Effect of Top and Bottom Boundary Conditions on Symmetric Instability under Full-Component Coriolis Force

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ABSTRACT

The linear stability of a zonal flow confined in a domain within horizontal top and bottom boundaries is examined under full consideration of the Coriolis force. The basic zonal flow is assumed to be in thermal wind balance with the density field and to be sheared in both vertical and horizontal directions under statically and inertially stable conditions. By imposing top and bottom boundary conditions in this framework, the number of wave modes increases to four, instead of two in an unbounded domain, as already reported in studies on internal gravity waves. The four modes are classified into two pairs of high- and low-frequency modes: the high modes are superinertial and the low modes are subinertial. The discriminant of symmetric instability is nevertheless determined by the sign of the potential vorticity of the basic zonal flow, as in the case of an unbounded domain. The solutions satisfying the top and bottom boundary conditions are interpreted as the superposition of incident and reflected waves, revealing that the neutral solutions consist of two neutral plane waves with oppositely directed vertical group velocities. This may explain why the properties of wave behavior, such as the instability criteria, remain the same in both the bounded and unbounded domains, although the manifestation of wave activity, such as the order of dispersion relation, is quite different in the two cases. Furthermore, the slope of the constant momentum surface, the slope of the isopycnic surface including the nontraditional effect of the Coriolis force, and the ratio between the frequencies of gravity and inertial waves form an essential set of parameters for symmetric motion. The combination of these dimensionless quantities determines the fundamental nature of symmetric motions, such as stability, regardless of boundary conditions with and without the horizontal component of the planetary vorticity.

1. Introduction

Symmetric instability is a type of instability in a shear flow under thermal wind balance in which perturbations normal to the flow direction become unstable when the baroclinicity of the shear flow becomes sufficiently strong in both statically and inertially stable states. Historically, this instability was investigated as the stability of a circular vortex responding to axisymmetric perturbations. Later, the problem was simplified by neglecting the curvature effects and was reformulated as the stability of a linear shear flow (Ooyama 1966). In this framework, the dispersion relation governing the stability condition is a quadratic equation with respect to the oscillation frequency, which depends only on the direction of motion (Ooyama 1966). In addition, when the potential vorticity of the shear flow is negative, motion is unstable in certain directions (Hoskins 1974). Such motion associated with symmetric instability consists of two linearly independent motions: inertial oscillation affected by buoyancy and buoyancy oscillation affected by ambient rotation (Xu and Clark 1985). Because of the nonnormality of the governing system of symmetric instability, there exist optimal growing perturbations that have a larger growth rate than the corresponding most unstable normal mode and that also have different directions of motion in the two types of solutions (Heifetz and Farrell 2008; Xu 2010).

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Symmetric instability has been studied, in most cases, under the “traditional approximation” (Eckart 1960), in which the horizontal component of the planetary vorticity is neglected. However, the properties of instability mentioned above are preserved even when the approximation is removed (Itano and Maruyama 2009). Apart from the nontraditional Coriolis effects, various factors affecting symmetric instability—including moist processes (e.g., Bennetts and Hoskins 1979; Emanuel 1982, 1983; Xu 1986), diffusivity (e.g., Emanuel 1979), and stability to three-dimensional perturbations (e.g., Stone 1966, 1970; Gu et al. 1998)—have been studied. A comprehensive review of symmetric instability and its applicability to actual atmospheric and oceanic phenomena is given by Schultz and Schumacher (1999). Among these factors, an enigmatic factor that deserves attention is the effect of boundary conditions. While a number of studies on symmetric instability were carried out in an unbounded domain (i.e., without imposing any boundary conditions), there are also a number of studies that consider top and bottom boundary conditions. The imposition of boundaries excludes pairs of free modes that do not match given boundary conditions, so that the allowable directions of motion are restricted. It is therefore unclear whether the role of the potential vorticity in determining the symmetric stability of the basic flow remains intact. Differences in solutions with and without boundary conditions can be seen, for example, in Kanehisa (2008), who obtained the solutions of symmetric instability by considering the instability as an initial value problem.

Concerning the effects of boundary conditions, another interesting conundrum derived by Xu (2007), who performed a detailed analysis of the nonnormal solutions of symmetric instability under the traditional approximation by imposing boundary conditions, is that when boundary conditions are imposed, the dispersion relation for the oscillation frequency becomes a quartic equation (instead of a quadratic equation in the unbounded case). A similar example of the increased order of the dispersion equation from quadratic to quartic due to the imposition of boundary conditions can be seen in studies of inertio-gravity waves without the traditional approximation (Thuburn et al. 2002; Kasahara 2003; Gerkmam and Shrira 2005). Since the motion associated with symmetric instability reduces to a special case of inertio-gravity waves when the shear of the zonal flow vanishes, there may be a connection between the two seemingly different phenomena, namely the phenomenon that could be unstable and occurs irrespective of the traditional approximation and the phenomenon that is completely stable and occurs exclusively under the nontraditional Coriolis effects. Thus far, however, the possible effects of boundary conditions on symmetric instability, in contrast to the case of inertio-gravity waves (Eckart 1960; Durran and Bretherton 2004; Kasahara 2004), have not been examined explicitly per se even though there have been studies involving boundary conditions (e.g., Jeffery and Wingate 2009).

