The Shape–Slope Relation in Observed Gamma Raindrop Size Distributions: Statistical Error or Useful Information?

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ABSTRACT

The three-parameter gamma distribution $n(D) = N_0 D^m \exp(-\Lambda D)$ is often used to characterize a raindrop size distribution (DSD). The parameters $m$ and $\Lambda$ correspond to the shape and slope of the DSD. If $m$ and $\Lambda$ are related to one another, as recent disdrometer measurements suggest, the gamma DSD model is simplified, which facilitates retrieval of rain parameters from remote measurements. It is important to determine whether the $m-\Lambda$ relation arises from errors in estimated DSD moments, or from natural rain processes, or from a combination of both statistical error and rain physics.

In this paper, the error propagation from moment estimators to rain DSD parameter estimators is studied. The standard errors and correlation coefficient are derived through systematic error analysis. Using numerical simulations, errors in estimated DSD parameters are quantified. The analysis shows that errors in moment estimators do cause correlations among the estimated DSD parameters and cause a linear relation between estimators $m$ and $\Lambda$. However, the slope and intercept of the error-induced relation depend on the expected values $m$ and $\Lambda$, and it differs from the $m-\Lambda$ relation derived from disdrometer measurements. Further, the mean values of the DSD parameter estimators are unbiased. Consequently, the derived $m-\Lambda$ relation is believed to contain useful information in that it describes the mean behavior of the DSD parameters and reflects a characteristic of actual raindrop size distributions. The $m-\Lambda$ relation improves retrievals of rain parameters from a pair of remote measurements such as reflectivity and differential reflectivity or attenuation, and it reduces the bias and standard error in retrieved rain parameters.

1. Introduction

Accurate characterization of raindrop size distribution (DSD) and the estimation of DSD parameters using remote measurements are needed for inferring rain microphysics. Because various factors contribute to the formation and evolution of rain DSDs, a single explicit functional form has not been found. Hence, simple functions have been used to model a rain DSD.

Historically, an exponential distribution with two parameters was used to characterize rain DSD. Special cases of exponential DSDs were determined by Marshall and Palmer (1948) and Laws and Parsons (1943). However, subsequent DSD measurements have shown that the exponential distribution does not capture “instantaneous” rain DSDs and a more general function is necessary.

Ulbrich (1983) suggested the use of the gamma distribution for representing rain DSD as

$$n(D) = N_0 D^m \exp(-\Lambda D).$$

The gamma DSD with three parameters ($N_0$, $m$, and $\Lambda$) is capable of describing a broader range of raindrop size distributions than an exponential distribution (a special case of the gamma distribution with $m = 0$). The three parameters of the gamma DSD can be obtained from three estimated moments. It was shown that the three parameters are not mutually independent (Ulbrich 1983; Chandrasekar and Bringi 1987; Kozu 1991; Haddad et al. 1997). Hence attempts were made to derive rain
Analysis of DSD data collected in Florida during the summer of 1998 revealed a high correlation between \( \mu \) and \( \Lambda \), suggesting that a useful \( \mu-\Lambda \) relation could be derived (Zhang et al. 2001). The resulting relation was used to retrieve rain DSDs from S-band polarization radar measurements of reflectivity \( (Z) \) and differential reflectivity \( (Z_{DR}) \). An independent analysis of DSD observations collected in Oklahoma also indicated the existence of a \( \mu-\Lambda \) relation similar to that observed in Florida (Brandes et al. 2003). Figure 1a shows a scatterplot between \( \mu \) and \( \Lambda \) for the Florida dataset. Figure 1b presents the relation for the cases with rain \( R > 5 \text{ mm h}^{-1} \) and large number of counts \( (C_r > 1000) \). The revised relation is

\[
\Lambda = 0.0365\mu^2 + 0.735\mu + 1.935. \tag{2}
\]

The relation also holds for the estimated DSD parameters using the truncated moment method (Ulbrich and Atlas 1998; Vivekanandan et al. 2003). It is noted that the scatterplot between \( \mu \) and \( \Lambda \) for the filtered rain cases in Fig. 1b has a higher correlation than that without filtering in Fig. 1a. Therefore, the relation (2) derived from the quality controlled dataset in Fig. 1b should have smaller error.

The median volume diameter \( (D_0) \) and the standard deviation of mass distribution \( (\sigma_m) \) are related to \( \mu \) and \( \Lambda \) (Ulbrich 1983). If the relation (2) holds, \( D_0 \) and \( \sigma_m \) depend only on \( \mu \) as shown in Fig. 2. Both \( D_0 \) and \( \sigma_m \) decrease as \( \mu \) increases, which means that rain with large (small) \( D_0 \) corresponds to a wide (narrow) distribution of rain DSD. As \( \mu \) changes from 10 to \(-2\), \( D_0 \) increases from 1.1 to 2.7 mm and \( \sigma_m \) increases from 0.3 to 2.4 mm, a range that includes most heavy and medium rain-rate cases. Thus, the \( \mu-\Lambda \) relation suggests that a characteristic size parameter such as \( D_0 \) and the shape of a raindrop spectrum are also related. The relation simplifies the three-parameter gamma distribution to a two-parameter constrained gamma DSD model.
However, since the η–Λ relation was reported, there has been concern as to whether the relation arises purely from statistical error or represents a physical property of rain.

In this paper, we describe a detailed analysis of error propagation from DSD moment estimators to DSD parameter estimators and evaluate the improvement in physical parameter retrievals using the η–Λ relation. The effects of moment estimator errors on the moments, the DSD parameters, and 2) why is it useful for retrieving rain parameters from remote measurements. In section 2, we derive the standard errors of estimated DSD parameters analytically. The effects of moment estimator errors on the estimated DSD parameters are then studied using numerical simulations in section 3. In section 4, the usefulness of the η–Λ relation is examined by comparing DSD retrievals using the η–Λ relation with those obtained using a fixed value of η. Finally, we summarize the work and discuss the findings in section 5.

