Modeling, Error Analysis, and Evaluation of Dual-Polarization Variables Obtained from Simultaneous Horizontal and Vertical Polarization Transmit Radar.  
Part I: Modeling and Antenna Errors

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ABSTRACT

In this two-part paper the biases of polarimetric variables from simultaneous horizontally and vertically transmitted (SHV) data are investigated. Here, in Part I, a radar-scattering model is developed and antenna polarization errors are investigated and estimated. In Part II, experimental data from the National Center for Atmospheric Research S-band dual-polarization Doppler radar (S-Pol) and the National Severe Storms Laboratory polarimetric Weather Surveillance Radar-1988 Doppler (WSR-88D) radar, KOUN, are used to illustrate biases in differential reflectivity ($Z_{\text{dr}}$). The biases in the SHV polarimetric variables are caused by cross coupling of the horizontally (H) and vertically (V) polarized signals. The cross coupling is caused by the following two primary sources: 1) the nonzero mean canting angle of the propagation medium and 2) antenna polarization errors. The biases are strong functions of the differential propagation phase ($\phi_{\text{dp}}$) and the phase difference between the H and V transmitted field components. The radar-scattering model developed here allows for the evaluation of biases caused by cross coupling as a function of $\phi_{\text{dp}}$, with the transmission phase difference as a parameter. Also, antenna polarization errors are estimated using solar scan measurements in combination with estimates of the radar system’s linear depolarization ratio (LDR) measurement limit. Plots are given that show expected biases in SHV $Z_{\text{dr}}$ for various values of the LDR system’s limit.

1. Introduction

The simultaneous transmission and reception of horizontally (H) and vertically (V) polarized waves (SHV) has become a very popular way to achieve dual-polarization measurements for weather radar. Previously, radars achieved dual-polarization measurements by either employing a fast, high-powered waveguide switch or by using two separate transmitters (Brunkow et al. 2000; Keeler et al. 2000). Both technologies incur significant costs to the operation and maintenance of the radar. The premise of the SHV technique is that as the transmitted wave propagates, no cross coupling occurs between the H and V electric field components. Mathematically, this requires that the propagation matrix be diagonal (Vivekanandan et al. 1991). An advantage of the SHV technique is that the expense of either a fast waveguide switch or two transmitters is avoided. There are other advantages of SHV mode (see Doviak et al. 2000 for details). Disadvantages of SHV mode are that 1) the linear depolarization ratio (LDR) is not measured, and, 2) if there is cross coupling of the H and V waves, there can be measurement biases. Thus, the viability of the SHV dual-polarization technique is based on having 1) a zero mean canting angle of the propagation medium and 2) negligible antenna polarization errors. If either condition is not met, cross coupling occurs between the H and V channels, which can cause measurement biases.

Measurement biases in the SHV mode have been investigated (Sachidananda and Zrnić 1985). Doviak et al. (2000) evaluated cross-coupling errors of SHV mode and concluded that since the mean canting angle of rain is close to zero, the biases were acceptable. More recently, Wang and Chandrasekar (2006) and Wang et al. (2006) investigated the biases in SHV polarization measurements resulting from cross-coupling errors caused by the radar system polarization errors as a function of $\phi_{\text{dp}}$ (differential propagation phase). They concluded that system isolation between the H and V channels must...
be greater than 44 dB in order to ensure that the $Z_{dr}$ bias is within 0.2 dB for worst-case errors.

Ryzhkov and Zrnić (2007) examined the effects of nonzero mean canting angle of the propagation medium on SHV mode measurements. Data gathered in SHV mode with KOUN, the National Severe Storms Laboratory’s (NSSL’s) S-band dual-polarization Doppler radar, displayed $Z_{dr}$ radial bias “stripes” after the radar waves passed through the ice phase of either convective cells or stratiform precipitation. They propose that the nonzero mean canting angle of the propagation medium produces coupling between the H and V polarized waves that causes the anomalous $Z_{dr}$ signatures.

All reflector-type antennas will introduce polarization errors to the desired H and V transmitted polarization states, causing cross coupling between the H and V channels. This will bias polarization measurements of precipitation unless the antenna interchannel isolation is extremely low. This paper investigates and quantifies the expected magnitude and phase of antenna polarization errors via an analysis of solar measurements and a known typical LDR system limit. Importantly, the antenna polarization errors are interpreted in terms of their tilt and ellipticity angles, which has not been done before. Such an analysis provides insight as to the nature of the antenna polarization errors. Specifically, we show that the antenna polarization errors are dominated by the ellipticity angle errors, at least for the National Center for Atmospheric Research (NCAR) dual-polarization Doppler radar (S-Pol). The radar model that is introduced by Hubbert and Bringi (2003) is modified to accommodate arbitrary transmit polarization states (refer to Fig. 1). The particles in the backscatter volume and the coherent propagation medium are independently modeled. The “steady” propagation medium is modeled via a 2 $\times$ 2 matrix that includes absolute attenuation ($A_h$), differential attenuation ($A_{dp}$), differential propagation phase ($\phi_{dp}$), and mean canting angle ($\theta$) as parameters. The resolution volume (or backscatter medium) is modeled as an ensemble of precipitation particles with gamma drop size distribution (DSD) and various spatial orientation distributions via the T-matrix method (Vivekanandan et al. 1991; Waterman 1969). Antenna polarization errors are modeled similar to McCormick (1981) and Bringi and Chandrasekar (2001) and are beam integrated so that a pair of complex numbers characterizes the H and V antenna polarization errors.

The paper is organized as follows. Section 2 describes the radar model of Hubbert and Bringi (2003) and how antenna polarization errors are accounted for. The model is then modified to permit arbitrary transmit polarizations, and model results are given. Section 3 shows how antenna polarization errors can be estimated from experimental data. The summary and conclusions are given in section 4. In Hubbert et al. (2009, hereafter Part II), experimental data from the NCAR S-Pol and from NSSL’s S-band research radar, KOUN, illustrate the theory established herein, in Part I.

2. The model

The radar scattering model is described in Hubbert and Bringi (2003), but is briefly reviewed here for convenience and clarity. The model is then expanded to accommodate arbitrary transmit polarization states (refer to Fig. 1). The particles in the backscatter volume and the coherent propagation medium are independently modeled. The “steady” propagation medium is modeled via a 2 $\times$ 2 matrix that includes absolute attenuation ($A_h$), differential attenuation ($A_{dp}$), differential propagation phase ($\phi_{dp}$), and mean canting angle ($\theta$) as parameters. The resolution volume (or backscatter medium) is modeled as an ensemble of precipitation particles with gamma drop size distribution (DSD) and various spatial orientation distributions via the T-matrix method (Vivekanandan et al. 1991; Waterman 1969). Antenna polarization errors are modeled similar to McCormick (1981) and Bringi and Chandrasekar (2001) and are beam integrated so that a pair of complex numbers characterizes the H and V antenna polarization errors.
errors. Our modeling of antenna errors is very similar to Wang and Chandrasekar (2006). The transmit polarization state is defined at the reference plane as shown in Fig. 1. The antenna system refers to the microwave path from the reference plane through the antenna dish.

The scattering geometry used is the backscatter alignment (BSA) convention (Bringi and Chandrasekar 2001). Canting angles are measured counterclockwise from the positive horizontal axis in the plane of polarization (i.e., the plane containing the H and V axis perpendicular to the propagation direction). Further details of the radar model are given in appendix A.

a. Modeling antenna polarization errors

The radar antenna and surrounding microwave circuitry introduce microwave cross coupling that gives rise to polarization errors so that pure H or V polarization are not transmitted. Polarization errors have been covered in detail (McCormick 1981; Metcalf and Ussails 1984; Bringi and Chandrasekar 2001; Hubbert and Bringi 2003; Wang and Chandrasekar 2006). Some of the sources of polarization error are nonideal feedhorn, nonideal parabolic reflector, antenna support struts, and edge effects. These polarization errors are distributed across the radar antenna patterns and thus can vary across the beam, especially where the cross-polarized lobes exist (Ussailis and Metcalf 1983; Bringi and Chandrasekar 2001). For distributed precipitation media, the resulting error is an integrated effect and we model these distributed errors with a $2 \times 2$ polarization error matrix.

