On the interaction of internal gravity waves with a magnetic field – I. Artificial wave forcing

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Accepted 2009 August 26. Received 2009 August 26; in original form 2009 July 7

ABSTRACT
We present results from numerical simulations of the interaction of internal gravity waves (IGW) with a magnetic field. In accordance with the dispersion relation governing IGW in the presence of magnetism and rotation, when the IGW frequency is approximately that of the Alfvén frequency, strong reflection of the wave occurs. Such strong reflection markedly changes the angular momentum transport properties of the waves. In these simple models a strong, time-independent shear layer develops, in contrast to the oscillating shear layer that develops in the purely hydrodynamic case.

Key words: MHD – Sun: interior – Sun: magnetic field.

1 INTRODUCTION
Helioseismic observations of the solar radiative interior have proven challenging to explain theoretically. The internal rotation of the Sun has been inferred from observations (Thompson et al. 1996, 2003) to be rotating differentially, with the equator spinning faster than the poles, throughout the convection zone. Conversely, the radiative interior is rotating relatively uniformly, in radius and latitude. The shear layer that mediates the transition between the differentially rotating convection zone and the uniformly rotating radiative interior has become known as the tachocline.

The convection zone appears to be rotating differentially through the action of Reynolds and Maxwell stresses. However, in order to break the restrictive Taylor–Proudman theorem, the incorporation of a subadiabatic tachocline is necessary (Rempel 2005; Miesch, Brun & Toomre 2006). The existence of the tachocline, however, owes itself to both the stresses that form the differentially rotating convection zone and the stresses which enforce uniform rotation in the radiative interior. Therefore, fundamental understanding of the rotation profile of the solar interior, from bottom to top, requires some comprehension of angular momentum transport processes in the radiative interior.

Two transport processes have been suggested to explain the uniform rotation of the solar radiative interior: internal gravity waves (IGW) and the Lorentz force associated with a primordial poloidal magnetic field. IGW have been known to be efficient transporters of angular momentum for decades. In the Earth’s stratosphere, such waves are responsible for the equatorial oscillating mean flows known as the quasi-biennial oscillation (QBO). Such wave-driven oscillations have also been observed in experiments (Plumb & McEwan 1978) and in Jupiter’s atmosphere (Leovy, Friedson & Orton 1991). IGW transport angular momentum in much the same way that turbulence does through the divergence of the Reynolds stress. However, because IGW can propagate over long ranges and transport angular momentum only where they are dissipated and/or attenuated, they are considered long-range transporters of angular momentum. Although they have yet to be conclusively observed in the Sun, they are inevitably generated at the convective–radiative interface. For these reasons, IGW have been invoked as a possible mechanism for long-range angular momentum transfer in the deep radiative interior.

Previous treatments of angular momentum transport by IGW have been almost exclusively done in the absence of magnetic fields. This is likely due to the fact that the early development of the treatment of angular momentum transport by IGW was done in the atmospheric science community, where magnetic fields are surely negligible. However, in the solar interior magnetic fields are hardly insignificant. At the very least, solar physicists expect a (strong) toroidal field at the base of the solar convection zone as the likely source of the sunspot cycle. This is precisely the region in which IGW are generated; therefore, such a field undoubtedly affects the propagation and hence the angular momentum transport by gravity waves. In this paper we investigate the interaction of IGW with magnetic fields in the simple scenario of a single, artificially imposed prograde and retrograde wave.

2 NUMERICAL MODEL

2.1 Equations
We solve the coupled magnetohydrodynamic (MHD) equations in the anelastic approximation. This approximation is employed in order to filter sound waves, thus allowing a larger numerical
time-step. It is valid when flow velocities, and the Alfvén speed, are significantly less than the speed of sound. A further constraint is that the thermodynamic perturbations are small relative to reference state values. These constraints are easily satisfied in the simulations and in the solar interior (below 0.98 $R_{\odot}$). We solve the following equations for the flow and field relative to the prescribed reference state, which is generated from a polynomial fit to the standard solar model, Model S (Christensen-Dalsgaard et al. 1996):

$$\nabla \cdot \mathbf{v} = 0$$  
(1)

$$\nabla \cdot \mathbf{B} = 0$$  
(2)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P - C\mathbf{\Omega} \times \mathbf{v} + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) + \nabla (\nu \nabla \cdot \mathbf{v})$$

$$+ \frac{1}{3} \nabla (\nabla \cdot \mathbf{v})$$  
(3)

$$\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T = -v_i \left( \frac{\partial T}{\partial r} - (\gamma - 1) T h_i \right)$$

$$+ (\gamma - 1) T h_i v_i + \gamma \kappa \left[ \nabla^2 T + (h_p + h_s) \frac{\partial T}{\partial r} \right]$$  
(4)