2. Theoretical analysis of error propagation

Disdrometer measurements of rain DSD are usually processed by calculating the statistical moments. The estimated moments of disdrometer observations are often used to estimate the DSD governing parameters such as N₀, μ, and Λ in (1), and then the DSD parameters are used to calculate physical parameters such as rain rate, characteristic size, and radar measurable. It is important to study how the errors associated with DSD measurements, the estimated DSD moments, and the DSD model propagate to the estimated DSD parameters and physical parameters. It is also worth pointing out that remote measurements of rain, such as reflectivity factor, attenuation, and phase shift at various wavelengths and polarizations, can be approximated as moments of rain DSD in a sampling volume. Errors in the determination of the moments occur both in the radar measurement and in the estimation of the moments from the radar data.

\[
\text{var}(\hat{\mu}) = \left(\frac{d\mu}{d\eta}\right)^2 \left(\frac{\partial\eta}{\partial M_2}\right)^2 \text{var}(\hat{M}_2) + \left(\frac{\partial\eta}{\partial M_4}\right)^2 \text{var}(\hat{M}_4) + \left(\frac{\partial\eta}{\partial M_6}\right)^2 \text{var}(\hat{M}_6) + 2 \frac{\partial\eta}{\partial M_2} \frac{\partial\eta}{\partial M_4} \text{cov}(\hat{M}_2, \hat{M}_4) \\
+ 2 \frac{\partial\eta}{\partial M_2} \frac{\partial\eta}{\partial M_6} \text{cov}(\hat{M}_2, \hat{M}_6) + 2 \frac{\partial\eta}{\partial M_4} \frac{\partial\eta}{\partial M_6} \text{cov}(\hat{M}_4, \hat{M}_6),
\]

\[
\text{var}(\hat{\Lambda}) = \left(\frac{d\Lambda}{d\mu}\right)^2 \text{var}(\hat{\mu}) + \frac{\partial\Lambda}{\partial M_2} \frac{\partial\Lambda}{\partial M_2} + 2 \frac{\partial\Lambda}{\partial M_2} \frac{d\mu}{d\eta} \frac{\partial\eta}{\partial M_2} \text{var}(\hat{M}_2) + \frac{\partial\Lambda}{\partial M_4} \frac{\partial\Lambda}{\partial M_4} + 2 \frac{\partial\Lambda}{\partial M_4} \frac{d\mu}{d\eta} \frac{\partial\eta}{\partial M_4} \text{var}(\hat{M}_4) \\
+ 2 \frac{\partial\Lambda}{\partial M_2} \frac{\partial\Lambda}{\partial M_4} + \frac{\partial\Lambda}{\partial M_2} \frac{\partial\Lambda}{\partial M_6} \frac{\partial\mu}{d\eta} \frac{\partial\eta}{\partial M_2} \frac{d\eta}{d M_4} \text{cov}(\hat{M}_2, \hat{M}_4) + 2 \frac{\partial\Lambda}{\partial M_2} \frac{\partial\Lambda}{\partial M_6} \frac{d\mu}{d\eta} \frac{\partial\eta}{\partial M_2} \frac{d\eta}{d M_6} \text{cov}(\hat{M}_2, \hat{M}_6)
\]

a. Error propagation from DSD moments to DSD parameters

For the gamma DSD, the three parameters can be estimated from any three moment estimators, \(\hat{M}_n\), with means of \(M_n = N_0 \Lambda^{(\mu/n+1)} \Gamma(\mu + n + 1)\). As an example, given the 2d, 4th, and 6th (\(M_2, M_4, M_6\)) moments, the DSD parameters (\(N_0, \mu, \Lambda\)) can be written as

\[
\mu = \frac{(7 - 11\eta) - (\eta^3 + 14\eta + 1)^{1/2}}{2(\eta - 1)},
\]

\[
\Lambda = \left[\frac{M_2 \Gamma(\mu + 5)}{M_4 \Gamma(\mu + 3)}\right]^{1/2}, \text{ and}
\]

\[
N_0 = \frac{M_6 \Lambda^{(\mu + n + 1)}}{\Gamma(\mu + n + 1)}, \quad (n = 2, 4, \text{ or } 6).
\]

where the ratio of the moments is \(\eta = M_2/(M_2 M_4)\). Obviously, the DSD parameters are nonlinear functions of the DSD moments.

Since the moment estimators (\(\hat{M}_2, \hat{M}_4, \text{ and } \hat{M}_6\)) contain measurement errors due to system noise or finite sampling, the estimated gamma DSD parameters (\(\hat{N}_0, \hat{\mu}, \hat{\Lambda}\)) also have error. Even if the moment estimators were determined precisely, the estimated DSD parameters would fluctuate due to the fact that natural rain DSDs may not exactly follow the assumed gamma distribution.

To understand the error propagation from the moment estimators to the retrieved DSD parameters, a detailed error analysis is described in appendix A, which is based on the first-order approximation (Papoulis 1965, section 7-3). The first-order theory is valid for small fractional moment errors (<20%) that are representative of quality-controlled datasets with sufficient drop counts (>1000) (Wong and Chidambaram 1985). Both the moment estimators and the DSD parameter estimators are written as sums of their means and fluctuations. The variances and covariance of DSD parameter estimators \(\hat{\mu}\) and \(\hat{\Lambda}\) are represented as.
\[ + 2 \frac{\partial \Lambda}{\partial \mu} \frac{\partial \Lambda}{\partial \eta} \frac{\partial \eta}{\partial \mu} \frac{\partial \eta}{\partial \mu} \text{cov}(\hat{M}_4, \hat{M}_6), \text{ and} \]

\[
\text{cov}(\hat{\mu}, \hat{\Lambda}) = \frac{d\mu}{d\eta} \frac{\partial \eta}{\partial \mu} \left( \frac{\partial \Lambda}{\partial \mu} + \frac{\partial \Lambda}{\partial \eta} \frac{\partial \eta}{\partial \mu} \right) \text{var}(\hat{M}_2) + \frac{d\mu}{d\eta} \frac{\partial \eta}{\partial \mu} \frac{\partial \eta}{\partial \mu} \text{var}(\hat{M}_4) + \frac{d\Lambda}{d\eta} \frac{\partial \eta}{\partial \mu} \frac{\partial \eta}{\partial \mu} \text{var}(\hat{M}_4) + \frac{2 \frac{\partial \Lambda}{\partial \mu} \frac{\partial \eta}{\partial \mu} \frac{\partial \eta}{\partial \mu}}{\frac{\partial \Lambda}{\partial \mu} \frac{\partial \eta}{\partial \mu} \frac{\partial \eta}{\partial \mu}} \text{cov}(\hat{M}_2, \hat{M}_4) + \frac{d\mu}{d\eta} \frac{\partial \eta}{\partial \mu} \frac{\partial \eta}{\partial \mu} \text{cov}(\hat{M}_2, \hat{M}_6) + \frac{2 \frac{\partial \Lambda}{\partial \mu} \frac{\partial \eta}{\partial \mu} \frac{\partial \eta}{\partial \mu}}{\frac{\partial \Lambda}{\partial \mu} \frac{\partial \eta}{\partial \mu} \frac{\partial \eta}{\partial \mu}} \text{cov}(\hat{M}_4, \hat{M}_6).
\]

(8)

where the derivatives and partial derivatives are given in appendix A.