In this study, the effects of horizontal top and bottom rigid boundaries on symmetric instability are investigated analytically, including the effects of both vertical and horizontal components of the planetary vorticity. Jeffery and Wingate (2009) have examined the instability of vertically sheared zonal flow in such a framework and obtained the corresponding dispersion relation and instability criterion. However, the properties of each solution were not determined and some mathematical treatments were incomplete. Thus, we supplement their analyses for the more generalized situation of a basic zonal flow that has both vertical and horizontal shears. This generalization, as seen in section 6, facilitates the physical interpretation of the stability condition of symmetric instability. The subsequent sections are organized as follows. After derivation of the governing equation for symmetric motions in section 2, we derive the dispersion equation of wave frequency in section 3 for the case of a bounded domain. Although the order of the dispersion equation in the bounded case is quartic, in contrast to quadratic in the unbounded case, the criterion of instability is unchanged regardless of the presence of boundary conditions. Detailed properties of the solutions of the quartic equation are also presented. Meanwhile, the solutions of the unbounded case obtained earlier by Itano and Maruyama (2009) are further examined in section 4. Then, the relationship between the properties of the bounded and unbounded solutions is analyzed in section 5 in the light of the incident and reflected wave dynamics discussed by Durran and Bretherton (2004) for the theory of inertio-gravity waves without the traditional approximation. In section 6, we identify the minimum set of dimensionless numbers that describe the properties of symmetric motion, such as the stability criterion, regardless of boundary conditions and the existence of the horizontal component of the planetary vorticity. Concluding remarks are presented in section 7.

2. Formulation

We use the two-dimensional Boussinesq equations in the Cartesian tangent plane as our basic equations. This set of equations includes both the vertical and horizontal components of the planetary vorticity (i.e., \( f_V \) and \( f_H \)) and describes the zonally symmetric motions on the meridional plane with the full Coriolis forces. They are given as follows:
\[ \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - f_y v + f_H w = 0, \]
\[ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + f_y u = -\frac{1}{\rho_0} \frac{\partial p}{\partial y}, \]
\[ \frac{\partial w}{\partial t} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} - f_y u = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} - \rho g, \]
\[ \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = 0, \]
\[ \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1} \]

Here, \( x, y, \) and \( z \) are directed eastward, northward, and upward, respectively, and \( u, v, \) and \( w \) indicate the velocity components in the corresponding directions; \( p \) indicates the dynamic pressure, \( \rho \) the density deviation from its reference value \( \rho_0, \) and \( g \) is the gravity constant.

Next, linearization is performed around the time-independent basic zonal flow \( U, \) pressure \( P, \) and density \( \rho, \) which are under a quasi-hydrostatic balance

\[ -f_H U = -\frac{1}{\rho_0} \frac{\partial P}{\partial z} - \frac{\bar{p}}{\rho_0} g \tag{2} \]

in the vertical direction, and the geostrophic balance

\[ f_y U = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} \tag{3} \]

in the horizontal direction. Thus, \( U \) satisfies the thermal wind relation

\[ f_y \frac{\partial U}{\partial z} + f_H \frac{\partial U}{\partial y} = g \frac{\partial \rho}{\partial y}, \tag{4} \]

which includes the presence of both the vertical and horizontal shears. Hereafter, \( U \) is assumed to be a linear function of \( y \) and \( z. \) The stratification is represented by the buoyancy frequency \( \overline{N} \) as

\[ \overline{N}^2 = -\frac{g}{\rho_0} \frac{\partial \bar{\rho}}{\partial z}, \tag{5} \]

and we assume that \( \overline{N}^2 \) is positive and constant. Then perturbations, indicated by primes, are governed by the following equations:

\[ \frac{\partial u'}{\partial t} + v' \frac{\partial u'}{\partial y} + w' \frac{\partial u'}{\partial z} - f_y v' + f_H w' = 0, \]
\[ \frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial y} + w' \frac{\partial v'}{\partial z} + f_y u' = -\frac{1}{\rho_0} \frac{\partial \rho'}{\partial y}, \]
\[ \frac{\partial w'}{\partial t} - f_H u' = -\frac{1}{\rho_0} \frac{\partial \rho'}{\partial z} - \frac{\rho'}{\rho_0} g, \]
\[ \frac{\partial \rho'}{\partial t} + \frac{\rho_0 g}{f_y} \left( f_v \frac{\partial U}{\partial z} + f_H \frac{\partial U}{\partial y} \right) v' - \frac{\rho \overline{N}^2 w'}{g} = 0, \]
\[ \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0. \tag{6} \]

The last equation in system (6) allows the use of a stream-function \( \psi \) defined as follows:

\[ v' = -\frac{\partial \psi}{\partial z}, \quad w' = \frac{\partial \psi}{\partial y}. \tag{7} \]

With the aid of (7), the five equations in (6) are reduced to the partial differential equation that determines the form of \( \psi: \)

\[ \frac{\partial^2 \psi}{\partial t^2} \left( \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + f_y \left( f_v - \frac{\partial U}{\partial y} \right) \frac{\partial^2 \psi}{\partial z^2} \]
\[ + 2f_v \left( f_H + \frac{\partial U}{\partial z} \right) \frac{\partial^2 \psi}{\partial y \partial z} \]
\[ + \left[ \overline{N}^2 + f_H \left( f_H + \frac{\partial U}{\partial z} \right) \right] \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{8} \]