The assumption is that the scattering medium is homogenous across the antenna pattern.

The polarization errors are included in the model by pre- and postmultiplication of $S$, where $S$ is the scattering matrix in Eq. (A4), by the error matrix $Y$,

$$S_v = Y^T S Y,$$

where

$$Y = \begin{bmatrix} i_h & \xi_v \\ \xi_h & i_v \end{bmatrix},$$

with constraints $i_h^2 + |\xi_h|^2 = i_v^2 + |\xi_v|^2 = 1$, where $i_h$ and $i_v$ are real. The polarization errors of the H and V channels are represented by the complex numbers $\xi_h$ and $\xi_v$, respectively. The polarization errors can also be equivalently represented in the following several ways: 1) the polarization ratio $\chi$ [see Eq. (A6)], 2) the geometric parameters of tilt angle $\alpha$ and ellipticity angle $\epsilon$, and 3) the phasor descriptors $\gamma$ and $\zeta$. For H errors

$$\chi = \xi_h / i_h,$$  

and for V errors

$$\chi = i_v / \xi_v,$$

because the definition of $\chi$ used here is the ratio of the V to H electric field components. Referring to the polarization ellipse of Fig. 2, the polarization error terms $\xi_h$ and $\xi_v$ can also be related to the geometric polarization ellipse parameters $\alpha$ and $\epsilon$, and the phasor descriptors, where $\gamma = \tan^{-1}(|E_v|/|E_h|)$ and $\arg(E_v/E_h) = \zeta = \zeta_v - \zeta_h$. The tilt angle $\alpha$ is measured from the positive horizontal axis to the major axis of the polarization ellipse and $\epsilon$ is defined as $\tan \epsilon = (\text{minor axis})/(\text{major axis})$.

Mathematically, these variables are related by (Azzam and Bashara 1989)

$$\tan 2\alpha = \frac{2\Re(\chi)}{1 - |\chi|^2},$$  

$$\sin 2\epsilon = \frac{2\Im(\chi)}{1 + |\chi|^2},$$

and

$$\cos 2\gamma = \cos 2\epsilon \cos 2\alpha, \quad \tan 2\alpha = \tan 2\gamma \cos \zeta,$$

$$\tan \zeta = \tan 2\epsilon / \sin 2\alpha, \quad \sin 2\epsilon = \sin 2\gamma \sin \zeta.$$

As can be seen from Eqs. (3) through (7), if the $\xi_h(\xi_v)$ is real then $\epsilon$ is zero and if $\xi_h(\xi_v)$ is imaginary then $\alpha$ is zero. If the errors are orthogonal,

$$\xi_v = -\xi_h^*,$$

then $Y$ is unitary and (1) represents an orthogonal change of polarization basis. Expressing the polarization
errors using the geometric descriptors gives a convenient and intuitive way to analyze polarization errors.

Given the tilt and ellipticity angles of the polarization state (or ellipse), the corresponding phasor parameters are easily found using the relations in Eqs. (7). Then, the antenna error matrix terms $\xi_h$ and $i_h$ are found as

\[
i_h = \frac{1}{\sqrt{1 + \tan^2 \gamma}},
\]

\[
|\xi_h| = \sqrt{1 - i_h^2},
\]

and finally

\[
\xi_h = |\xi_h| \exp^{i \xi_h}.
\]

Similarly, the vertical polarization error $\xi_v$ is found from $\alpha_v$ and $\epsilon_v$ (remembering that the polarization ratio is defined as $\chi = i_v / \xi_v$) as

\[
i_v = \frac{\tan \gamma}{\sqrt{1 + \tan^2 \gamma}},
\]

\[
|\xi_v| = \sqrt{1 - i_v^2},
\]

and finally

\[
\xi_v = |\xi_v| \exp^{-i \xi_v}.
\]

It is also useful to express the polarization ratio in terms of its real and imaginary parts as (Tragl 1990)

\[
\chi = \frac{\cos2\epsilon \sin2\alpha + j \sin2\epsilon}{1 + \cos2\epsilon \cos2\alpha}.
\]

b. Modeling simultaneous H and V transmission

The model thus far was constructed so that the transmit and receiving polarization states are the same according to radar polarimetry theory. The covariance matrix of Eq. (A5) is a convenient form for covariance analysis and for polarization basis transformations; however, it is not a transmission matrix. It does not express a transfer relationship between an arbitrary input polarization and the resultant output covariances, as does the Mueller matrix (Azzam and Bashara 1989). To model the H and V receiving covariances that result from arbitrary transmitting polarizations, a $4 \times 4$ covariance matrix is formed using the feature vector

\[
\mathbf{\Omega}^T = [S_{HH} \ S_{VH} \ S_{HV} \ S_{VV}].
\]

Taking the outer product of the feature vector yields the $4 \times 4$ covariance matrix in the H–V basis as

\[
\Sigma'_0 = \begin{bmatrix}
\langle S_{HH}^2 \rangle & \langle S_{HH}^* S_{VH} \rangle & \langle S_{HH}^* S_{HV} \rangle & \langle S_{HH}^* S_{VV} \rangle \\
\langle S_{VH}^* S_{HH} \rangle & \langle S_{VH}^2 \rangle & \langle S_{VH}^* S_{HV} \rangle & \langle S_{VH}^* S_{VV} \rangle \\
\langle S_{HV}^* S_{HH} \rangle & \langle S_{HV}^* S_{VH} \rangle & \langle S_{HV}^2 \rangle & \langle S_{HV}^* S_{VV} \rangle \\
\langle S_{VV}^* S_{HH} \rangle & \langle S_{VV}^* S_{VH} \rangle & \langle S_{VV}^* S_{HV} \rangle & \langle S_{VV}^2 \rangle 
\end{bmatrix},
\]

where $\langle \cdot \rangle$ denotes ensemble (spatial) or temporal averages (which are equivalent because of the assumption of ergodicity). Note that the covariance matrix is Hermitian. It can be shown that the matrix of Eq. (17) may be transformed to the Mueller matrix (Azzam and Bashara 1989), and thus the covariance matrix of Eq. (17) can also be used as a transfer function matrix

\[
\mathbf{J}_t = \Sigma_0 \mathbf{J}_i,
\]

where $\mathbf{J}_i$ and $\mathbf{J}_o$ are $4 \times 1$ input and output coherency matrices. In terms of the desired polarization characteristics of the incident polarization, namely, tilt angle ($\alpha$) and ellipticity angle ($\epsilon$), $\mathbf{J}_i$ becomes

\[
\mathbf{J}_i = \begin{bmatrix}
J_{i,1} \\
J_{i,2} \\
J_{i,3} \\
J_{i,4}
\end{bmatrix} = \begin{bmatrix}
1 + \cos2\alpha \cos2\epsilon \\
\sin2\alpha \cos2\epsilon - j \sin2\epsilon \\
\sin2\alpha \cos2\epsilon + j \sin2\epsilon \\
1 - \cos2\alpha \cos2\epsilon
\end{bmatrix}.
\]

If linear, a slant of $45^\circ$ transmitting polarization is desired (i.e., SHV mode), and then $\alpha = 45^\circ$ and $\epsilon = 0^\circ$. SHV variables of interest can then be calculated, for example, as

\[
Z_{dr}^{shv} = 10 \log_{10}(J_{o,1} / J_{o,4}),
\]

\[
\Psi_{dp}^{shv} = \tan^{-1}[\Im(J_{o,3})/\Re(J_{o,3})],
\]

\[
\rho_{hv}^{shv} = \frac{|J_{o,2}|}{\sqrt{|J_{o,1}|^2 + |J_{o,4}|^2}},
\]

where the superscript “shv” denotes SHV variables. In this way, the radar model is modified to allow for arbitrary transmitting polarizations (but still receiving H and V) by putting the covariances of Eq. (A5) into the matrix form of Eq. (17). The input vector is then controlled by Eq. (19) via the tilt and ellipticity angles of the desired transmitting polarization state.