The covariance terms among the moment estimators are included in (6)–(8). However, the correlations among errors in the moment estimators depend on what type of error source is dominant in the dataset. For the DSD measurements, sampling errors among the moment estimates tend to be correlated. The closer the two moments, the higher the correlation. A high-order moment estimator (e.g., \( \tilde{M}_6 \)) may have little correlation with a lower-order moment estimator (\( \tilde{M}_2 \)). For remote measurements, the errors in reflectivity measurements at two different frequency channels are uncorrelated, while those at a dual polarization are partially correlated depending on how the signals are collected. The errors due to system noise are uncorrelated. Detailed study of the correlations among the moment errors is beyond the scope of the present study but has been studied in detail by Chandrasekar and Bringi (1987). Consequently, a number of correlation coefficients such as 0, 0.5, and 0.8 are assumed for this study.

The square roots of (6) and (7) give the standard deviations of \( \hat{\mu} \) and \( \hat{\Lambda} \), which are denoted as std(\( \hat{\mu} \)) and std(\( \hat{\Lambda} \)), respectively. The standard deviations are plotted for various parameters in Figs. 3 and 4. Figure 3 shows the std(\( \hat{\mu} \)) and std(\( \hat{\Lambda} \)) as a function of the relative standard error of moment estimators, std(\( \tilde{M}_n/\tilde{M}_n \)). The correlation coefficients among the moment estimators used for the calculations are fixed at \( \rho(\tilde{M}_2, \tilde{M}_4) = 0.5 \), \( \rho(\tilde{M}_4, \tilde{M}_6) = 0.5 \), and \( \rho(\tilde{M}_2, \tilde{M}_6) = 0.25 \). As expected, the standard deviations of \( \hat{\mu} \) and \( \hat{\Lambda} \) increase as the errors in moments increase. It is noted that the standard errors also depend on their expected values. The errors for large values of \( \mu \) and \( \Lambda \) can be many times larger than those for small values of \( \mu \) and \( \Lambda \). This might be the reason for the large scatter on the right side of Fig. 1a.

Figure 4 shows the standard errors and correlation coefficient of \( \hat{\mu} \) and \( \hat{\Lambda} \) as a function of the correlation coefficients among the moment errors for fixed relative errors in moment estimators of 5%. For convenience in showing the results, the correlation coefficients among

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**Fig. 3. Analytical results of the standard errors of DSD parameter estimators as a function of the relative error of the moment estimators for fixed correlations among moment errors of \( \rho(\tilde{M}_2, \tilde{M}_4) = 0.5 \), \( \rho(\tilde{M}_4, \tilde{M}_6) = 0.5 \), and \( \rho(\tilde{M}_2, \tilde{M}_6) = 0.25 \): (a) std(\( \hat{\mu} \)) and (b) std(\( \hat{\Lambda} \)).**
the moment estimators are chosen such that $\rho(M_z, \tilde{M}_z) = \rho(M_s, \tilde{M}_s)$ and $\rho(M_2, \tilde{M}_4) = \rho(M_4, \tilde{M}_2)$. As shown in Figs. 4a and 4b, the standard errors in $\bar{\mu}$ and $\bar{\Lambda}$ estimates decrease as the correlations among moment estimators increase, a consequence of the fact that correlated moment errors tend to cancel each other in the retrieval of $\bar{\mu}$ and $\bar{\Lambda}$ values. Such error cancellation is obvious in the ratio $\tilde{\eta} = M_2/(\bar{M}_4 \bar{M}_6)$. The standard error of $\tilde{\eta}$ follows the same trend as that of $\bar{\mu}$. The correlation coefficient between $\bar{\mu}$ and $\bar{\Lambda}$, $\rho = \text{cov}(\bar{\mu}, \bar{\Lambda})/\sqrt{\text{var}(\bar{\mu}) \text{var}(\bar{\Lambda})}$, calculated from (6)–(8), is shown in Fig. 4c and is very high ($>0.9$).

Further analysis shows that the estimated $\bar{\mu}$ is highly sensitive to the change in $\tilde{\eta}$ due to errors in the moment estimators; as a result, the variance of $\bar{\mu}$ is large. For the same reason, the first term in (7), the variance of $\bar{\mu}$, is the dominant term in the variance of $\tilde{\eta}$. Therefore, $\bar{\mu}$ and $\bar{\Lambda}$ are highly correlated. The high correlation leads to a linear relation between the standard deviations of $\bar{\mu}$ and $\bar{\Lambda}$, $\text{std}(\Lambda) \approx \text{std}(\mu)$. This approximate relation between the estimation errors is given by

$$\delta \bar{\Lambda} = \frac{\partial \Lambda}{\partial \mu} \delta \bar{\mu} = \frac{\Lambda (\mu + 3.5)}{(\mu + 4) (\mu + 3)} \delta \bar{\mu}. \quad (9)$$

A simple way to study the relation between the error of estimators $\bar{\mu}$ and $\bar{\Lambda}$ is to start from $\Lambda = (\mu + 3.67)/D_0$. Taking the differential gives $\delta \Lambda = (1/D_0) \delta \mu - (\mu + 3.67/D_0^2) \delta D_0$. Since the error in $D_0$ tends to be small (any good fitting procedure would have small error in estimating physical parameters), neglecting the second term yields

$$\delta \Lambda \approx \frac{1}{D_0} \delta \mu = \frac{\Lambda}{(\mu + 3.67)} \delta \mu. \quad (10)$$