The same form of the equation is used by Itano and Maruyama (2009), who obtained the dispersion relation of symmetric motion in an unbounded domain by using the equation. In this analysis, however, we attempt to impose boundary conditions on the vertical, and the effect of the top and bottom boundaries on symmetric motions within the zonal flow is considered, including the full Coriolis forces. As the boundary conditions, we assume that \( w = 0 \) at the top \((z = H)\) and bottom \((z = 0)\) of the model domain. These conditions are expressed in terms of \( \psi \) by

\[ \psi = 0 \quad \text{at} \quad z = 0, H. \tag{9} \]

To facilitate the following discussion, and to clarify its connection with previous studies, we rewrite the coefficients in (8) with the following notation that has been adopted customarily:

\[ F^2 = f_v \left( f_v - \frac{\partial U}{\partial y} \right), \]
\[ S^2 = f_v \left( f_H + \frac{\partial U}{\partial z} \right), \]
\[ \overline{N}^2 = \overline{N}^2 + f_H \left( f_H + \frac{\partial U}{\partial z} \right). \tag{10} \]

Here, \( F \) indicates the inertia frequency including the effect of the horizontal shear of the basic flow, \( S \) is the measure of baroclinicity, and \( N \) is the buoyancy frequency including the nontraditional Coriolis effect. Then, (8) is written as

\[ \frac{\partial^2 \psi}{\partial t^2} \left( \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + F^2 \frac{\partial^2 \psi}{\partial z^2} + 2S \frac{\partial^2 \psi}{\partial y \partial z} + \overline{N}^2 \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{11} \]
Here, the horizontal component of the planetary vorticity is present in $N^2$ and $S^2$. Therefore, although the nontraditional Coriolis parameter is considered, the equation formally takes the same expression as in the case of the traditional approximation.

3. Symmetric motion bounded by two horizontal rigid boundaries

a. Dispersion relation

Because the solution of (11) is assumed to be periodic in time and in the horizontal direction, it can be expressed as

$$\psi = \Psi(z) \exp[i(my - \omega t)].$$  \hspace{1cm} (12)

where $\Psi(z)$ is the vertical structure function. Here, $m$ and $\omega$ indicate the wavenumber in the $y$ direction and the angular frequency, respectively. Substitution of (12) into (11) gives the following second-order ordinary differential equation in $\Psi(z)$:

$$\frac{d^2 \Psi}{dz^2} = \frac{2mS^2}{\omega^2 - F^2} \frac{d^2 \Psi}{dz^2} + m^2N^2 - \omega^2 \Psi = 0.$$  \hspace{1cm} (13)

To eliminate the first-order term in (13), we introduce the transformation

$$\Psi = \eta(z) \exp(i\Gamma z),$$  \hspace{1cm} (14)

where

$$\Gamma = \frac{mS^2}{\omega^2 - F^2} = \frac{mf_V(f_H + \partial U/\partial z)}{\omega^2 - f_V(f_V - \partial U/\partial y)}.$$  \hspace{1cm} (15)

Then, the following constant coefficient ordinary differential equation for $\eta(z)$ is obtained:

$$\frac{d^2 \eta}{dz^2} + \left(\Gamma^2 + m^2 \omega^2 - N^2\right) \eta = 0.$$  \hspace{1cm} (16)

To satisfy the boundary conditions (9), $\eta$ should take the form

$$\eta \propto \sin\left(n \frac{\pi}{H} z\right),$$  \hspace{1cm} (17)

where $n = 1, 2, 3, \ldots$ is an integer indicating the number of modes in the vertical. Therefore, the wavenumber $k$ can be defined as

$$k = n \frac{\pi}{H}.$$  \hspace{1cm} (18)

The dispersion relation is obtained by substituting (17) and (18) into (16). This step gives the following quartic dispersion relation:

$$\omega^4 - \left[2F^2 + (N^2 - F^2) \frac{m^2}{k^2 + m^2}\right] \omega^2 + \frac{F^4}{k^2 + m^2} k^2 + m^2 \omega^2 + (F^2N^2 - S^4) \frac{m^2}{k^2 + m^2} = 0.$$  \hspace{1cm} (19)

Since (19) is independent of the magnitude of the vector $(m, k)$, we can introduce the following quantities to decrease the number of variables in the dispersion relation:

$$\sin \chi = \frac{m}{\sqrt{k^2 + m^2}}, \quad \cos \chi = \frac{k}{\sqrt{k^2 + m^2}}.$$  \hspace{1cm} (20)

More details on the role of $\chi$ will be given later. Now (19) is rewritten as follows:

$$\omega^4 - \left[2F^2 + (N^2 - F^2) \sin^2 \chi\right] \omega^2 + \frac{F^4 \cos^2 \chi}{k^2 + m^2} + (F^2N^2 - S^4) \sin^2 \chi = 0.$$  \hspace{1cm} (21)

This is a generalized form of the quartic equations obtained earlier in the case of $\partial U/\partial y = 0$ by Xu (2007) with the traditional approximation and those obtained by Jeffery and Wingate (2009) without the approximation. In the limit of no wind shear (i.e., $\partial U/\partial y = \partial U/\partial z = 0$), it is reduced to the dispersion relation of pure inertia-gravity waves, as obtained by Thuburn et al. (2002) and Kasahara (2003), although motions in the present case are restricted to two dimensions.

