual geomagnetic flux tubes, or to identify unambiguously individual flux tubes where there is net ionospheric-dynamo generation of electromagnetic energy, places only a modest limit on its value as a diagnostic tool. The EBPF Theorem indicates that an area integral of $S_{\parallel}$ does achieve these diagnostic goals, but only for a collection of flux tubes bounded on the side by an equipotential surface. Instantaneous determinations of equipotentials are difficult to achieve, but with increasing radar coverage of the polar regions the estimation of potential contours is becoming more feasible [e.g., Shepherd and Ruohoniemi, 2000]. Coupled with increasingly detailed observation of magnetic perturbations over the polar regions [e.g., Anderson et al., 2008; Kumar, 2008], instantaneous two-dimensional estimates of $S_{\parallel}$ are becoming possible. In any case, statistical means of $S_{\parallel}$ over the entire polar region can be constructed from large collections of satellite data, or satellite plus radar data, of which the area integrals accurately represent the mean integrated electromagnetic energy dissipation over the entire polar region, since the midlatitude boundary of this region can usually be approximated as an equipotential [Kelley et al., 1991].

Accurately knowing where the electromagnetic energy is being converted to heat and kinetic energy is
important for understanding how the ionosphere and thermosphere will respond. Examples 3 and 4 show how it is possible for \( S_p \) to be substantial on field lines that have low \( E \)-region conductivity, for which the field line integral of the conductivity is dominated by \( F \)-region conductivity. An example of this is seen in the empirical model of \( S_p \) presented by Deng et al. [2009], where non-negligible \( S_p \) is seen over the dark portion of the polar cap. If it were assumed that the electromagnetic energy represented by \( S_p \) must be dissipated on the same field line, even though the \( E \)-region conductivity is low, then much of the energy would have to be dissipated in the upper thermosphere, possibly causing very large temperature increases because of the low density there. If, on the other hand, the observed \( S_p \) in low-conductance regions is due largely to geometrical spreading, as in Examples 3 and 4, then the local energy dissipation in low-conductance regions may be considerably lower than \( S_p \), and the thermospheric consequences would be more moderate. We need additional information not only about the horizontal distribution of the dissipation within the volume bounded on the sides by an equipotential surface, but also about the height distribution of the dissipation. \( S_p \) alone cannot provide this information. Additional observations, as well as simulation modeling, can help provide the needed information.

[30] Strict though the requirement may be that the EBPF Theorem applies only to regions bounded by equipotentials on the sides, we can nevertheless apply it to different components of the electric field, components having different patterns of equipotential contours. Only certain ways of dividing the potential into components are useful, however. In particular, let us break \( \Phi \) into large- and small-scale components \( \Phi_l \) and \( \Phi_s \),

\[
\Phi = \Phi_l + \Phi_s.
\]

The definition of \( \Phi_l \) is not critical; it may be considered to be a spatially low-pass filtered version of \( \Phi \), containing scales sizes comparable to those of empirical models of \( \Phi \) like that of Weimer [2005]. \( \Phi_s \) is then the residual, or difference between \( \Phi \) and \( \Phi_l \). Because \( \Phi_s \) is composed of relatively small-scale features, it is likely to have most of its contours close over relatively small regions. For each of these contours we can apply the EBPF Theorem to the component of \( S_p \) calculated from \( \mathbf{E}_s \times \delta \mathbf{B}/\mu_0 \), where \( \mathbf{E}_s = -\nabla \Phi_s \), meaning that the field-aligned transfer of energy associated with small-scale features of the electric field will tend to occur over relatively localized regions. Observations [e.g., Sugiura et al., 1982; Golovchanskaya and Maltsev, 2004] indicate that the small-scale structures of \( \delta \mathbf{B} \) tend to correlate with those of \( \mathbf{E} \), in such a way that the associated component of \( S_p \) is mostly downward. The large-scale structures of \( \delta \mathbf{B} \) also contribute to this component of \( S_p \), producing contributions that are either upward or downward, depending on the direction of \( \mathbf{E}_s \), but often not making a large net contribution to the area-integrated perturbation Poynting flux.

5. Conclusions

[31] Application of Poynting’s Theorem to electromagnetic energy transfer between the magnetosphere and ionosphere, based on observations of the perturbation Poynting vector \( S_p \) above the ionosphere, gives an accurate quantitative measure of this transfer in a spatially integrated sense. However, a local measurement of the geomagnetic field-aligned component of \( S_p \), \( S_{p||} \), does not necessarily equal the geomagnetic field line integral of energy dissipation, contrary to what has often been assumed. For quasi-static electric fields the Equipotential Boundary Poynting Flux (EBPF) Theorem gives an alternative to this assumption: when a volume of the ionosphere is bounded on the sides by an equipotential surface and on the bottom by the base of the ionosphere, then the integral of the downward normal component of \( S_p \) over the top of the volume exactly equals the energy dissipation within. Similar statements can be made regarding the total Poynting vector \( \mathbf{S} \) and the quantity \( \mathbf{J}(\Phi - \Phi_p) \) in place of \( S_p \). If the perturbation Poynting flux is determined sufficiently high above the ionosphere that contributions from Hall currents are negligible, then its use to estimate field line-integrated energy dissipation can be valid when the horizontal ionospheric electric field and current spread out from their field-aligned current sources in a manner similar to the way magnetic fields spread geometrically in space around the field-aligned currents, as approximated by Weimer [2005]. \( S_{p||} \) is also a good measure of energy dissipation for two-dimensional current configurations like paired current sheets closed by horizontal ionospheric current. However, for a three-dimensional current system the associated perturbation Poynting flux can possibly be very different from the integrated energy dissipation below. The flux tends to underestimate the dissipation in regions of large Pedersen conductance, and can significantly overestimate the dissipation in regions of low conductance. The downward perturbation Poynting flux associated with Hall currents also fails to represent energy dissipation, although it often tends to be small, because \( \mathbf{E} \) and the associated horizontal component of \( \delta \mathbf{B} \) often tend to be approximately parallel or antiparallel. Furthermore, local regions of upward \( S_{p||} \) may not necessarily correspond to regions of net ionospheric generation of electromagnetic energy, unless the integral of \( S_{p||} \) over an area normal to \( \mathbf{B}_0 \) that is bounded by an equipotential shows net upward flux.

[32] A vertical variation of \( S_{p||} \), whether associated with Hall currents or with horizontal variations of the Pedersen conductance, is connected to a divergent or convergent horizontal component of \( S_p \), which in turn is associated with the vertical component of \( \delta \mathbf{B} \). Analysis of observations of the vertical component of \( \delta \mathbf{B} \) in relation to \( \mathbf{E} \) could be useful for estimating how important the horizontal redistribution of perturbation Poynting flux is for realistic ionospheric conditions, as opposed to the idealized examples presented here.

[33] The EBPF Theorem can be applied to separate components of the electric potential, such as a division into large- and small-scale components. For such a separation, the theorem indicates that the perturbation Poynting flux associated with small-scale structures of the electric potential is indeed dissipated relatively locally. In contrast, the dissipation of perturbation Poynting flux associated with large-scale features of the electric potential may occur some distance away from the location of the measured \( S_{p||} \), although it is constrained to occur within the boundary of an enclosing equipotential contour.
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References


Kumar, M. (2008), New project will measure electric currents in near-earth space, Space Weather, 6, S09002, doi:10.1029/2008SW000435.


Poynting, J. H. (1884), On the transfer of energy in the electromagnetic field, Philos. Trans., 175, 343–361.


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