It can be seen that (9) and (10) are very similar. We replace the errors ($\delta \bar{\mu}$, $\delta \bar{\Lambda}$) in (10) with the differences of their estimators ($\mu(\bar{\mu}, \bar{\Lambda})$ and expected values ($\mu$, $\Lambda$) in (10), respectively, and obtain an artifact linear relation between $\bar{\mu}$ and $\bar{\Lambda}$ estimators, as given by

$$\bar{\Lambda} \approx \frac{\Lambda}{(\mu + 3.67)} (\bar{\mu} - \mu) + \Lambda. \quad (11)$$

The $\mu$–$\bar{\Lambda}$ relation (11) looks like the $\mu$–$\Lambda$ relation (2) as they both have a positive roughly similar slope. However, they are essentially different in terms of both physics and functional relation, although we acknowledge that statistical errors and physical variations are difficult to separate. Equation (2) is a general relation for the mean $\mu$ and $\Lambda$ values, whereas (11) is an artificial relation between the estimators $\bar{\mu}$ and $\bar{\Lambda}$ for a pair of expected values ($\mu$, $\Lambda$) because of the introduced errors. The slope and intercept of (11) depend on the expected values ($\mu$, $\Lambda$), and the overall relation between $\mu$ and $\Lambda$ remains unspecified. The difference will be studied further through simulations in the next section.
b. Error propagation from DSD parameters to physical parameters

When gamma rain DSD parameters \((N_0, \mu, \Lambda)\) are known, physical characteristics of rain, such as rain rate \((R)\) and median volume diameter \((D_o)\), can be easily calculated with

\[
R = 7.125 \times 10^{-4} N_0 \Lambda^{-\mu + 4.67} \Gamma(\mu + 4.67)\] and \((12)\)

Since the DSD parameters contain errors inherited from the moment estimates or remote measurements, the rain-rate estimators \(\hat{R}\) and median volume diameter estimators \(\hat{D}_o\) also have error components. The variances of \(\hat{R}\) and \(\hat{D}_o\) are derived in appendix B as

\[
\text{var}(\hat{R}) = \frac{R^2}{N_0^2} \text{var}(\hat{N}_0) + \left[ \frac{\Gamma'(\mu + 4.67)}{\Gamma(\mu + 4.67)} - \ln(\Lambda) \right]^2 R^2 \text{var}(\hat{\mu}) + \frac{(\mu + 4.67)^2}{\Lambda^2} R^2 \text{var}(\hat{\Lambda})
\]

\[
+ 2 \frac{R^2}{N_0} \left[ \frac{\Gamma'(\mu + 4.67)}{\Gamma(\mu + 4.67)} - \ln(\Lambda) \right] \text{cov}(\hat{N}_0, \hat{\mu}) - 2 \frac{R^2(\mu + 4.67)}{N_0 \Lambda} \text{cov}(\hat{N}_0, \hat{\Lambda})
\]

\[
\text{var}(\hat{D}_o) = \frac{1}{\Lambda^2} \text{var}(\hat{\mu}) + \frac{D_o^2}{\Lambda^2} \text{var}(\hat{\Lambda}) - 2 \frac{D_o}{\Lambda} \text{cov}(\hat{\mu}, \hat{\Lambda}).
\]

It is noted that there is a negative sign in front of the last covariance term, \(\text{cov}(\hat{\mu}, \hat{\Lambda})\), in \((14)\) and \((15)\). Therefore, the positive correlation between the estimation errors of \(\hat{\mu}\) and \(\hat{\Lambda}\) reduces the errors in \(\hat{R}\) and \(\hat{D}_o\) estimators for given errors in DSD parameter estimators.

If the approximate relation \((10)\) is used in the derivation for the variances of \(\hat{R}\) and \(\hat{D}_o\), approximate expressions are obtained as

\[
\text{var}(\hat{R}) = \frac{R^2}{N_0^2} \text{var}(\hat{N}_0) + \left[ \frac{\Gamma'(\mu + 4.67)}{\Gamma(\mu + 4.67)} - \ln(\Lambda) - \frac{\mu + 4.67}{\mu + 3.67} \right]^2 R^2 \text{var}(\hat{\mu})
\]

\[
+ 2 \frac{R^2}{N_0} \left[ \frac{\Gamma'(\mu + 4.67)}{\Gamma(\mu + 4.67)} - \ln(\Lambda) - \frac{\mu + 4.67}{\mu + 3.67} \right] \text{cov}(\hat{N}_0, \hat{\mu}) \quad \text{and}
\]

\[
\text{var}(\hat{D}_o) = \frac{1}{\Lambda^2} \left[ 1 - \left( \frac{\mu + 3.67}{\Lambda D_o} \right)^2 \right] \text{var}(\hat{\mu}) = 0.
\]

In this case, the standard error of rain-rate estimates is further reduced as compared with that by uncorrelated errors in DSD parameters, and the standard deviation of the estimator \(\hat{D}_o\) is minimized. It is also noted that the mean values of retrieved physical parameters should not be biased by the fluctuation errors in the DSD parameters. In other words, the artifact linear relation between estimators \(\hat{\mu}\) and \(\hat{\Lambda}\) is the requirement of unbiased moments and it leads to minimum error in rain parameters. This becomes even clearer in the simulation results in the next section.

3. Numerical simulations

In support of the error analysis described in the previous section, a numerical simulation was performed to study the standard errors in the estimates of the DSD parameters \(\hat{\mu}\) and \(\hat{\Lambda}\). The steps involved in the simulation are as follows.