b. Stability criterion

The condition of symmetric instability in the bounded domain is derived as follows. First, in order to satisfy the boundary conditions given by (9), the eigensolutions of (16) must be oscillatory (Brekhovskikh and Goncharov 1985). Therefore, the coefficient of the second term in (16) must be positive; that is,

$$\Gamma^2 + m^2 \omega^2 - N^2 \frac{F^2}{k^2 + \omega^2} > 0.$$  \hspace{1cm} (22)

By substituting the definition (15) of $\Gamma$ into (22), we get the inequality

$$\omega^4 - (N^2 + F^2) \omega^2 + N^2 F^2 - S^4 < 0.$$  \hspace{1cm} (23)

This inequality yields the possible range of $\omega^2$ between $\omega^2_{\text{max}}$ and $\omega^2_{\text{min}}$ as given by
\[ \omega_{\text{max}}^2 = \frac{1}{2} \left[ N^2 + F^2 \pm \sqrt{(N^2 - F^2)^2 + 4S^4} \right], \] (24)

where \( \omega_{\text{max}}^2 \) takes the plus sign in front of the square root and \( \omega_{\text{min}}^2 \) takes the minus sign. It should be noted that these extrema agree with those in (10) of Itano and Maruyama (2009) in the case of an unbounded domain.

It is also useful to rewrite (24) as

\[ \omega_{\text{max}}^2 = \frac{1}{2} \left[ N^2 + F^2 \pm \sqrt{(N^2 + F^2)^2 + 4D} \right], \] (25)

where \( D \) is defined by

\[ D = S^4 - N^2 F^2 = f_V \left( f_H + \frac{\partial U}{\partial z} \right)^2 \]
\[ - f_V \left( f_H - \frac{\partial U}{\partial y} \right) N^2 f_V + f_H \left( f_H + \frac{\partial U}{\partial z} \right). \] (26)

This is the same notation \( D \) introduced by Itano and Maruyama (2009), who showed that \( D \) is proportional to the inner product of the absolute vorticity \([=0, f_H + \partial U/\partial z, f_V - \partial U/\partial y]\) and the gradient of the basic state density \([=(\rho_0/\rho)(0, f_H/\partial z + f_V/\partial y, -N^2)]\), thus the potential vorticity of the basic flow.

Note that if \( D < 0 \), both \( \omega_{\text{max}}^2 \) and \( \omega_{\text{min}}^2 \) are positive so that the basic flow is stable regardless of the direction of the perturbation. However, if \( D > 0 \), \( \omega_{\text{min}}^2 \) is always negative. Hence, there exists a range of \( \chi \) for which the perturbation flow becomes unstable. Therefore, symmetric stability under the top and bottom boundary conditions is again discriminated by the sign of \( D \). In short, \( D \) is the discriminant of symmetric instability in both the bounded and unbounded cases and the condition of symmetric instability is \( D > 0 \) irrespective of the boundary conditions.

c. Subinertial and superinertial modes

Since the dispersion relation given by (21) is quadratic in \( \omega^2 \), we can get high- and low-frequency solutions, \( \omega_{\text{c}}^2 \) and \( \omega_{\text{s}}^2 \), as

\[ \omega_{\text{c}}^2 = \frac{F^2 + N^2 - F^2 \sin^2 \chi}{2} \]
\[ \pm \sqrt{\frac{(N^2 - F^2 \sin^2 \chi)^2}{4} + S^4 \sin^2 \chi}. \] (27)

Note that the inside of the square root is always positive, so that both \( \omega_{\text{c}}^2 \) and \( \omega_{\text{s}}^2 \) are real.

Here, \( \omega_{\text{c}}^2 = F^2 \) is a singular point of the governing equation (13). In the following, we will show that this singular point splits the range of \( \omega^2 \) into two categories: in one set \( \omega^2 \) is larger than \( F^2 \) whereas in the other \( \omega^2 \) is smaller than \( F^2 \).

Differentiating (27) with respect to \( |\sin \chi| \), we obtain

\[ \frac{\partial \omega_{\text{c}}^2}{\partial |\sin \chi|} = \frac{\sqrt{[N^2 - F^2 \sin^2 \chi + 2S^4]^2 - 4S^8 + (N^2 - F^2)^2 \sin^2 \chi + 2S^4}}{\sqrt{(N^2 - F^2)^2 \sin^2 \chi + 4S^4}}. \] (28)

Therefore, by introducing a new symbol \( L \) defined by

\[ L = (N^2 - F^2)^2 \sin^2 \chi + 2S^4, \]

we can rewrite (28) as

\[ \frac{\partial \omega_{\text{c}}^2}{\partial |\sin \chi|} = \frac{\sqrt{L^2 - 4S^8 + L}}{\sqrt{L + 2S^4}} \begin{cases} > 0 \text{ (for } \omega_{\text{c}}^2 \text{)} & \text{ or } \omega_{\text{c}}^2 \text{,} \\ < 0 \text{ (for } \omega_{\text{s}}^2 \text{)} & \text{ or } \omega_{\text{s}}^2 \text{.} \end{cases} \] (29)