The advantages of the presented radar model are that 1) no approximations are made in terms of the relative significance of the various error terms and 2) transmit, propagation, backscatter, and antenna model parameters are all independently set so that the effect of varying
each parameter on the polarization variables of $Z_{dr}$, $\phi_{dp}$, etc., can be investigated.

3. Estimating antenna polarization errors

The estimation of the complex error terms $\xi_h$ and $\xi_v$ for antennas is difficult, and they are not typically supplied by antenna manufacturers. There are ways, however, to estimate the magnitude of the error terms and to generally qualify their character. Two generally available and measurable quantities are LDR and passive solar scan measurements.

For well-designed radars with parabolic, center-fed antennas, the dominant cross-correlation factor is the antenna (Bringi and Chandrasekar 2001). The radar system lower limit of LDR can be estimated by measurement in drizzle where raindrops are considered circular so that the backscatter (and propagation) medium cause no cross coupling, and thus intrinsic LDR is $-\infty$ dB. Theoretically, LDR can be expressed as a function of the polarization errors. S-Pol collects solar data by performing a “box scan” of the sun in the passive mode. The sun here is considered to be an unpolarized radio frequency (RF) source subtending a solid angle of about 0.53° (Jursa 1985; Tapping 2001). The dimension of the box scan is approximately 3° high (elevation angle) by 7° wide (in azimuth), and the radar scan elevation steps are 0.2°. The scanning rate is 1° s$^{-1}$ so that one complete solar box scan requires about 2 min. Noise samples are collected while the radar is pointing away from the sun so that the thermal background noise can be estimated and used to correct the measured sun data. S-Pol’s sensitivity is $-113$ dBm and the sun’s measured power is about $-100$ dBm when the main antenna beam is centered on the sun. The data are interpolated to a square 2° × 2° grid in 0.1° intervals. The data are first corrected for sun movement and for distortion caused by scanning in elevation and azimuth angle rather than in a rectangular grid.

Shown in Fig. 3 in the top panels are the H and V “pseudo”-antenna patterns, respectively, obtained from S-Pol data gathered on 19 May 2008 during the Terrain-Influenced Monsoon Rainfall Experiment (TIMREX) in southern Taiwan. The powers are uncalibrated. These are termed pseudo-antenna patterns because the sun is not a point source and thus the given antenna patterns are a convolution of the antenna beam pattern of S-Pol with the solar disk. The complex H and V time series data can be used to create a cross-channel correlation antenna pattern. The simultaneously received voltage time series from a single dwell angle, $V_h(i)$ and $V_v(i)$ for the horizontal and vertical channels, respectively, are correlated in usual fashion as

$$\omega = \frac{\sum_{i=1}^{N} V_h(i)V_v^{*}(i)}{\sqrt{\sum_{i=1}^{N} V_h^{2}(i)} \sum_{i=1}^{N} V_v^{2}(i)}.$$ (24)

Thus, $\omega$ gives the pointwise (spatial) correlation from temporal averages. This correlation data from all dwell angles is interpolated to a grid. The resulting magnitude and phase of the correlation product of Eq. (24) are given in Fig. 3 in the bottom two panels. If solar radiation is unpolarized, then the correlation of data between any two orthogonal receiving polarization channels is zero by definition. The correlation magnitude in Fig. 3 (bottom left) shows two principal “lobes” in the lower two quadrants where the correlation increases to about 0.07. These large areas of increased correlation coefficient are manifestations of the antenna polarization errors. The antenna errors are obviously a function of azimuth and elevation angle and are not constant across the 2° × 2° antenna patterns shown. The areas of maximum correlation do, however, fall outside the 3-dB beamwidth of the antenna (which is about 0.9° (Keeler et al. 2000)), which helps reduce the magnitude of the cross coupling. Figure 3d shows...
the complex behavior of the phase of the correlation product, with the phases being fairly constant in the regions of the highest correlation. For the lower-left quadrant, this phase is \(-100^\circ\), while the lower right quadrant phase is about \(+60^\circ\). These antenna pattern correlations can be integrated to obtain a single complex correlation coefficient and this is discussed later in the text.

The radar model presented above represents the antenna polarization errors as a single complex number for the H and V polarizations, that is, the polarization errors are integrated. Even though antenna errors are distributed, this is a useful approximation that simplifies analysis and permits a realistic numerical evaluation and simulation of polarization errors. The assumption is that there is a homogeneous distribution of scatterers across the antenna beam.

It can be shown that for small polarization errors (see appendix B)

\[ \Omega = \xi_h^* + \xi_v, \]  

where \( \Omega \) is the pattern-integrated correlation coefficient.

b. General observations

Solving Eqs. (23) and (25) simultaneously yields

\[ \Imag(\xi_v) = \frac{\Imag(\Omega) \pm \sqrt{\Imag^2(\Omega) + \text{LDR}_t - |\Omega|^2}}{2}, \]  

(26)

\[ \Imag(\xi_h) = \frac{-\Imag(\Omega) \pm \sqrt{\Imag^2(\Omega) + \text{LDR}_t - |\Omega|^2}}{2} \]  

(27)
where LDR is LDR in linear units. The real parts are not solvable, but obviously \( \Re(\xi_h + \xi_v) = \Re(\xi_h^* + \xi_v) \).

Starting with Eqs. (23) and (25) several interesting observations are possible. The theoretical conditions for making LDR minus infinity and \( \Omega \) zero are \( \xi_v = -\xi_h \), and \( \xi_v = -\xi_h^* \), respectively. In terms of the geometric polarization quantities, the system LDR limit minima condition is

\[
\alpha_v = \alpha_h + 90^\circ, \\
\epsilon_v = \epsilon_h, \\
\text{and the minimum condition for } \Omega \text{ is}
\]

\[
\alpha_v = \alpha_h + 90^\circ, \\
\epsilon_v = -\epsilon_h
\]

Again, for low solar cross correlation, the errors must be near orthogonal. This is consistent with the observation that the cross correlation of passive solar measurements from two orthogonal polarization states is, by definition, zero if the solar radiation is unpolarized, as is assumed. It is interesting to note that for minimum LDR, the H and V tilt angles are orthogonal while the ellipticity angles are not. This is a direct artifact of the S-Pol antenna errors and the H and V tilt angles are orthogonal while the ellipticity angles are not. This is a direct artifact of radar polarimetry theory and the radar voltage equation (Kennaugh 1949–1954; Hubbert 1994; Hubbert and Bringi 1996). Because the polarization basis of the radar is transformed from linear H and V polarizations to the circular polarization basis by increasing \( \epsilon \) from 0° to 45°, the depolarization ratio from a circular scatterer (e.g., a circular raindrop) goes from −\( \infty \) to +\( \infty \) dB [e.g., see Fig. 7 from Hubbert (1994) or Fig. 3.10 of Bringi and Chandrasekar (2001)]. Thus, for increasing orthogonal ellipticity errors, the LDR system limit will increase while \( \Omega \) is unaffected. Also, note that the orthogonal tilt angle errors will not increase the LDR system limit (as determined from measurements in drizzle) nor will they increase \( \Omega \). Therefore, neither LDR drizzle measurements nor solar correlation measurements will detect orthogonal tilt antenna errors. For example, these measurements would not reveal whether a radar was un leveled, that is, whether the desired H polarization state of the antenna was not parallel to the earth. Equivalently, it can be shown that if the H and V errors are orthogonal, this implies the phase relationship \( \xi_v = \pi + \xi_h \), where \( \xi_h = \arg(\xi_h/l_h) \) and \( \xi_v = \arg(l_v/\xi_v) \). If the antenna errors meet the LDR minimum criteria, that is, \( \xi_v = -\xi_h \), then \( \xi_v = \pi - \xi_h \).