1) Assign gamma DSD parameters with a set of specific values (inputs) for \(N_0, \mu, \text{ and } \Lambda\).
2) Calculate the expected moments \((M_2, M_4, \text{ and } M_6)\) for the specified gamma DSD parameters.
3) Randomize the moments with given standard deviations as \(\hat{M}_n = M_n + \delta M_n (n = 2, 4, 6)\). The fluctuations \(\delta M_n\) of the moments represent error and are assumed to be uniformly distributed with a zero mean (a Gaussian error distribution makes almost no difference in the result). Calculate the estimated DSD parameters \((\hat{N}_0, \hat{\mu}, \hat{\Lambda})\) from the randomized moments using Eqs. \((3)-(5)\).
errors in the estimated physical parameters \( \hat{R} \) and \( \hat{D}_o \). Rather, the standard errors of \( \hat{R} \) and \( \hat{D}_o \) are very small. The relative standard deviation of \( \text{std}(\hat{R})/R = 5.21\% \) is close to the introduced errors in moments of 5% and that of \( \text{std}(\hat{D}_o)/\hat{D}_o = 2.64\% \) is even smaller than the relative moment errors. This shows that the moment fitting procedure does not significantly amplify the moment errors. More importantly, the means of the estimated DSD parameters, \( \langle N_s \rangle = 8395.4, \langle \mu \rangle = 0.071, \) and \( \langle \Lambda \rangle = 1.967 \) are very close to the input values. The mean rain rate and median volume diameter are \( \langle \hat{R} \rangle = 38.58 \text{ mm h}^{-1} \) and \( \langle \hat{D}_o \rangle = 1.897 \text{ mm} \) and are essentially identical to the input values of 38.62 \text{ mm h}^{-1} and 1.897 mm.

Figure 5b shows the simulation result when the errors of moment estimators are taken to be correlated but with the same relative standard deviation of 5% as that in Fig. 5a. The correlation coefficients used for the simulation are \( \rho(M_2, M_4) = 0.8, \rho(M_4, M_6) = 0.8, \) and \( \rho(M_2, M_6) = 0.64. \) As shown in Fig. 5b, the standard errors of \( \hat{\mu} \) and \( \hat{\Lambda} \) are reduced considerably from those when the moment errors are not correlated (Fig. 5a). This shows that correlated moment errors cause smaller errors in estimated gamma DSD parameters and have less effect on the \( \mu-\Lambda \) relation than uncorrelated errors as shown in Figs. 4a and 4b. Again, the means of the estimated DSD parameters, \( \langle \hat{N}_s \rangle, \langle \hat{\mu} \rangle, \) and \( \langle \hat{\Lambda} \rangle, \) as well as the means of rain rate \( \langle \hat{R} \rangle \) and median volume diameter \( \langle \hat{D}_o \rangle \), are almost the same as the input values. The relative standard errors of \( \hat{R} \) and \( \hat{D}_o \) are also very small as std(\( \hat{R} \))/\( \hat{R} \) = 5.13% and std(\( \hat{D}_o \))/\( \hat{D}_o \) = 1.32%, respectively.

The estimates \( \hat{\mu} \) and \( \hat{\Lambda} \) scatter mainly along a line centered at the input \( \mu = 0.0 \) and \( \Lambda = 1.935 \). There is a high correlation between \( \hat{\mu} \) and \( \hat{\Lambda} \) due to the added errors in the estimated moments. The correlation leads to an artifact linear relation between \( \hat{\mu} \) and \( \hat{\Lambda} \) as shown in (11). Note, however, that the slope and intercept of the relation depend on the input values \( \mu \) and \( \Lambda \), that is, the means of the estimated moments rather than the fluctuation errors. Also, the scatterplot in Fig. 5 is for one pair of \( (\mu, \Lambda) = (0.0, 1.935) \) rather than a dataset consisting of various pairs of \( (\mu, \Lambda) \) values. Therefore, the artifact relation in Fig. 5 is different from the \( \mu-\Lambda \) relation (2) derived from the measurements with quality-controlled data in Fig. 1b. Instead, the mean relations among the gamma DSD parameters should have a physical cause and not due to purely statistical error (Chandrasekar and Bringi 1987).

Realizing the difficulty of separating statistical errors and physical variations in measurements (e.g., Fig. 1), simulations with various pairs of \( (\mu, \Lambda) \) were performed. Pairs of \( (\mu, \Lambda) \) were generated both with a random number generator and the constrained relation (2). Results are shown in Fig. 6. Relative standard errors of 5% were introduced in the moment estimates with the correlation coefficients of \( \rho(M_2, M_4) = 0.5, \rho(M_4, M_6) = 0.5, \) and \( \rho(M_2, M_6) = 0.25. \)

First, a hundred pairs of \( (\mu, \Lambda) \) were randomly gen-
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Fig. 6. Numerical simulations of DSD parameters determined from randomized moments for various pairs of $m$ and $L$ inputs. Relative standard errors of 5% were introduced in the moment estimates with the correlation coefficients of $\rho(M_{2}, M_{4}) = 0.5$, $\rho(M_{4}, M_{6}) = 0.5$, and $\rho(M_{2}, M_{6}) = 0.25$: (a) 100 random pairs of $\mu$ and $\Lambda$, (b) random pairs of $\mu$ and $\Lambda$ with a threshold of $1.0 < D_{0} < 3.0$ mm, and (c) $\mu$–$\Lambda$ relation pairs.

Erated with $\mu$ between $-2$ and $10$ and $\Lambda$ between $0$ and $15$. The relative random errors are added to each set of moments to generate 50 sets of moment and DSD parameter estimates. The simulation results are shown in Fig. 6a. The estimated $\bar{\mu}$ and $\bar{\Lambda}$ are scattered along a line centered on the input $\mu$ and $\Lambda$ (red circles). The estimated $\bar{\mu}$ and $\bar{\Lambda}$ values cover the whole domain. The scattered points show little correlation between $\bar{\mu}$ and $\bar{\Lambda}$, even when errors are added in the moment estimators. It is noted that for heavy and medium rains $D_{0}$ is typically in the range from $1.0$ to $3.0$ mm. Applying such thresholds, we obtain Fig. 6b. The estimated $\bar{\mu}$ and $\bar{\Lambda}$ are now in a confined region because of the threshold $1.0 < D_{0} < 3.0$ mm. This shows that physical constraints (not only errors) determine the pattern of estimated $\bar{\mu}$ and $\bar{\Lambda}$. The scatter increases as $\mu$ and $\Lambda$ increase, which is similar to that in Fig. 1a containing measurement errors. But, it is not the same as that shown in Fig. 1b leading to the $\mu$–$\Lambda$ relation (2).