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$  
(5)

Equation (1) represents the mass continuity equation within the anelastic approximation. $\mathbf{v}$ represents the reference state density and $\mathbf{v}$ is the velocity. Magnetic effects are incorporated through the Lorentz force, $\mathbf{J} \times \mathbf{B}$, where $\mathbf{J}$ represents the current density and $\mathbf{B}$ the magnetic field. Equation (3) represents the momentum equation, with $P$ being the pressure, $C$ the co-density (Braginsky & Roberts 1995; Rogers & Glatzmaier 2006), $g$ the gravitational acceleration, $\mathbf{\Omega}$ the rotation rate and $\nu$ the viscous diffusivity. Equation (4) is the energy equation written as a temperature equation, where $T$ represents the temperature, $v_i$ is the radial component of the velocity, $h_i$ is the inverse density scaleheight and $\gamma$ is the adiabatic index. Equation (2) represents the conservation of magnetic flux, while equation (4) represents the magnetic induction equation with $\eta$ representing the magnetic diffusivity, which we assume is constant. Reference state variables such as density and temperature are denoted by overbars.

For numerical convenience and to ensure strict adherence to (1), we solve equation (3) using a vorticity-streamfunction formulation. Taking the curl of equation (3), the vorticity equation in the $\hat{z}$ direction ($\omega = \nabla \times \mathbf{v}$) in cylindrical geometry becomes

$$\frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla) \omega = 2(\Omega + \omega) h_i v_i - \frac{\tau}{T} \frac{\partial T}{\partial \phi} - \frac{1}{\rho T} \frac{\partial T}{\partial r} \frac{\partial \rho}{\partial \phi}$$

$$+ \frac{1}{\rho} ((\mathbf{B} \cdot \nabla) J - h_j J B_j) + \tau \nabla^2 \omega,$$  
(6)

where the streamfunction $\psi$ is defined by

$$\nabla \psi = \mathbf{v} \times \psi$$  
(7)

resulting in the following relation between the streamfunction and vorticity:

$$\omega = \frac{\partial^2 \psi}{\partial r^2} + \left( \frac{1}{r} - h_j \frac{\partial}{\partial r} \right) \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2}.$$  
(8)

Similarly, if the magnetic field is represented as the curl of the magnetic vector potential $A$, such that

$$\mathbf{B} = \nabla \times \mathbf{A}$$  
(9)

then, assuming a Coulomb gauge, the current density $\mathbf{J}$ is written as

$$\mathbf{J} = -\frac{1}{\mu} \nabla^2 \mathbf{A}$$  
(10)

and the induction equation (4) becomes

$$\frac{\partial \mathbf{A}}{\partial t} = -((\mathbf{v} \cdot \nabla) \mathbf{A} + \eta \nabla^2 \mathbf{A}).$$  
(11)

### 2.2 Numerical technique

Equations (4), (6), (8) and (11) are solved in two dimensions in cylindrical coordinates, representing an equatorial slice of the Sun. The computational domain extends from 0.05 to 0.70 $R_{\odot}$, representing the bulk of the radiative interior. The radial domain is discretized using a finite difference scheme; in the longitudinal direction, the variables are expanded in a Fourier series. The solution is advanced in time using the second-order Adams–Bashforth method for the non-linear terms and an implicit Crank–Nicolson method for the linear terms. All models have 1000 uniform grid points in radius and 512 longitudinal grid points. 1 The temperature boundary conditions are isothermal on both the bottom and top boundaries. On the bottom boundary the velocity boundary conditions are impermeable and stress free, while at the top boundary the velocity conditions are stress free and the wave forcing is accomplished through the other boundary condition (Rogers, MacGregor & Glatzmaier 2008). The model is parallelized using Message Passing Interface (MPI).

In all magnetic models, we specify a toroidal magnetic field between 0.90 $R_{\odot}$ and 0.95 $R_{\odot}$ with the following form:

$$A(r) = A_e \exp^{-0.1925 (r - 0.925 R_{\odot})^2 / 0.0002},$$  
(12)

where $R_{\odot}$ denotes the radius of the radiation zone, so that the toroidal magnetic field changes sign at mid-depth of the magnetized layer and $A_e$ represents the magnitude of the magnetic field. $A_e$ is varied substantially, leading to field strengths between 10$^2$ and 10$^3$ G for the various models (see Table 1).