Thus, the cross-correlation \( \Omega \) can be either zero or very low, indicating that the receive polarization states are orthogonal or nearly orthogonal, but this does not necessarily mean that the LDR system limit is low.

Conversely, the LDR system limit can be low, indicating that the H and V tilt errors are nearly orthogonal and that the ellipticity errors are nearly equal, but \( \Omega \) could be relatively high.

From the S-Pol solar data of Fig. 3, the integrated solar correlation coefficient is calculated to be \( \Omega = 0.0038 + j0.00088 \). There is very likely system phase offset included in this number, such as the differential phase from the reference plane to the I and Q time series samples (see Fig. 1), so that the phase of \( \Omega \) is not an accurate estimate of the phase of \( \xi_h + \xi_v^* \). However, the magnitude of \( \Omega \) should be an accurate estimate of the magnitude of \( \xi_h + \xi_v^* \). The magnitude of \( \Omega \) is about 0.0039, and from the measurements in drizzle S-Pol’s LDR limit is about −31 dB. Using Eqs. (23) and (25), we can write

\[
|\Omega| = |\xi_h + \xi_v^*| = 0.0039,
\]

\[
10^{\text{LDR/20}} = |\xi_h + \xi_v| = 0.028,
\]

where 0.028 = 10\(^{-31 \text{dB/20}}\). The real parts of the arguments of the absolute values in Eqs. (32) and (33) are equal, and therefore there must be significant canceling of the imaginary parts of \( \xi_h \) and \( \xi_v^* \) in Eq. (32) to account for the large magnitude difference between \( 10^{\text{LDR/20}} \) and |\( \Omega \)|. Interpreting these measurements in conjunction with Eqs. (3)–(7) gives some insight as to the nature of the S-Pol antenna errors. Orthogonal tilt angle errors (with no ellipticity angle errors) require that \( \alpha_v = \alpha_h + 90^\circ \), \( \Re(\xi_h) = -\Re(\xi_v) \), and \( \Im(\xi_h) = \Im(\xi_v) = 0 \). Under this condition both LDR, and \( \Omega \) are both zero, so obviously orthogonal tilt angle errors cannot produce the observed measurement. Additionally, nonorthogonal tilt angle errors will increase both LDR, and \( \Omega \) and therefore they cannot account for \( LDR \ll |\Omega|^2 \). For orthogonal ellipticity errors (with no tilt angle errors), \( \Re(\xi_v) = \Re(\xi_h) = 0 \) and \( \Im(\xi_h) = \Im(\xi_v) \). Therefore, for increasing orthogonal ellipticity errors, LDR increases while \( \Omega \) is not affected. It follows that S-Pol antenna errors are likely dominated by ellipticity errors and the H and V ellipticity errors are such that

\[
|\xi_h + \xi_v^*| \gg |\xi_h + \xi_v|.
\]

Orthogonal tilt errors are detected by neither the system limit LDR nor the \( \Omega \) measurements, and they could be significant. We assume, however, that physically the feed horn is well aligned (leveled) and this then would make these errors negligible. Under these conditions, \( \Re(\xi_h) = \Re(\xi_v) \approx 0 \), and then

\[
\Im(\xi_h) - \Im(\xi_v) = 0.0039,
\]

\[
\Im(\xi_h) + \Im(\xi_v) = 0.028.
\]
Using Eqs. (3), (4), and (6), the H and V ellipticity angles are found to be $\epsilon_h = -0.91^\circ$ and $\epsilon_v = 0.69^\circ$. These antenna error values are used in the model in Part II. Summarizing, if LDR is significantly increased, the antenna error values are used in the model in Part II. that is, $\epsilon_v \approx -\epsilon_h$, rather than close to equal, that is, $\epsilon_v \approx \epsilon_h$.

The functional relationship between the antenna errors and SHV $Z_{dr}$ is found by calculating SHV mode $Z_{dr}$ in drizzle (see appendix C) as

$$Z_{dr} = 10 \log_{10} \left[ \frac{1 + \xi_h^2 + e^{i\beta}(\xi_v + \xi_h)}{e^{i\beta}(1 + \xi_v^2 + \xi_v + \xi_h)} \right]^2, \tag{37}$$

where $\beta$ is the differential transmit phase $\arg(E_h/E_v)$. Equation (37) shows that the first-order H and V antenna error terms appear in the form $\xi_h + \xi_v$, as they do in the LDR system limit in Eq. (23). This shows that the sum of the antenna errors appears both in SHV $Z_{dr}$ and LDR, and antenna errors that increase LDR will also, in general, increase SHV $Z_{dr}$ bias. However, as shown in appendix C, $Z_{dr}$ is a strong function of differential transmit phase and differential propagation phase. The conclusion is that the LDR system limit of a radar is a good indicator of the expected SHV $Z_{dr}$ bias. Later, we present curves that relate the LDR system limit to the SHV $Z_{dr}$ bias.

4. Model results

Next, the model is used to examine biases in the SHV mode $Z_{dr}(Z_{dr})$, $K_{dp}^{shv}$, and $\rho_{hv}^{shv}$ for the 1) transmit errors, 2) nonzero mean propagation canting angle, and 3) antenna polarization errors. Again, the radar variables are plotted as a function of principal plane $\phi_{dp}(\phi_{dp})$. Because $\phi_{dp}$ is the independent variable and $K_{dp}$ is of more meteorological interest than $\phi_{dp}$, the normalized SHV mode $K_{dp}$ is expressed as

$$K_{dp}^{shv} = \frac{K_{dp}^{shv}}{K_{dp}^{ref}}, \tag{38}$$

where $K_{dp}^{shv}$ is normalized SHV $K_{dp}$, $K_{dp}^{shv}$ is SHV $K_{dp}$, and $K_{dp}^{ref}$ is the principal plane $K_{dp}$ [see Eq. (A2) and surrounding text for clarification of principal plane]. For the relatively small antenna polarization errors examined here, $0.97 < K_{dp}^{shv} < 1.03$ and therefore the errors in $K_{dp}^{shv}$ are small.