From the sign of the derivative in (29), we find that \( \omega_{\text{c}}^2 \) (\( \omega_{\text{s}}^2 \)) increases (decreases) monotonically with \( |\sin \chi| \) from \( \omega_{\text{c}}^2 = F^2 \) at \( |\sin \chi| = 0 \) and reaches its maximum (minimum)—that is, \( \omega_{\text{max}}^2 (\omega_{\text{min}}^2) \)—at \( |\sin \chi| = 1 \). We can state this observation as follows:

\[ \omega_{\text{min}}^2 \leqslant \omega_1^2 < F^2 < \omega_2^2 \leqslant \omega_{\text{max}}^2. \] (30)

This inequality indicates that \( \omega_{\text{c}}^2 \) is always larger than \( F^2 \) whereas \( \omega_{\text{s}}^2 \) is smaller than \( F^2 \). Therefore, it is appropriate to refer to \( \omega_{\text{c}}^2 \) as the superinertial branch of the solution and \( \omega_{\text{s}}^2 \) as the subinertial branch.

As mentioned in the introduction, the governing equation (13) for the problem of symmetric instability in the bounded domain has the same form of that in the theory of the inertia-gravity wave in the bounded domain when the horizontal component of the planetary vorticity is included (Gerkmans and Shrirra 2005). The resultant formalism of (30) is valid for both the problems of symmetric instability and inertia-gravity waves. Nevertheless, there is a notable distinction between the two problems: in the case of inertia-gravity waves, the subinertial solution \( \omega_{\text{s}}^2 \) is always positive, whereas in the case of symmetric instability, the branch \( \omega_{\text{c}}^2 \) can be
negative when the discriminant $D$ as defined by (26) becomes positive.

4. Further investigation of the unbounded solution

To understand what happens in symmetric motion between the two horizontal boundaries, the solution of the unbounded case investigated by Itano and Maruyama (2009) is examined here. By substituting a plane wave solution of the form

$$\psi \approx \exp[i(my + kz - \omega t)]$$

(31)

into (11), we obtain the dispersion relation for the unbounded case as

$$\omega^2 = F^2 \cos^2 \chi' + N^2 \sin^2 \chi' + 2S^2 \sin \chi' \cos \chi'$$

$$= \sin^2 \chi'(F^2 \cot^2 \chi' + 2S^2 \cot \chi' + N^2),$$

(32)

where $\chi'$ is defined in the same way as $\chi$ in (20) and denotes the elevation angle of the streamline measured clockwise from the negative $y$ axis. It is also regarded as the zenith angle of the wavenumber vector $(m, z)$, which is perpendicular to the streamline. Note that unlike the bounded case, here the dispersion relation is quadratic on $\omega$.

Figure 1 shows a typical graph of $\omega^2$ for the unbounded solution given by (32) as a function of $\chi'$ and that for the bounded solution given by (27) as a function of $\chi$. In this example, we specified $(S/F)^2 = (N/F)^2 = 2$ and plotted the curves of $(\omega/F)^2$ in the vertical axis and $\chi$ or $\chi'$ in the horizontal axis. The thick and thin solid lines indicate the superinertial and subinertial branches of the bounded solution, respectively, and the dashed line represents the unbounded solution. All lines pass through $(\omega/F)^2 = 1$ at $\chi = \chi' = 0$, but the profiles of the bounded and unbounded cases are quite different: whereas the bounded case for superinertial $\omega_i^2$ and subinertial $\omega_s^2$, branches, respectively. Dashed line indicates the unbounded case.

in which $\cot \chi_{\text{max}}$ takes the plus sign in front of the square root and $\cot \chi_{\text{min}}$ takes the minus sign. The directions of motion that give the maximum and minimum of $\omega^2$ are perpendicular to each other. This is shown from the relation

$$\cot \chi_{\text{max}} \cdot \cot \chi_{\text{min}} = -1,$$

(34)

which indicates that

$$\chi_{\text{max}} = \chi_{\text{min}} \pm 90^\circ.$$  

(35)

One example of this situation is seen in Fig. 1. Ooyama (1966) derived the same relation under the traditional approximation.

Next, let $D > 0$ for the case of possible instability. In this case, symmetric motion becomes unstable when the elevation angle $\chi'$ of the streamline is within the range of $\chi_0 < \chi' < \chi_1$ where

$$\cot \chi_0 = \frac{-S^2 \pm \sqrt{S^4 - N^2 F^2}}{F^2} = -\frac{S^2 \pm \sqrt{D}}{F^2}.$$  

(36)

Here, $\cot \chi_0$ takes the plus sign in front of the square root and $\cot \chi_1$ takes the minus sign. The two angles of $\chi_0$ and $\chi_1$ indicate the directions where the motion becomes marginally unstable. In Fig. 1, they are located at the two intersections of the dashed curve with the horizontal axis. With reference to these angles, we can show that

$$\cot(\chi_0 + \chi_1) = \cot(2\chi_{\text{max}}) = \cot(2\chi_{\text{min}})$$  

(37)

by applying the addition and double-angle formulas of cotangent to (36) and (33), respectively, and comparing them with each other. This relation indicates that
\[ |x_0 - x_{\text{max}}| = |x_1 - x_{\text{max}}|, \]
\[ |x_0 - x_{\text{min}}| = |x_1 - x_{\text{min}}|. \]

and it implies that the direction of the two neutral motions is symmetrical relative to the direction of the motions that gives the extrema of \( \omega^2 \). The situation mentioned above is illustrated in Fig. 2.