Second, pairs of $\mu$ and $\Lambda$ values were generated with $\mu$ varying between $-2$ and $15$ with steps of $1$, and $\Lambda$ calculated from (2) for each $\mu$. The simulation results are shown in Fig. 6c. The estimated $\bar{\mu}$ and $\bar{\Lambda}$ lie along a line for each pair of input $\mu$ and $\Lambda$, and are centered at the inputs with scatter. For the reason discussed in the previous section, the larger the input values of $\mu$ and $\Lambda$, the broader the variation in the estimated $\bar{\mu}$ and $\bar{\Lambda}$. This feature is similar to that in Fig. 1a. However, the means of $\bar{\mu}$ and $\bar{\Lambda}$ depend on the input values of $\mu$ and $\Lambda$ rather than the added errors in the moment estimates. The moment errors have little effect on estimates $\bar{\mu}$ and $\bar{\Lambda}$ for the small values of $\mu$ and $\Lambda$ (heavy $\mu$–$\Lambda$ rain cases). In contrast, the threshold data in Fig. 1b do not exhibit variations in $\bar{\mu}$ and $\bar{\Lambda}$ that increase as their mean values increase. Therefore, the relation in Fig. 1b is believed to represent the actual physical nature of rain DSD rather than purely statistical error.

The $\mu$–$\Lambda$ relation (2) derived from measurements should not be confused with the linear relationship shown in Fig. 5, which is a result of statistical errors. It is true that the estimates $\bar{\mu}$ and $\bar{\Lambda}$ exhibit a high correlation and have a linear relation when moment estimators contain random errors for a pair of $\mu$ and $\Lambda$. However, the mean values of estimated DSD parameters and physical parameters are not biased by fluctuation errors in moment estimators as shown in Figs. 5 and 6. Furthermore, since each pair of $\mu$ and $\Lambda$ has its own error-induced linear relation, the overall relation between $\bar{\mu}$ and $\bar{\Lambda}$ remains unknown. Consequently, the $\mu$–$\Lambda$ relation (2), obtained by a second-order polynomial fitting of estimated $\bar{\mu}$ and $\bar{\Lambda}$ for a quality-controlled dataset, is believed to be related to the physical nature of rain DSD and contains useful information for the following reasons:

(i) The pattern of the scatter shown in Fig. 6 depends mainly on the expected values ($\mu$, $\Lambda$) rather than on statistical error alone. The slope and intercept of a linear relation associated with moment error depend on the expected values ($\mu$, $\Lambda$), while the mean values of the DSD parameters and physical parameters are not biased by fluctuation errors in the moments. Furthermore, relation (2) exhibits a
quadratic rather than the linear form associated with characteristics of the moment errors.

(ii) The moment errors have very little effect on the $\mu$–$\Lambda$ relation when the rain rate is greater than 5 mm h$^{-1}$ and when the total drop count is large (>1000). For fixed relative errors in moment estimators, the errors in the estimated $\bar{\mu}$ and $\bar{\Lambda}$ values are large for large $\mu$ and $\Lambda$ values (light rains), while the errors are small for small $\mu$ and $\Lambda$ values (heavy rains). This is shown in Fig. 6, which is similar to measured data with sampling errors shown in Fig. 1a. The quality-controlled dataset shown in Fig. 1b, which was used to derive relation (2), does not exhibit such increased spreading with expected values ($\mu$, $\Lambda$).

(iii) The relation (2) predicts that the raindrop spectrum is wide when large drops are present (Fig. 2). This is consistent with raindrop spectra observed by video disdrometer. It is important to note that the relation (2) and the correlation between $\mu$ and $\Lambda$ are not a consequence of the relation $\Delta D = \mu + 3.67$ since it is theoretically possible for $\mu$ and $\Lambda$ to be uncorrelated in particular and any two of the three parameters uncorrelated. Even though, in practice, $\mu$ and $\Lambda$ are somewhat correlated because $D_o$ usually varies in a limited range between 1 and 3.0 mm for most heavy rain events, the correlation between $\mu$ and $\Lambda$ shown in Fig. 6b does not lead to the relation (2). Therefore, we contend that Eq. (2) is partially a consequence of the physical nature of the rain DSDs and not a consequence of the $\Delta D = \mu + 3.67$ relation.

(iv) Even though the $\mu$–$\Lambda$ relation (2) is influenced to some extent by the errors in moment estimates, it is a useful relation in that it reduces the errors in rain parameter retrieval. The relation (2) simplifies the gamma DSD model and enables rain DSD retrievals from two independent remote measurements. Nevertheless, remote measurements, which correspond approximately to moments of the DSD in the radar sampling volume, contain measurement error. As such, the retrieved $\bar{\mu}$ and $\bar{\Lambda}$ values from remote measurements will contain some spurious correlation. Nevertheless, there is almost no bias in the mean values of the DSD parameters and in physical parameters such as rain rate $\langle R \rangle$ and $\langle D_o \rangle$.

4. Retrieval of DSD parameters from two moments

In a real remote measurement, the number of independent measurable is generally limited. For example, in a dual-wavelength or dual-polarization radar technique, only two statistical moments are measured rather than the three required to determine the three gamma DSD parameters. In practice, the problem is how to retrieve unbiased physical parameters, such as rain rate and median volume diameter with two remote measurables.

Some DSD retrieval algorithms assume one of the DSD parameters when only two independent remote observations are available. For example, $\mu$ is fixed so that $\Lambda$ and $N_0$ can be retrieved from reflectivity and attenuation, such as in the Tropical Rainfall Measuring Mission (TRMM) algorithm (Kozu and Nakamura 1991; Iguchi et al. 2000; Meneghini et al. 2001). The scatter in Fig. 1 would seem to preclude such an approach. When a $\mu$–$\Lambda$ relation (2) is used, then two parameters can be determined from two measurements such as reflectivity and differential reflectivity (Zhang et al. 2001).

In this section, a comparison between the rain retrieval using the $\mu$–$\Lambda$ relation and that with fixed $\mu$ is presented using moment pairs that correspond to S-band polarization radar measurements. With the dual-polarization and dual-wavelength radar techniques, the moment pair might correspond to the 5th and 6th or 3d and 6th moments, respectively. The DSD retrievals are evaluated based on numerical simulations and measurements.