### 2.3 Wave driving

For this first investigation, a single prograde wave and a retrograde wave of a specified frequency and wavenumber are driven at the top boundary through the boundary condition on the vertical velocity. This driving, while artificial, allows for more direct comparison with the analytic model in MacGregor & Rogers (2009) and, therefore, simplified interpretation. The details of the artificial wave driving are found in Rogers et al. (2008). In a forthcoming paper, wave driving by a simulated convection zone will be studied.

The control parameters, such as the wave frequency, wavenumber and amplitude as well as the field strength and diffusivities, are varied in order to investigate the effect of the field on angular momentum transport. The models and their parameters are listed in Table 1.

### 3 THE INTERACTION OF INTERNAL GRAVITY WAVES AND MAGNETIC FIELD

In Barnes, MacGregor & Charbonneau (1998), it was argued that the presence of a magnetic field limited the wavevectors and frequencies of vertically propagating IGW. This idea was expanded

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1 Model M4 was tested with 1500 radial grid points and 1024 longitudinal grid points, yielding the same results as the lower resolution model.
Clearly, the Alfvén–196 cm s (ωs cm Gc m G. is the horizontal wavenumber of the driven wave, κ × are C Model parameters. η ω (10 ≈ 10 (A is the amplitude described in equation (12); note that values of 1.66 and (10 mf cm s × cm ́s 2009 RAS, MNRAS A is the amplitude of the driven A refers to the Alfvén ω´401, A is the amplitude of the driven gravity wave. A strong field will be taken to indicate ω (10 ⋅ 10 mf cm s × cm ́s 2009 The Authors. Journal compilation ω is the frequency of the driven wave and ω is the frequency of the driven wave. M33 15 10 0.8 0.5530 1 0.3 1 OSC M32 15 10 0.6 0.8300 1 0.1 1 OSC M31 15 10 1 NF 1 1 NA OSC M30 18 10 1 1.6600 1 1 1 STA M29 15 12 1 1.6600 1 1 1 STA M28 15 10 1 1.6600 1 1 0.8 STA M27 20 10 1 1.6600 1 1 1 STA M26 15 10 1 1.6600 1 1 0.8 STA M25 10 10 1 1.6600 1 1 0.8 STA M24 15 10 1 0.1660 1 1 1 OSC M23 15 20 1 1.6600 1 1 1 STA M22 15 10 1 1.6600 0.8 1 1 STA M21 15 10 1 1.6600 2 0.5 1 OSC M20 15 10 1 0.320 1 1 1 OSC M19 15 10 0.5 0.1660 1 0.1 1 STA M18 15 10 0.1 0.1660 0.01 0.01 1 STA M17 15 10 0.1 0.1660 0.1 0.1 1 STA M16 15 10 0.1 1.6600 0.1 0.1 1 STA M15 15 10 0.6 1.6600 1 1 1 OSC M14 15 10 0.8 1.6600 1 1 1 STA M13 15 10 1 1.6600 2 1 1 OSC M12 15 10 1 0.1660 2 0.5 1 OSC M11 15 10 1 0.8300 1 1 1 OSC M10 15 10 1 1.6600 2 0.5 1 OCC M9 15 10 1 1.6600 1 2 1 OSC M8 15 10 1 1.6600 1 1 2 OSC M7 5 10 1 1.6600 1 1 1 STA M6 10 10 1 1.6600 1 1 1 STA M5 15 10 1 1.6600 1 1 1 STA M4 15 10 1 1.6600 1 1 1 STA M3 15 10 1 0.1660 1 1 1 OSC M2 15 10 1 0.0166 1 1 1 OSC M1 15 10 1 NF 1 1 NA OSC

in MacGregor & Rogers (2009), where it was argued that gravity waves could be reflected in the presence of a magnetic field because of the vertically varying properties. It was further argued that this reflection, and the subsequent duct that could be set up between the magnetized layer and the base of the convection zone, could contribute to the angular momentum transport and helicity in the region. In the simulations presented here, we investigate these ideas using numerical simulations which can account for non-linear interactions and momentum transport.