Finally, the group velocity \( c_g = (c_{gy}, c_{gz}) \) is presented. By differentiating (32) by each component of the wavenumber vector \((m, k)\), we obtain, after considerable algebraic calculations,

\[ c_{gy} = \frac{\partial \omega}{\partial m} = -\frac{1}{c_z} \sqrt{\left(F^2 - N^2\right)^2 + 4N^4 \sin^2(2\chi' - x_{\text{max}})} \frac{2(k^2 + m^2)}{2(k^2 + m^2)}, \]
\[ c_{gz} = \frac{\partial \omega}{\partial k} = \frac{1}{c_y} \sqrt{\left(F^2 - N^2\right)^2 + 4N^4 \sin^2(2\chi' - x_{\text{max}})} \frac{2(k^2 + m^2)}{2(k^2 + m^2)}, \]

where \( c_y = (\omega/m) \) and \( c_z = (\omega/k) \) indicate the horizontal and vertical components of the phase velocity, respectively. From (39), we can verify that

\[ (m, k) \cdot (c_{gy}, c_{gz}) = 0. \]

This indicates that the directions of the phase velocity and the group velocity are orthogonal. Since the wavenumber vector is perpendicular to the streamline, the direction of the group velocity becomes parallel to the streamline and the motion.

5. Incident and reflected wave dynamics

In this section, we give an interpretation of the bounded solution obtained in section 3 by decomposing it into two plane waves. By combining (12) with (14), the solution imposed by top and bottom rigid boundary conditions is expressed in the form

\[ \psi \propto \sin(kz) \exp \left\{ i \left[ m \left( y + \frac{S^2}{\omega^2 - F^2} z \right) - \omega t \right] \right\} \]

and the dispersion relation is given by the quartic equation (21). The solution of the form (41) can be represented as a superposition of upward and downward propagating plane waves that form a pair of incident and reflected waves at the top and bottom boundaries (Eckart 1960; Durran and Bretherton 2004; Kasahara 2004; Xu 2007). Accordingly, (41) can be rewritten as

\[ \cot \chi \frac{\omega^2 - F^2}{\omega^2} \pm \frac{k}{m} \]

where the subscript A refers to the case of plus in front of the square root and the subscript B the case of minus. The last square root term in (43) was obtained by rewriting \( k/m = (\cot \chi) \) with the aid of the quartic dispersion relation (21).

With regard to (42), the elevation angle of the streamline of wave A, \( \chi_A(\omega) \), and that of wave B, \( \chi_B(\omega) \), are given as

\[ \psi \propto \exp \left\{ i \left[ m \left( y + \frac{S^2}{\omega^2 - F^2} m + k \right) z - \omega t \right] \right\} \]

\[ - \exp \left\{ i \left[ m \left( y + \frac{S^2}{\omega^2 - F^2} m - k \right) z - \omega t \right] \right\}. \]
\[ \cot(x_A + x_B) = \cot(2x_{\text{max}}) = \cot(2x_{\text{min}}). \]  

This relation indicates that

\[ |x_A - x_{\text{max}}| = |x_B - x_{\text{max}}| \quad \text{and} \quad |x_A - x_{\text{min}}| = |x_B - x_{\text{min}}|. \]  

Thus, the direction of the motions associated with waves A and B is symmetrical relative to the direction of the motions that gives the extrema of \( \omega^2 \). This leads to the fact that the group velocity of waves A and B is, as reported by Durran and Bretherton (2004) in the case of internal gravity waves, opposite in the vertical direction since, according to (39) and (35), the signs of \( c_{gz} \) whose elevation angle is in the range of \( x_{\text{max}} < \chi' < x_{\text{min}} \) and \( x_{\text{min}} < \chi' < x_{\text{max}} \) are different to each other because of the presence of \( \sin[2(\chi' - x_{\text{max}})] \) in (39). This situation is illustrated in Fig. 3.

Now, considering the facts derived above, we evaluate the elevation angle of the motions for waves A and B when the square of its frequency becomes its maximum and minimum (i.e., \( \omega^2 = \omega^2_{\text{max}}, \omega^2_{\text{min}} \)). By substituting the expression (24) of \( \omega^2_{\text{max}} \) and \( \omega^2_{\text{min}} \) into \( \omega^2 \) of (43) and comparing the results with (33), we find that

\[ \cot x_A(\omega_{\text{max}}) = \sqrt{(N^2 - F^2)/2S^2} + 1 - \frac{N^2 - F^2}{2S^2} = \cot x_{\text{max}}, \]

\[ \cot x_A(\omega_{\text{min}}) = -\sqrt{(N^2 - F^2)/2S^2} + 1 - \frac{N^2 - F^2}{2S^2} = \cot x_{\text{min}}. \]  

Therefore,

\[ x_A(\omega_{\text{max}}) = x_B(\omega_{\text{max}}) = x_{\text{max}}, \]

\[ x_A(\omega_{\text{min}}) = x_B(\omega_{\text{min}}) = x_{\text{min}}. \]

This means that when the square of the frequency takes its extremum value, the angles \( x \) of the two plane waves that constitute the bounded solution coincide at the angle \( \chi' \) where the square of the frequency of the unbounded solution gives its corresponding extremum value. This result explains why \( \omega^2_{\text{max}} \) and \( \omega^2_{\text{min}} \) are invariant regardless of the boundary conditions.