### Table 1. DSD parameter retrieval from the 5th and 6th moments.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Method 1: $\mu$–$\Lambda$ relation</th>
<th>Method 2: fixed $\mu = 2.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_0$</td>
<td>$8000$</td>
<td>$8350.7$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$0.0$</td>
<td>$2.0$</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>$1.935$</td>
<td>$2.582$</td>
</tr>
<tr>
<td>$R$</td>
<td>$38.62$</td>
<td>$35.80$</td>
</tr>
<tr>
<td>$D_0$</td>
<td>$1.897$</td>
<td>$2.207$</td>
</tr>
</tbody>
</table>

#### a. Gamma DSD parameter retrieval from the 5th and 6th moments

Dual-polarization radar measures reflectivity at horizontal and vertical polarizations. The 5th and 6th moments are chosen because they are close to the reflectivity at vertical polarization ($Z_V \sim M_5$) and at horizontal polarization ($Z_H \sim M_6$). The moment estimators are generated using the procedure described in the previous section for a set of input DSD parameters having a relative standard error of 5% (0.21 dB) and without correlation among the moments. We have

\[ \Lambda = \frac{M_5(\mu + 6)}{M_6} \]

(18)

\[ N_0 = \frac{M_6\Lambda^{\mu+7}}{\Gamma(\mu + 7)}. \]  

(19)

Expressions (18) and (19) can be solved in two ways: using a $\mu$–$\Lambda$ relation such as (2) or assuming a fixed $\mu$. With the first method, (18) is solved jointly with (2) for $\mu$ and $\Lambda$ from the estimated moments $M_5$ and $M_6$. Then $N_0$ is found from (19). With the second method, $\mu$ is fixed at 2. Then $\Lambda$ and $N_0$ are solved from (18) and (19). The means and standard deviations of the DSD parameter estimators and rain rate as well as median volume diameter were calculated and are listed in Table 1. The means of the retrieved parameters with the $\mu$–
Table 2. DSD parameter retrieval from the 3d and 6th moments.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Method 1: ( \mu-L ) relation</th>
<th>Method 2: fixed ( \mu = 2.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_0 )</td>
<td>8000</td>
<td>8363.7</td>
</tr>
<tr>
<td>( \mu )</td>
<td>1.935</td>
<td>1.964</td>
</tr>
<tr>
<td>( \Lambda )</td>
<td>38.62</td>
<td>38.52</td>
</tr>
<tr>
<td>( R )</td>
<td>1.897</td>
<td>0.024</td>
</tr>
</tbody>
</table>

A relation are in better agreement with expected values than those with fixed \( \mu \). It is true that the bias of \( N_0 \) and \( \Lambda \) depend on the \( \mu \) bias. But, the bias of rain parameters should be comparable, which are smaller when the \( \mu-L \) relation is used. The standard deviations are also smaller except for \( \mu \), which was specified.

b. Gamma DSD parameter retrieval from the 3d and 6th moments

Dual-wavelength radar measures reflectivity and attenuation using reflectivity measurements where at least one of the channels is attenuating. Because the attenuation coefficient is proportional to the 3d moment for Rayleigh scattering, the DSD parameter retrieval from the 3d and 6th moments is studied. Using the moment estimators \( M_3 \) and \( M_6 \) and the gamma DSD model, the DSD parameter can be written as a function of the estimated moments

\[
\Lambda = \left[ \frac{M_5(\mu + 6)(\mu + 5)(\mu + 4)}{M_6} \right]^{1/3}
\]  

As in the previous section, two methods are used to retrieve the DSD parameters. In the first method, Eq. (20) is jointly solved in conjunction with (2) for \( \mu \) and \( \Lambda \) from \( M_3 \) and \( M_6 \). The estimate \( N_0 \) is then found from (19). In the second method, \( \mu \) is fixed at 2; \( \Lambda \) and \( N_0 \) are then solved from (20) and (19) accordingly. The means and standard deviations of the DSD parameter estimators and rain rate, as well as median volume diameter are listed in Table 2. Again, the retrieved parameters are much less biased when the \( \mu-\Lambda \) relation is used than that when \( \mu \) is fixed. The standard errors are comparable. However, in the case of a fixed \( \mu \) approach, the actual standard error is a function of \( \mu \) and the error could be larger and retrieved parameters could be biased significantly. In contrast, rain parameters \( R \) and \( D_0 \) are almost unbiased when the \( \mu-\Lambda \) relation is used.

c. Rain parameter retrieval from the S-Pol measurements

As mentioned previously, polarization radar measures reflectivity \( Z \) and differential reflectivity \( Z_{DR} \), which can be used for retrieving rain DSD parameters. The measured reflectivity at S band is not exactly the 6th moment of DSD when large raindrops are present in the sampling volume. The scattering amplitudes are numerically calculated using the T-matrix method (Oguchi 1983; Vivekanandan et al. 1991), and then the reflectivity and differential reflectivity are computed for assumed rain DSDs. Following the procedure in Zhang et al. (2001), Vivekanandan et al. (2003), and Brandes et al. (2003), \( \mu \) and \( \Lambda \) are determined from \( Z_{DR} \) and the \( \mu-\Lambda \) relation. Here \( N_0 \) is estimated from \( Z \). The DSD parameters are also retrieved for a fixed \( \mu \). Rain rate and median volume diameter are calculated.

Data used for this study were collected on 17 September 1998 in Florida during the PRECIP98 project. Measurements were available from NCAR’s S-Pol radar and a disdrometer operated by University of Iowa (Brandes et al. 2002). The results of disdrometer measurements and the radar-retrieved values using (i) the \( \mu-\Lambda \) relation and (ii) a fixed \( \mu \) are shown in Fig. 7. The fixed \( \mu \) method overestimates the rain (almost by a factor of 2 for \( \mu =
values of associated with moment error depend on the particular ments. The slope and intercept of the linear relation from a quality-controlled dataset of rain DSD measure-

such as that observed between and for a single pairÃ

should not be equated to the relation agrees well with the disdrometer measurement. By using the relation, the retrievals of median volume diameter (Fig. 7b) agree more closely with the disdrometer measurements than by using a fixed  

5. Summary and discussion

In this paper, detailed analyses of error propagation from moment estimators to the estimated gamma DSD parameters were performed. A mathematical approach was used to quantify the effects of errors on moments on DSD parameters and on and retrievals. The retrievals using the relation were compared with the fixed approach. The relation is believed to capture a mean physical characteristic of raindrop spectra and is useful for retrieving unbiased DSD parameters when only two independent remote measurements are available such as and attenuation.