In Fig. 1 we show the angular velocity as a function of time and radius, for three models: one without a magnetic field (a, M1), one with a weak magnetic field (b, M3) and, finally, one with a strong magnetic field (c, M4). In all cases, a ‘weak’ magnetic field will be taken to mean a field strength for which ωA ≪ ωIGW, where ωA refers to the Alfvén frequency and ωIGW is the frequency of the driven gravity wave. A strong field will be taken to indicate a field in which ωA ≈ ωIGW.2 One can readily see that the weak-field case is virtually identical to the case without a magnetic field. Equally recognizable is the significant difference between those models and the strong-field case (c). Several differences are worth noting. First, in the strong-field case, there is no oscillatory be-

2 Clearly, the Alfvén frequency varies substantially over the magnetized region; the value intended is the maximum Alfvén frequency in the region.

haviour. Second, the amplitude of the shear flow in (c) is significantly larger than the maximum value obtained in (a) or (b). Finally, the strong magnetic case does not show a double-peaked shear layer. Balancing this strong shear is an oppositely directed flow at the bottom of the computational domain, with the weaker amplitude being dictated by its larger density. Furthermore, it appears that the sign of the stationary shear that develops is random. This marked discrepancy between the no-field and strong-field cases indicates a strong wave–magnetic-field interaction, which contributes to the angular momentum transport and which we will presently describe.

We attribute the discrepancy between weak- and strong-field cases to two main factors. First, the strong field increases the reflection coefficient discussed in MacGregor & Rogers (2009) allowing for more efficient reflection. This reflection causes a large divergence (convergence) of the Reynolds stress contributing significantly to the strong shear that develops. Second, in the strong-field case, despite a large horizontal flow in which the angular velocity is greater than the phase speed of the driven IGW, no non-linear transfer of energy to higher harmonics ensues. In the non-magnetic models presented in Rogers et al. (2008) it is found that the non-linear transfer of energy to higher harmonics, associated with a critical layer, facilitates the reversal of the mean flow. In the magnetic models presented here waves with opposite sense to the mean flow are continually shifted to higher frequencies, causing them to
Figure 1. Mean angular velocity as a function of radius and time for models M1 (a), M3 (b) and M4 (c). Only the outer 50 per cent in radius of our simulated stable region is depicted here. Horizontal red lines represent the initial position of the toroidal field. Prograde motion is represented by red, while retrograde motion is represented by blue. (a) and (b) reproduce the main features of a gravity-wave-driven mean flow oscillation, while (c) is markedly different, showing a stationary strong shear flow. It should be noted that the colour table is saturated at $\pm 10^{-6}$ rad s$^{-1}$, with the peak angular velocity in (a) and (b) being $\approx 4 \times 10^{-6}$ rad s$^{-1}$. However, the peak angular velocity in (c) is $\approx 10^{-5}$ rad s$^{-1}$.

propagate to the deep interior and dissipate there hence causing an acceleration at the bottom boundary.

In Fig. 2 we show contours of the vertical velocity, as a function of time and radius, for the same three models shown in Fig. 1 (although over a reduced time-span). The horizontal lines overlaid on these plots are the initial position of the magnetic field (in those models which included a magnetic field). One can promptly see the difference in the models. In (a) and (b), contours of a constant phase move upwards in time, as expected of pure gravity waves in which the phase and group velocity are perpendicular. However, in Fig. 2(c), one sees a remarkable difference. Near the centre of the original magnetic field, where the magnetic field passes through zero, the phase propagation changes sign and becomes upwards. Simultaneously, the sign of the vertical velocity changes. Both of these indicate flow reflection. Moving upwards, we see that a standing wave is set up between the top of the field and the top of the domain. This is precisely the ducting behaviour discussed in MacGregor & Rogers (2009).

The strong reflection in the magnetic layer causes a large divergence of the Reynolds stress. Fig. 3 shows the divergence of the Reynolds and Maxwell stresses for the weak and strong magnetic field cases. In (a), the Reynolds stresses for the weak (dotted line) and strong (solid line) cases are shown. While both models show fairly strong Reynolds stress at the very top of the domain, only the strong-field case shows significant Reynolds stress in the region of the magnetic field (denoted by the vertical lines). The Maxwell stress in both the weak- and the strong-field cases is significantly lower, by at least an order of magnitude. Therefore, it is the reflection of the flow which causes an increased divergence of the Reynolds stress and subsequent enhanced angular momentum transport.

Integrated in time, the Reynolds and Maxwell stresses maintain the mean flow shown in Fig. 1. For the weak-field case, the flow eventually reaches an amplitude which is equal to or greater than the horizontal phase speed of the driven wave, typically described as a critical layer. In models without strong magnetic fields, such a critical layer induces energy transfer from the driven wave to higher harmonics, as discussed in Rogers et al. (2008). However, for the strong-field case, despite the large mean angular velocity attained, no transfer to higher harmonics ensues and therefore no reversals occur.