Next, the neutral case is considered. Substituting \( \omega^2 = 0 \) into (43) gives

\[ \cot x_A(0) = \frac{-S^2 \pm \sqrt{S^4 - N^2F^2}}{F^2} = \cot x_0. \]

This indicates that the two plane waves that constitute the neutral solution, satisfying the top and bottom boundary conditions, are both neutral and both \( x_A \) and \( x_B \) coincide with one of the two neutral angles of the unbounded solution. In this context, when the incident wave is stable, the reflected wave is also stable, and vice versa. This fact, however, is not surprising since the frequencies of incident and reflected waves do not change before and after reflection.

An example of the neutral solution \( (\omega = 0) \) under the top and bottom boundary conditions and its constituent plane waves is shown in Fig. 4. As indicated by (42), the neutral solution becomes a superposition of two neutral plane waves whose horizontal wavenumber is the same but vertical wavenumber is different. The slopes of streamlines of the two constituent plane waves indicate the two neutral directions of motion given by the unbounded solution (Fig. 4, middle and bottom). When these two waves are superposed, the resultant streamlines become closed and their ridges and troughs are also inclined (Fig. 4, top). Note that the inclination of contours in the superposed solution (Fig. 4, top), which may be represented by the inclination of zero lines, is different from the neutral angles indicated by the direction of the streamlines of the two plane waves (Fig. 4, middle and bottom).

![Fig. 3. A schematic diagram illustrating the direction of motion of the two plane waves \( x_A \) and \( x_B \), which constitute the bounded solution. The sign of the vertical component of the group velocity \( c_{gz} \), for a northward phase velocity (i.e., \( c_y > 0 \)) is shown here. Thick arrows in the upper right indicate the directions of the phase velocity \( c \) and the group velocity \( c_g \) for plane wave A, whereas the arrows in the lower left indicate those for plane wave B. When \( c_y < 0 \), the sign of \( c_{gz} \) and the directions of the thick arrows are reversed.](image)
6. Dimensionless numbers describing symmetric instability

In the above analyses, we have shown that the fundamental characteristics of symmetric motions, such as the instability criterion and the maximum and minimum of the frequency squared, are unchanged whether the boundary conditions are imposed or not. Moreover, in the governing equation of symmetric instability \((11)\), the effect of the horizontal component of the planetary vorticity is included in the parameters \(N^2\) and \(S^2\). Therefore, the properties of symmetric instability are solely determined by the three parameters \(N, F, \) and \(S\) irrespective of boundary conditions and with and without the horizontal component of the planetary vorticity.

We now consider the dimensionless numbers as the ratio of these three parameters:

\[
\left( \frac{F}{S} \right)^2, \quad \left( \frac{N}{F} \right)^2, \quad \left( \frac{S}{N} \right)^2.
\]  

(50)

The first parameter \((F/S)^2\) indicates the slope of the constant momentum surface by defining

\[
M = U - f_V y + f_H z
\]  

(51)
as the angular momentum of the basic zonal flow. It is shown that

\[
\left( \frac{F}{S} \right)^2 = \frac{f_V (f_V - \partial U/\partial y)}{f_V (f_H + \partial U/\partial z)} = \frac{f_V - \partial U/\partial y}{f_H + \partial U/\partial z} = -\frac{\partial M/\partial y}{\partial M/\partial z} = \left( \frac{dz}{dy} \right)_M.
\]  

(52)

The second parameter \((N/F)^2\) indicates the ratio between the frequencies of the gravity and inertial waves, including the effect of vertical and horizontal shears of the basic flow.

The third parameter \((S/N)^2\) is interpreted as the slope of the “effective” isopycnic surface (i.e., the constant density surface including the effect of the nontraditional Coriolis force). To demonstrate this, the new variable \(\Phi\) is introduced:

\[
\Phi = \frac{-\bar{p}}{\rho_0} + f_H M = \frac{-\bar{p}}{\rho_0} + f_H (U - f_V y + f_H z).
\]  

(53)

Here, the first and second terms on the right-hand side indicate the buoyancy and the nontraditional Coriolis effect acting on the basic zonal flow, respectively. From this, it is shown that

\[
\left( \frac{S}{N} \right)^2 = \frac{f_V (f_H + \partial U/\partial z)}{N^2 + f_H (f_H + U/\partial z)} = -\frac{\bar{p}}{\rho_0} \frac{\partial \Phi/\partial y}{\partial \Phi/\partial z} = \left( \frac{dz}{dy} \right)_\Phi
\]  

(54)

with the aid of (4) and (5). Because of the above result, it seems reasonable to conclude that \((S/N)^2\) indicates the slope of the constant density surface including the nontraditional effect of the Coriolis force. Note that this surface reduces to the original isopycnic surface under the traditional approximation \((f_H = 0)\).