The backscatter medium is modeled as rain with a zero mean canting angle, a standard deviation of canting angles of $5^\circ$, $Z_{dr} = 2.8$ dB, and LDR = $-35$ dB. For this study, the characteristics of the backscatter medium are relatively unimportant in terms of characterizing antenna polarization errors and the effect of nonzero mean canting angle of the propagation medium on $Z_{dr}$. Thus, the mean canting angle of the backscatter medium is always $0^\circ$. For the propagation medium, typical S-band values of absolute attenuation $A_h = 0.0165$ dB $\circ^{-1}$ (of $\phi_{dp}$) and differential attenuation $A_{dp} = 0.0035$ dB $\circ^{-1}$ (Bringi and Chandrasekar 2001) are used unless otherwise stated.

a. Transmit errors

Here we examine SHV $Z_{dr}$ bias caused by errors in the transmit polarization state. The transmit polarization state is defined by the transmit H and V electric fields $E_h$ and $E_v$ at the reference plane shown in Fig. 1. For the SHV mode, ideally $E_h = E_v$. Figure 4 shows $Z_{dr}^{shv}$ for several values of $|E_h^\prime/|E_h|$ versus $\phi_{dp}$. Note that the tilt angle of the transmit polarization state is defined as $\tan^{-1}(|E_h^\prime/|E_h|)$ (V-to-H, angles are given in the plot for each curve), whereas $Z_{dr}$ is H-to-V power. The mean canting angle of the particles in the propagation medium is zero (i.e., diagonal propagation matrix) and the antenna polarization errors are zero, that is, $\xi_h = \xi_v = 0$. The $Z_{dr}^{shv}$ bias is independent of the phase difference between $E_h$ and $E_v$. The slope of the curves is caused by $A_{dp} = 0.0035$ dB $\circ^{-1}$. The dashed nominal line is considered ideal. As can be seen, the biases are constant as compared to the nominal curve, and such biases could be corrected via radar calibration. Even though the phase difference between $E_h$ and $E_v$ does not affect $Z_{dr}^{shv}$, this

![Fig. 4. SHV $Z_{dr}$ as a function of principal plane $\phi_{dp}$ with unbalanced transmit power as a parameter. For SHV mode ideally $E_h = E_v$, and this nominal curve is shown (dashed line). The errors are independent of the phase difference between $|E_h^\prime|$ and $|E_v^\prime|$](image-url)
phase difference is very important in the sections below. We note that Wang and Chandrasekar (2006) show the effect of differential antenna error phase on the $Z_{dr}^{sv}$ bias. Here we show how the differential H and V transmit phase (as well as antenna phase error) affects the $Z_{dr}^{sv}$ bias. The differential transmit phase and differential antenna phase error are distinctly different.

b. Nonzero mean propagation canting angle

The model is now used to illustrate $Z_{dr}^{sv}$ bias caused by nonzero mean canting angle of the propagation medium $\theta$, which causes cross coupling between the H and V components of the electric field. Antenna errors are zero and the transmit errors are zero, that is, $E_h^t = E_v^t$. Figure 5 shows $Z_{dr}^{sv}$ bias as a function of $\phi_{dp}^{p}$ with $\theta$ as a parameter. The $Z_{dr}^{sv}$ bias is defined as the difference between the modeled $Z_{dr}^{sv}$ and the error-free, nominal $Z_{dr}$ shown in Fig. 4. As $|\theta|$ increases, the $|Z_{dr}^{sv}|$ bias increases. If $\phi_{dp}^{p} < 10^\circ$, the biases are kept to within about 0.1 dB. Figure 6 is similar to Fig. 5, except that the phase difference between $E_h^t$ and $E_v^t$ is 90$^\circ$, that is, the transmit polarization is circular. The $|Z_{dr}^{sv}|$ bias now increases much more rapidly as a function of $\phi_{dp}^{p}$ as compared to Fig. 5. The $\phi_{dp}^{p}$ value must be less than about 1$^\circ$ in order for $Z_{dr}^{sv}$ bias to be less than about 0.1 dB. This then shows that if $\theta$ is not near zero, very little $\phi_{dp}^{p}$ needs to accumulate in order to cause significant $Z_{dr}^{sv}$ bias. The phase difference of the H and V transmitted waves controls the amount of the constructive–destructive interference between the main copolar wave and the biasing cross-coupled component. This illustrates the importance of the phase difference between the transmit electric field components $E_h^t$ and $E_v^t$, and corroborates and explains the SHV $Z_{dr}$ bias in the ice regions of storms reported by Ryzhkov and Zrnić (2007).

When the backscatter medium is changed to ideal drizzle, that is, small spherical drops, the $Z_{dr}^{sv}$ bias curves of Figs. 5 and 6 change very little, and thus are not given. Also, the propagation medium parameters can be changed to simulate C band with $A_{dp} = 0.018$ dB $\circ^{-1}$ and $A_h = 0.075$ dB $\circ^{-1}$, and to simulate X band with $A_{dp} = 0.03$ dB $\circ^{-1}$ and $A_h = 0.28$ dB $\circ^{-1}$. The corresponding $Z_{dr}^{sv}$ bias curves are given in Figs. 7a,b, respectively. These bias curves are very similar to the S-band errors curves in Fig. 6. Thus, the primary difference between the S-, C-, and X-band $Z_{dr}^{sv}$ biases here is that for the same propagation medium, $\phi_{dp}^{p}$ will accumulate faster at C band and still faster at X band, as compared to S band. This means that the $Z_{dr}^{sv}$ biases discussed here will be more problematic at C band and even more so at X band. We give one note of observational interest: If ice particles have a mean canting angle of around 45$^\circ$, $Z_{dr}^{sv}$ will be a constant as a function of range because it propagates through such a medium even though $\phi_{dp}^{p}$ can be significant so that $Z_{dr}^{sv}$ becomes biased.

c. Antenna errors

Antenna errors are quantified in the model by the complex numbers $\xi_h$ and $\xi_v$. These errors can be equivalently...
defined by the tilt and ellipticity angles, $\alpha_{hv}$ and $\epsilon_{hv}$ respectively, of the polarization ellipse as described above. First, the model is used to illustrate the different effects that tilt and ellipticity angle errors individually have on $Z_{dr}$ bias. Figure 8 shows $Z_{drh}$ for $1^\circ$ orthogonal polarization tilt errors (top panel) and $1^\circ$ orthogonal polarization ellipticity errors (bottom panel), both versus $\phi_{dp}$. The $\theta$ is zero and $E_v = E_h$. The magnitude of these errors, that is $|\xi_h + \xi_v|$ corresponds to an LDR system limit of about $-30$ dB. The solid straight lines represent nonbiased $Z_{dr}$ that would be measured in fast alternating H and V transmit mode. The figure shows that $Z_{dr}$ bias is significant with a maximum error of about $0.6$ dB when $\phi_{dp} = 180^\circ$.

It is likely that true antenna errors will be some combination of tilt and ellipticity angle errors, and thus we present the following again as an illustrative example of how antenna errors can affect $Z_{dr}$ bias. Figure 9 shows $Z_{drh}$ for $1^\circ$ orthogonal polarization angle errors given in Table 1. The antenna errors are orthogonal [Eq. (8)]. The figure shows that the character of the $Z_{dr}$ bias is quite different for each curve, with a maximum bias of about $0.4$ dB. These antenna errors all correspond to about a $-31$-dB LDR system limit.

These same antenna errors from Table 1 are used again in Fig. 10, but for circular transmit polarization. The $Z_{dr}$ biases curves have changed dramatically and demonstrate the importance of the phase difference between the H and V components of the transmit wave. Again the transmit wave is defined here at the reference plane given in Fig. 1.

As shown above, S-Pol’s antenna polarization errors are fairly well characterized by orthogonal ellipticity angles and by H and V tilt angles of $0^\circ$ and $90^\circ$, respectively (i.e., no tilt angle errors). Using this restriction and given an LDR system limit value, the ellipticity error angle can be calculated. Table 2 gives the ellipticity error angles corresponding to several LDR system limit values. The corresponding values for the Im($\xi_h$) (or equivalently $\epsilon_h$ in radians) are also given. The information in Table 2 is used below.

We next examine SHV $K_{dp}$ biases caused by polarization errors given in Table 3. These antenna polarization errors correspond to an LDR system limit of $-25$ dB.

---

**Fig. 7.** As in Fig. 4, but with the mean canting angle of the propagation medium as a parameter. Here, $|E_h| = |E_v|$, but there is a $90^\circ$ phase difference; that is circular polarization is transmitted. (a) C band and (b) X band. There is very little difference between these two error plots and the analogous one for S band in Fig. 6.