3.1 Transmission

As demonstrated above, when a strong magnetic field is present IGW can be effectively reflected, with subsequent efficient angular momentum transport. If a substantial fraction of the wave energy is reflected, little is left to be transmitted to the deeper radiative region below the field. In Fig. 4, we show the kinetic energy spectrum of waves in model M1 (no field) compared with the strong-field case M4. Clearly, in the no-field case, energy is transmitted to lower radii, with mainly lower frequencies being dissipated. However, in the strong-field case a substantial amount of energy is dissipated and/or transferred, particularly at high wavenumbers. Notably, the highest wavenumbers are also those that likely have Alfvén frequencies in excess of the IGW frequency, leading to strong reflection. One can also see that the strong field induces mode leakage, whereby energy spreads into neighbouring wavemodes, rather than staying restricted to the harmonics of the driven wavenumber.
Wave–field interaction

3.2 Dependencies

In MacGregor & Rogers (2009), the reflection coefficient strongly depends on the Alfvén frequency compared to the driven gravity wave frequency. When the Alfvén frequency is approximately that of the gravity wave frequency the reflection coefficient is 1; it is possible to have reflection away from this matching, albeit at a lower efficiency. Here we have confirmed strong interaction when $\omega_A \approx \omega_{\text{IGW}}$ and shown that the angular momentum transfer can be remarkably altered in the presence of such interaction. Two-dimensional turbulence studies have shown that hydrodynamic behaviour is recovered in MHD simulations when the magnetic Reynolds number is small (Tobias, Diamond & Hughes 2007). In light of these results, and the general dependence of dynamic behaviour on diffusivities of all varieties, we investigate these dependencies presently.

We ran a total of 32 magnetic models, listed in Table 1. Of the models run roughly, half demonstrated non-oscillating shear flows. In addition to the requirement that the Alfvén frequency be approximately that of the IGW frequency for strong reflection, we find additional dependencies on the wavenumber and amplitude of the driven IGW as well as on the diffusivities. Empirically, we find that the ratio

$$R_w = \frac{u u_A c_p R^3}{\nu \kappa \eta}$$

must be above approximately $10^{16}$. Where $u$ is the driven IGW amplitude, $u_A$ is the Alfvén velocity and $c_p$ is the phase velocity of the IGW ($\omega/m$). Fig. 5 shows this ratio for each of the models run, with stars representing stationary shear flows, crosses representing oscillating shear flows and diamonds representing models in which $\omega_{\text{IGW}} > \omega_A$. The IGW–field interaction is strongest when $\omega_A > \omega_{\text{IGW}}$, as described in MacGregor & Rogers (2009). Additionally, diffusive effects must be small enough so that Alfvén waves and IGW can propagate a substantial distance without being dissipated.

It is difficult to estimate what the ratio in (13) would be for the Sun. The diffusivities are substantially lower in the solar interior,
indicating a large ratio and strong shear flows. However, the amplitudes of the waves are significantly lower as well. Furthermore, it is not clear what effect the broad spectrum of waves generated at the base of the solar convection zone will have. Despite this uncertainty, it is clear that some subset of IGW generated at the base of the convection zone will interact with the field and likely be reflected and/or strongly attenuated. Detailed simulations of wave generation in the presence of a strong toroidal field will be presented in Paper II.

4 DISCUSSION

We have conducted numerical experiments of the interaction of IGW with an imposed large-scale magnetic field to understand the effect of that field on the propagation, dissipation and angular momentum transport by such waves. Like previous work, we have found that when the field strength is such that the Alfvén frequency is similar to or greater than the IGW frequency, waves are efficiently reflected. The reflection causes the divergence of the Reynolds stress to increase dramatically with a corresponding increase in the angular momentum transport. Unlike purely hydrodynamic cases, a strong field appears to hinder non-linear transfer of energy to higher harmonics when the wave speed is equivalent to the angular velocity (the classic critical layer). The lack of energy transfer prevents the shear from reversing and therefore the shear continues to grow until the field decreases in amplitude due to resistive decay.

While it appears that the main determinant of the IGW–field interaction is the similarity of the Alfvén and IGW frequency, we also find that non-linear and diffusive effects play a role in the interaction that leads to altered angular momentum transport. These simple models study the interaction of only a single prograde and retrograde wave. The solar convection zone generates a spectrum of waves and the effect of a strong field on wave propagation under these circumstances will be investigated in a forthcoming paper.

ACKNOWLEDGMENTS

We thank Gary Glatzmaier and D. Gough for helpful discussions. Support for this research was provided by a NASA grant NNG06GD44G. TR is supported by an NSF ATM Faculty Position in Solar Physics grant under award number 0457631. Computing resources were provided by NAS at NASA Ames.

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