These three dimensionless numbers in (50) constitute the fundamental parameters that describe the properties of symmetric instability. An example of the stability diagram, indicating the maximum instability described with such pairs of dimensionless numbers, is shown in Fig. 5. Here, \(\omega_{\text{min}}^2\) normalized by \(F^2\)
as shown in (15) of Itano and Maruyama (2009). This equation indicates the tendency of external forces acting on the plane normal to the zonal direction. Here, the first parameter \((F/S)^2 = (dz/dy)_M\) and the third parameter \((S/N)^2 = (dz/dy)_\Phi\) are found as the ratio of the first column to the second column in the first and second row of the matrix in (57), respectively. Therefore, the equation indicates the northward and vertical components of the external forces for \(-f_Vu'\) and \(f_Hu' - \rho'g/\rho_0\) are constant along the constant \(M\) and \(\Phi\) surfaces, respectively.

It should be noted that the discriminant \(D\) of symmetric instability given by (26) is the determinant of the matrix in (57) and the condition of symmetric instability \(D > 0\) can be rephrased as

\[
\left(\frac{dz}{dy}\right)_{\Phi} > \left(\frac{dz}{dy}\right)_M
\]

with the aid of (52) and (54). This is the generalization of the result obtained under the traditional approximation, which states, as another expression of the instability criterion based on the potential vorticity, that the zonal flow is symmetrically unstable when the isopycnic surface is steeper than the constant momentum surface (e.g., Lilly 1986; Holton 1992; Schultz and Schumacher 1999).

7. Further remarks and conclusions

The linear instability seen in the geophysical flows can be classified into two categories. The first type of instability occurs as a result of resonance between two waves propagating in opposite directions (Sakai 1989). For this type of instability to occur, boundary conditions need to be imposed on the basic flow. Barotropic instability, baroclinic instability, Kelvin–Helmholtz instability, etc., belong to this category. The second type of instability occurs when the restoring force acting on a fluid parcel is insufficient in certain environmental conditions. This type of parcel instability does not require any boundary conditions. Static instability and inertial instability belong to this category. Symmetric instability, whose mathematical expression is isomorphic to that of both static and inertial instability (Xu and Clark 1985), also belongs to this category. Accordingly, it basically requires no boundary conditions. Nevertheless, there are also a number of studies that have investigated symmetric instability by imposing boundary conditions. Here the question arises: what is the role of boundary conditions on symmetric instability?

One of the notable consequences resulting from the imposition of boundary conditions is that the dispersion
relation becomes quartic with respect to the angular frequency, although it is quadratic in the case of no boundary conditions. The quartic equation contains two pairs of high-frequency and low-frequency solutions: the former solutions are superinertial, whereas the latter are subinertial; it is the latter solutions alone that could become unstable when the basic state is inertially stable. Despite these apparent differences in the manifestation of wave activity, appearing in the order of the dispersion relation, the properties of wave behavior, such as the stability criterion and the frequency extrema, are identical in the two cases (with and without boundary conditions). With regard to this conundrum, Durran and Bretherton (2004) gave an insightful perspective on such modes originating from the imposition of boundaries. They make a physical distinction between modes such as edge waves, which are linearly independent of free modes in unbounded solutions, and those emerging as “new modes” (Thuburn et al. 2002) when the angles of incidence and reflection differ from each other. Unlike the former modes, the latter modes consist of a pair of free modes in unbounded solutions. The subinertial modes considered here apparently belong to the latter, where the individual plane waves that constitute the solution in the case of imposition of boundary conditions behave as if they are the solutions in an unbounded domain. Also, since the frequencies of incident and reflected waves are invariant before and after reflection, the frequency or the growth rate (if instability occurs) of the subinertial modes is identical to that of their constituent plane waves. Therefore, even though boundary conditions are imposed, the mechanism of instability does not change. In this respect, the type of instability remains in the second category of parcel instability.

Based on the above results, we identified the minimum set of parameters that describe the properties of symmetric motion. The slope of the constant momentum surface of the basic zonal flow, the slope of the isopycnic surface including the nontraditional effect of the Coriolis force, and the ratio between the frequency of the gravity wave and that of the inertial wave determine the essential nature of symmetric motion, such as stability and the maximum growth rate, irrespective of the boundary conditions and the presence of the horizontal component of planetary vorticity.

Stone (1966, 1970) examined the Eady (1949) problem without making the geostrophic approximation (i.e., the stability of a zonal shear flow bounded by two horizontal rigid boundaries) and investigated symmetric instability as a special case of vanished zonal wavenumber. According to the descriptions in his articles, the study was planned to reveal the factor that determines the dominant types of instabilities that appear in planets. In this kind of study, however, the horizontal component of the planetary vorticity, as was considered here, becomes important, especially for planets like Jupiter, whose convections are possibly deep. In this respect, the present analysis may have some contribution to the understanding of motion on such planets. On the other hand, there arises another question of whether the present result is applicable under the influence of another important factor, viscosity: there is a possibility of significant modification or alteration to the stability criterion, the dispersion relation, and the incident and reflected wave dynamics, which were derived here, since the order of governing equations including the viscous term is higher than that of the inviscid counterparts and since their solutions are singular with regard to the diffusion parameters (McIntyre 1970; Emanuel 1979). The extension of the present analyses to cover the viscous case will be an important subject in the future.

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