---

**Fig. 8.** SHV mode $Z_{dr}$ for $1^\circ$ antenna polarization errors. (top) $\pm 1^\circ$ tilt errors and (bottom) $\pm 1^\circ$ ellipticity errors.
Shown in Fig. 11 is $K_{dp}^{shv}/K_{dp}^{dp}$ as a function of principal plane $\phi_{dp}$. The $K_{dp}$ bias is fairly small, always being less than 3%. If the LDR system limit is less than $2^{30}$ dB, the $K_{dp}$ error is within 2%.

The biases of SHV $\rho_{hv}$ for the LDR system limits as high as $2^{31}$ dB are less than 1% and are not given.

d. SHV $Z_{dr}$ as a function of LDR system limit

As shown in appendix C, the antenna error terms appear as $\xi_h + \xi_v$ in the expression for the LDR system limit and SHV $Z_{dr}$ in drizzle. Thus, the LDR system limit for a radar can be related to the SHV $Z_{dr}$ bias as a function of $\phi_{dp}^{\rho}$, with differential transmit phase as a parameter. Based on the antenna errors for S-Pol, the antenna errors are modeled as orthogonal ellipticity angles with no tilt angle errors. This is shown in Figs. 12a,b for slant 45$^\circ$ linear transmit polarization (i.e., $E_h = E_v$) and circular transmit polarization, respectively. The shown $\epsilon$ denotes the sign of the H polarization ellipticity angle. The values of the ellipticity angle corresponding to each curve are given in Table 2. Note how not only the shape of bias curves changes, but also the maximum $Z_{dr}$ bias increases significantly for circular transmit polarization. The model shows that the most stringent cross-polar isolation criteria results from the circular polarization transmit condition. As can be seen, if the SHV $Z_{dr}$ bias is to be kept under 0.2 dB, the LDR system limit needs to be about $2^{40}$ dB. Practically, if one of the circular transmit bias curves characterized a radar, the $Z_{dr}$ bias at $\phi_{dp} = 0$ would likely be detected by the user and a $Z_{dr}$ offset correction factor would be used. Then, the maximum $Z_{dr}$ bias would occur for $\phi_{dp} = 180^\circ$.

5. Summary and conclusions

Biases in simultaneous H and V transmit (SHV) radar data were investigated via a radar-scattering model. The

| Table 1. The H and V tilt and ellipticity error angles corresponding to Fig. 9. |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| H tilt | H ellipticity | V tilt | V ellipticity |
| (°)   | (°)           | (°)   | (°)           |
| A     | −0.5          | −0.7  | 89.5          | 0.7                        |
| B     | 0.5           | −0.7  | 90.5          | 0.7                        |
| C     | −0.5          | 0.7   | 89.5          | −0.7                       |
| D     | 0.5           | 0.7   | 90.5          | −0.7                       |

| Table 2. Antenna polarization errors as a function of system LDR limit. The antenna errors are assumed to be orthogonal and elliptical. |
|-----------------------------|-----------------------------|-----------------------------|
| LDR (dB) | $\Im(\xi_h)$ | $E$ (rad) | $\epsilon$ (°) |
| A     | −25       | 0.0281    | 1.61                    |
| B     | −30       | 0.016     | 0.91                    |
| C     | −35       | 0.009     | 0.509                   |
| D     | −40       | 0.005     | 0.286                   |
| E     | −45       | 0.003     | 0.161                   |
model includes the transmit errors, antenna polarization errors, propagation medium, and backscatter medium. The validity of the SHV technique is based on the following two assumptions: 1) a zero mean canting angle of the propagation precipitation medium and 2) negligible antenna polarization errors. Violation of either condition will cause cross coupling of the H and V electric field components, which in turn biases polarimetric variables. Differential reflectivity (Z_{dp}) is particularly sensitive to the cross coupling. How the cross-coupled signal combines (constructively versus destructively) with the primary copolar field component depends on their relative phase, and thus the biases are strong functions of differential propagation phase \( \phi_{dp} \) and differential transmit phase.

The zero mean canting angle is a good approximation for rain but not for ice particles. The Z_{dr} bias resulting from oriented ice crystals has been reported before (Ryzhkov and Zrnić 2007), and the results reported here support that work. Demonstrated here for the first time was the impact of differential transmit phase on SHV Z_{dr}. It was also shown that the phase difference between the H and V transmit signals at the plane of reference (see Fig. 1) will significantly shape the Z_{dr} errors as a function of principal plane \( \phi_{dp} \). For example, consider the ice phase of storms where ice crystals can be aligned by electric fields so that their mean canting can be significantly away from 0°. If the transmit polarization state is circular, only 2° or 3° of \( \phi_{dp} \) need accumulate, which in turn will cause several tenths of a decibel bias in SHV Z_{dr}. This bias will be difficult to correct and can have deleterious effects on particle classification schemes that rely on accurate Z_{dr} estimates.

Antenna error–induced cross coupling was also investigated. The model indicates that antenna errors of the magnitude to be expected from well-designed, center-fed parabolic reflector antennas will cause significant SHV Z_{dr} bias in rain after sufficient \( \phi_{dp} \) accumulation. The model shows that if Z_{dp} bias is to be kept below 0.2 dB, the LDR system limit must be reduced to about -40 dB. This is largely in agreement with Wang and Chandrasekar (2006) who quote a similar requirement of a -44-dB LDR system limit for worst-case antenna errors.

Table 3. The H and V tilt and ellipticity error angles corresponding to Fig. 11. The corresponding LDR system limit is 25 dB.

<table>
<thead>
<tr>
<th></th>
<th>H tilt (°)</th>
<th>H ellipticity (°)</th>
<th>V tilt (°)</th>
<th>V ellipticity (°)</th>
<th>Transmission polarization</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>1.61</td>
<td>90</td>
<td>-1.61</td>
<td>Linear 45°</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>-1.61</td>
<td>90</td>
<td>1.61</td>
<td>Linear 45°</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>1.61</td>
<td>90</td>
<td>-1.61</td>
<td>Circular</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>-1.61</td>
<td>90</td>
<td>1.61</td>
<td>Circular</td>
</tr>
</tbody>
</table>

SHV Z_{dr} bias in rain will also affect Z_{dr} measurements in the ice phase of storms. Any SHV Z_{dr} bias acquired in the rain will be carried forward into the subsequent ice phase along that radar radial; that is, if 0.3 dB of Z_{dp} bias were accumulated in rain before the radar signal reached the ice phase, then the Z_{dr} measurements in the ice phase would also manifest the 0.3-dB bias.

Estimates of the magnitude and type (tilt versus ellipticity angle) of antenna polarization errors for well-designed, center-fed parabolic reflector antennas, such as NCAR’s S-Pol, were estimated. Describing antenna errors with the geometric polarization ellipse parameters of tilt and ellipticity angles help provide insight as to why the antenna errors. If a radar has an LDR measurement limit of -30 dB (determined from measurements in drizzle), the corresponding magnitude of either tilt or ellipticity angle polarization error is about 1°. If the cross correlation of H and V solar measurements are also considered, then the relative values of the tilt and ellipticity angle error can be better estimated. For S-Pol it was shown that the H and V antenna errors are characterized by nearly orthogonal ellipticity angles. The ellipticity angles were estimated based on knowledge of the S-Pol LDR system limit and solar scan measurements. These antenna error estimates were used in the model and were found to predict the experimentally observed S-Pol SHV Z_{dr} bias and the behavior of LDR measurements through a rain medium. This is further addressed in Part II.

Achieving an LDR limit figure of -40 dB is very difficult and the authors are not aware of any S-band weather radars that are capable of this, except for the new
Colorado State University–University of Chicago–Illinois State Water Survey (CSU–CHILL) dual-offset-fed an-
tenna. It may not be cost effective to design a radar with
a center-feed parabolic antenna that achieves such a level
of cross-polar isolation, and this cost must be considered
against the above-mentioned benefits of implementing
the SHV mode dual-polarization radar systems.

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APPENDIX A

Radar Model Details

The details of the radar model can also be found in
Hubbert and Bringi (2003). Because forward scatter is co-
herent (van de Hulst 1957), the propagation medium can be
completely described via a $2 \times 2$ scattering matrix $\mathbf{P}$ as

$$\mathbf{P} = \mathbf{R}(-\theta)\mathbf{P}_0 \mathbf{R}(\theta),$$  \hspace{1cm} (A1)

where $\mathbf{R}$ is the Cartesian rotation matrix and $\mathbf{P}_0$ is the
principal plane propagation matrix

$$\mathbf{P}_0 = \begin{bmatrix} e^{i\lambda_1 z} & 0 \\ 0 & e^{i\lambda_2 z} \end{bmatrix},$$ \hspace{1cm} (A2)

where $\lambda_1$ and $\lambda_2$ are the propagation constants along
the principal planes of the propagation medium and $z$ is
the distance along the direction of propagation. It can
be shown that specific differential attenuation is

$$\Delta \alpha \equiv \frac{\partial}{\partial f} \Delta \beta \equiv \frac{2}{8.686 \times 10^3} \cdot \frac{(\lambda_1^2 - \lambda_2^2)}{r^2},$$ \hspace{1cm} (A3)

where $\phi_{dp}$ is principal plane $\phi_{dp}$. The propagation me-
dium is coupled to the backscatter medium via the radar-
scattering matrix (Kennaugh 1949–1954; Sinclair 1950)

$$\mathbf{S} = \begin{bmatrix} S_{hh} & S_{hv} \\ S_{hv} & S_{vv} \end{bmatrix} = \mathbf{P}^T \begin{bmatrix} S_{hh} & S_{hv} \\ S_{hv} & S_{vv} \end{bmatrix} \mathbf{P},$$ \hspace{1cm} (A4)

where $S_{hh}$, $S_{vv}$, and $S_{hv}$ are backscatter amplitudes in the
H–V polarization basis. The propagation-modified co-
variance matrix is then formed as (Tragl 1990)
\[
\Sigma_v = \begin{bmatrix}
\langle S_{aa}^2 \rangle & \sqrt{2} \langle S_{aa} S_{ab}^* \rangle / 2 & \langle S_{aa} S_{bb}^* \rangle / 2 \\
\sqrt{2} \langle S_{ab} S_{ab}^* \rangle / 2 & \langle S_{ab}^2 \rangle & \langle S_{ab} S_{bb}^* \rangle / 2 \\
\langle S_{bb}^2 \rangle / 2 & \sqrt{2} \langle S_{bb} S_{ab}^* \rangle / 2 & \langle S_{bb}^2 \rangle / 2
\end{bmatrix},
\]  
(A5)

where \( \langle \cdot \rangle \) denotes ensemble average and * signifies complex conjugation. Note that ensemble averaging only applies to the particles in the resolution volume (i.e., backscatter medium). Each matrix member in (A5) consists of propagation terms and backscatter covariances of the form \( \langle S_{x1,y1} S_{x2,y2}^* \rangle \), \((x_{1,2}, y_{1,2} = h, v)\). The H–V basis backscatter covariance matrix \( \Sigma_v \) is simulated via the T-matrix method. An arbitrary mean canting angle \( \alpha \) and mean ellipticity angle can be given to the backscatter covariance matrix \( \Sigma_v \) by rotating this simulated covariance matrix in the plane of polarization with (Tragl 1990)

\[
\Sigma_{hv}(\chi) = T^{-1}(\chi) \Sigma_v T(\chi),
\]  
(A6)

where

\[
T(\chi) = \Gamma^2 \begin{bmatrix}
q^2 & \sqrt{2} q \sqrt{2} \chi & \sqrt{2} \chi^2 \\
\sqrt{2} q \sqrt{2} \chi & 2(1 - q^2 \chi^2) & \sqrt{2} \chi^2 \\
\sqrt{2} \chi^2 & \sqrt{2} \chi^2 & q^2
\end{bmatrix},
\]  
(A7)

and \( \chi \) is the polarization ratio defined as the ratio of the V-to-H electric field components (or electric field phasors) as \( E_v/E_h \) and \( \Gamma = (1 + \chi^2)^{0.5} \). As shown in Hubbert (1994), the phase term \( q = e^{-\tan^{-1}(\tan \alpha \tan \epsilon)} \) is necessary to maintain a constant phase difference between the elliptic basis polarization vectors. The \( \chi \) can be expressed in terms of the mean tilt (or canting) angle \( \alpha \) and mean ellipticity angle \( \epsilon \) as (Azzam and Bashara 1989)

\[
\chi = \frac{\tan \alpha + j \tan \epsilon}{1 - j \tan \alpha \tan \epsilon}.
\]  
(A8)

In this paper, cross coupling caused by antenna polarization errors and by the nonzero mean canting angle of the propagation medium are the focus and the backscatter medium is of secondary importance. Therefore, the mean tilt angle of the backscatter medium is assumed to be zero and this implies that the co- to cross-covariance terms in the backscatter 3 \( \times \) 3 covariance matrix are zero.

**APPENDIX B**

Cross Correlation of Horizontal and Vertical Sun Measurements

Antenna beam-integrated passive solar measurements can be represented as

\[
\begin{bmatrix}
V_h \\
V_v
\end{bmatrix} =
\begin{bmatrix}
i_h \\
i_v
\end{bmatrix}
\begin{bmatrix}
\xi_h & \xi_v \\
\xi_v & \xi_v
\end{bmatrix}
\begin{bmatrix}
E_h^* \\
E_v^*
\end{bmatrix},
\]  
(B1)

where \( \xi_h \) and \( \xi_v \) are the complex antenna polarization errors of the H and V channels, respectively, and \( E_h^*, v \) are the incident electric fields and received voltages. Radar gain and proportionality constants are nonessential and are omitted. We wish to find the correlation

\[
\Omega = \frac{\langle V_h V_v^* \rangle}{\sqrt{\langle |V_h|^2 \rangle \langle |V_v|^2 \rangle}},
\]  
(B2)

where \( \langle \cdot \rangle \) indicates time average. Expanding the numerator yields

\[
\langle V_h V_v^* \rangle = \langle i_h \xi_h^* |E_h^*|^2 \rangle + \langle i_v \xi_v^* |E_v^*|^2 \rangle
\]

\[
+ \langle i_h \xi_h |E_v^*|^2 \rangle \xi_v^* E_v^* E_h^* + \langle i_v \xi_v |E_v^*|^2 \rangle \xi_h^* E_h^* E_v^*.
\]  
(B3)

Because solar radiation is unpolarized, the \( E_h \) to \( E_v \) cross correlations are zero so that

\[
\langle V_h V_v^* \rangle = \langle i_h \xi_h^* |E_h^*|^2 \rangle + \langle i_v \xi_v^* |E_v^*|^2 \rangle.
\]  
(B4)

Similarly,

\[
\langle |V_h|^2 \rangle = \langle |E_h|^2 \rangle \xi_h^2 + \langle |\xi_h|^2 \rangle |E_h|^2,
\]  
(B5)

\[
\langle |V_v|^2 \rangle = \langle |E_v|^2 \rangle \xi_v^2 + \langle |\xi_v|^2 \rangle |E_v|^2.
\]  
(B6)

If the errors are small so that \( \xi_h \ll i_h \) and \( \xi_v \ll i_v \), then

\[
\langle |V_h|^2 \rangle \approx \langle \xi_h^2 |E_h|^2 \rangle,
\]  
(B7)

\[
\langle |V_v|^2 \rangle \approx \langle \xi_v^2 |E_v|^2 \rangle.
\]  
(B8)

Using Eqs. (B4), (B7), and (B8), the correlation can be written as

\[
\Omega = \frac{\langle i_h \xi_h^* |E_h|^2 \rangle + \langle i_v \xi_v^* |E_v|^2 \rangle}{\sqrt{\langle \xi_h^2 |E_h|^2 \rangle \langle \xi_v^2 |E_v|^2 \rangle}}.
\]  
(B9)

The H and V solar radiation power should be equal, that is, \( |E_h|^2 = |E_v|^2 \). Assuming that \( |\xi_h| \) and \( |\xi_v| \) are small, then \( i_h \approx i_v \approx 1 \). Then, Eq. (B9) can be approximated with

\[
\Omega = |\xi_h + \xi_v|.
\]  
(B10)

Finally, \( \Omega = \xi_h + \xi_v \) because by definition \( \xi_{h,v} \) are beam-integrated, constant antenna errors.
APPENDIX C

An Estimate of System Limit LDR and SHV Z_{dr}

a. LDR in drizzle

To derive radar covariances in drizzle we begin with equations for scattering from a single particle, taking into account antenna errors (Bringi and Chandrasekar 2001). Using Eqs. (1) and (2), the received voltages can be modeled in drizzle as

\[
\begin{bmatrix}
V^h_x \\
V^v_x
\end{bmatrix} = \begin{bmatrix}
i_h & 0 \\
i_v & 1
\end{bmatrix} \begin{bmatrix}
s_{hh} & 0 \\
s_{vv} & s_{hv}
\end{bmatrix} E^0
\]

\[
\begin{bmatrix}
V^h_v \\
V^v_v
\end{bmatrix} = \begin{bmatrix}
i_h & 0 \\
i_v & 1
\end{bmatrix} \begin{bmatrix}
s_{hh} & 0 \\
s_{vv} & s_{hv}
\end{bmatrix} E^0
\]

\[\text{Eq. (C1)}\]

where \(s_{hh}\) and \(s_{vv}\) are the backscatter coefficient amplitudes, \(\xi_{h,v}\) are the H and V antenna polarization errors, and \(E^0\) are the transmitted H and V electric fields. Radar gain and proportionality constants are nonessential and are omitted. Executing the matrix multiplication Eq. (C1) yields

\[
\begin{bmatrix}
V^h_x \\
V^v_x
\end{bmatrix} = \begin{bmatrix}
\hat{v}^h_{hh} + \xi^h_{hh} s_{hv} \\
\hat{v}^h_{vv} + \xi^h_{hv} s_{hv}
\end{bmatrix} E^0
\]

\[\text{Eq. (C2)}\]

Next, the appropriate covariance products are then taken and the products are integrated over the particle distribution and over the antenna pattern. Here we assume beam-integrated antenna errors. The LDR for the transmit state \(E^0 = 1\), \(E^0 = 0\) is

\[
\text{LDR}_t = \frac{\langle V^0_x \rangle}{\langle V^0_h \rangle} = \frac{\langle i_h \xi_{hh} s_{hv} + i_v \xi_{hv} s_{hv} \rangle}{\langle \hat{v}^h_{hh} + \hat{v}^h_{vv} \rangle}
\]

\[\text{Eq. (C3)}\]

where \(\langle \cdot \rangle\) stands for ensemble average.

Because we are interested in isolating the effects of antenna errors, several simplifying observations and assumptions are made. First, we assume that the antenna errors are small so that \(\xi_{h,v} \ll i_h\) and \(i_h \approx i_v \approx 1\). For example, if we assume that \(\xi_{h,v} \approx i_h\), then for a LDR limit of \(-30\) dB, \(i_h = i_v = 0.99975\). Eq. (C3) reduces to

\[
\text{LDR}_t = \frac{\langle |\xi_{hh} s_{hv} + i_v \xi_{hv} s_{hv}|^2 \rangle}{\langle |\hat{v}_{hh} |^2 \rangle}
\]

\[\text{Eq. (C4)}\]

b. SHV \(Z_{dr}\) bias and antenna errors

We begin with Eq. (C1). Propagation effects can be included in the model by attaching the coefficient \(e^{i\beta}\) to the \(s_{hv}\) term, where \(\nu\) is a complex constant. The voltages can then be expressed as

\[
\begin{bmatrix}
V^h_x \\
V^v_x
\end{bmatrix} = \begin{bmatrix}
\hat{v}^h_{hh} + \xi^h_{hh} s_{hv} + i_v \xi_{hv} s_{hv} \\
\hat{v}^h_{vv} + \xi^h_{hv} s_{hv}
\end{bmatrix} E^0
\]

\[\text{Eq. (C6)}\]

Without loss of generality, divide through the scattering matrix by \(s_{vv}\) and let \(\varsigma = s_{hh}/s_{hv}\). For simultaneous H and V transmissions, let \(E^0 = 1\) and \(E^0 = e^{i\beta}\), where \(\beta\) represents the phase difference between the H and V transmission signals. Then,

\[
Z_{dr}^{s_h} = 10 \log_{10} \frac{\langle |i_h \xi_{hh} s_{hv} + i_v \xi_{hv} s_{hv}|^2 \rangle}{\langle |\hat{v}^h_{hh} + \hat{v}^h_{vv} |^2 \rangle}
\]

\[\text{Eq. (C7)}\]

The magnitude-squared operation could be executed, ensemble averages taken, and antenna errors separated from particle ensemble averages. Here we retain the argument of the magnitude-squared operation in order to examine the effects of antenna errors. Notice that the cross-polar term \(e^{i\beta} (i_h \xi_{hv} s_{hv} + i_v \xi_{hh} s_{hv})\) appears in the numerator of LDR and also appears in both the numerator and denominator of SHV \(Z_{dr}\). Thus, in general, as the LDR system limit increases because of an increase in \(\xi_{hh} + \xi_{hv}\) the maximum bias in SHV \(Z_{dr}\) also will increase. Where the maximum SHV \(Z_{dr}\) bias occurs depends upon the phase of \(e^{i\beta}\) and the magnitude and phase of \(s_{hv}\). Thus, because the LDR system limit of a radar is typically a known performance characteristic of a radar, it is useful to plot SHV \(Z_{dr}\) bias using the LDR system limit as a parameter as is done in this paper. Again, to simplify, for small antenna polarization errors, \(i_h = i_v \approx 1\) and the second-order \(\xi\) are dropped. Note that the model used in the paper retains all of these terms. Equation (C7) simplifies to

\[
Z_{dr}^{s_h} = 10 \log_{10} \frac{\langle |\xi_{hv} e^{i\beta} (\hat{v}_{hv} + \xi_{hh} s_{hv})|^2 \rangle}{\langle |\hat{v}_{hv} + \hat{v}_{hv} |^2 \rangle}
\]

\[\text{Eq. (C8)}\]

This equation shows how differential propagation phase (embedded in \(\nu\)) and differential transmission phase (\(\beta\)) affect SHV \(Z_{dr}\). Importantly, it shows that for small \(\varsigma\) and a phase of \(e^{i\beta} = 0\), the antenna errors approximately appear as \(\xi_{hv} + \xi_{hv}\), which is the same form as found in the expression for the system LDR limit (as opposed to
$\xi_h + \xi_v$). Finally, if the precipitation medium is drizzle, then $\zeta = 1$ and $\upsilon = 0$ and Eq. (37) results.

REFERENCES