Theoretical analyses and numerical simulations con-
firm that errors in moment measurements (estimates) can cause high correlations in gamma DSD parameters such as that observed between and for a single pair of expected values in Fig. 5. This error effect, however, should not be equated to the relation derived from a quality-controlled dataset of rain DSD measurements. The slope and intercept of the linear relation associated with moment error depend on the particular values of and , whereas the mean values of retrieved DSD and physical parameters are not biased by fluctuation errors in the moments. The moment errors have little effect on the relation for rain rates that contribute most to rain accumulations. The relation is consistent with observation whereby heavy rain is represented with large drops having a broad distribution. Compared to the gamma distribution with a fixed , the constrained gamma distribution with the relation is more flexible in representing a wide range of instantaneous DSD shapes.

Recognizing the difficulty of separating statistical errors and physical variations, we believe the errors in DSD parameter estimators should not be considered meaningless; rather they should be studied further for the following reasons.

1) The errors in the estimated DSD parameters are linked to the functional relations between DSD parameters and moments. The correlations among the estimated gamma DSD parameters due to moment errors are a result of DSD fitting (moment method), and a requirement of unbiased moments and physical parameters.

2) Natural rain DSD may not be the same as the math-

ematically modeled gamma distribution. In the model we have used here, the difference between actual DSD and assumed gamma distribution can be attributed to errors in the moment estimators.

3) “Fluctuation” is a more appropriate description than “error” in characterizing the differences of DSD parameters or moments from their expected values since each realization could be a real physical event. The DSD parameters should be allowed to vary as in Zhang et al. (2002). It is very difficult to separate the physical variations from statistical errors.

4) Nevertheless, measurements always contain errors and as a result the correlation between and may be strengthened. If such a correlation can improve retrievals such that the bias and standard error in physical parameters are minimized, it can be a valuable addition to the retrieval process.

It has been shown that rain DSD retrievals from radar measurements that use the relation agree with the in situ measurements better than those obtained with fixed . The relation is thought to capture the physical nature of rain DSDs and should provide a way to improve DSD estimation by dual-parameter radar retrieval techniques.

We derived the relation from video disdrometer measurements in Florida during the summer of 1998 for moderate and heavy rain case ( ) to minimize the sampling error effect. The relation should be extendable to rain rates smaller than mm h . The relation is also valid for the observations collected in Oklahoma. It is possible that the relation changes depending on climatology and rain type. If that is true, a tuned relation based on local DSD observations should be derived and used for accurate rain DSD estimation.

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APPENDIX A

Derivation of Variances of Estimated DSD Parameters due to Moment Errors

To understand the error propagation in the estimation of DSD parameters based on the moment method, the moments and the DSD parameters can be expressed as sums of their respective means and fluctuations:
\[ \dot{M}_s = M_n + \delta M_n \quad \text{and} \]
\[ \dot{N}_0 = N_0 + \delta N_0; \quad \mu = \mu + \delta \mu; \quad \Lambda = \Lambda + \delta \Lambda. \quad (A2) \]

Here we analyze the errors of DSD parameters as determined from the three (2d, 4th, and 6th) moment estimators [Eqs. (5)–(7)]. The analysis of error propagation gives relations between the errors of the estimated DSD parameters and those of the moment estimators as

\[
\delta \mu = \frac{d \mu}{d \eta} \left( \frac{\partial \eta}{\partial M_2} \delta M_2 + \frac{\partial \eta}{\partial M_4} \delta M_4 + \frac{\partial \eta}{\partial M_6} \delta M_6 \right),
\]
\[ (A3) \]

\[
\delta \Lambda = \left( \frac{\partial \Lambda}{\partial \mu} \delta \mu \right) + \left( \frac{\partial \Lambda}{\partial M_2} \delta M_2 + \frac{\partial \Lambda}{\partial M_4} \delta M_4 \right),
\]
\[ (A4) \]

\[
\delta N_0 = \left( \frac{\partial N_0}{\partial M_n} \delta M_n + \frac{\partial N_0}{\partial \Lambda} \delta \Lambda + \frac{\partial N_0}{\partial \mu} \delta \mu \right).
\]
\[ (A5) \]

The variance of a DSD parameter estimator is simply an ensemble average of the square of the fluctuation error because the mean of the fluctuation is zero. Hence the variances of \( \bar{\mu} \) and \( \bar{\Lambda} \) are

\[
\text{var}(\bar{\mu}) = \langle (\delta \mu)^2 \rangle = \left[ \left( \frac{d \mu}{d \eta} \left( \frac{\partial \eta}{\partial M_2} \delta M_2 + \frac{\partial \eta}{\partial M_4} \delta M_4 + \frac{\partial \eta}{\partial M_6} \delta M_6 \right) \right)^2 \right],
\]
\[ (A6) \]

\[
\text{var}(\bar{\Lambda}) = \langle (\delta \Lambda)^2 \rangle = \left( \left( \frac{\partial \Lambda}{\partial \mu} \delta \mu \right) + \left( \frac{\partial \Lambda}{\partial M_2} \delta M_2 + \frac{\partial \Lambda}{\partial M_4} \delta M_4 \right)^2 \right).
\]
\[ (A7) \]

Similarly, their covariance is

\[
\text{cov}(\bar{\mu}, \bar{\Lambda}) = \langle (\delta \mu \delta \Lambda) \rangle = \left( \left( \frac{\partial \Lambda}{\partial \mu} \frac{\partial \mu}{\partial M_2} \delta M_2 + \frac{\partial \Lambda}{\partial \mu} \frac{\partial \mu}{\partial M_4} \delta M_4 + \frac{\partial \Lambda}{\partial \mu} \frac{\partial \mu}{\partial M_6} \delta M_6 \right) \right),
\]
\[ (A8) \]

Performing the algebra in (A6)–(A8) and expressing them in terms of the variances and covariance of the moment estimator errors, we obtain expressions for the variances and covariance of \( \bar{\mu} \) and \( \bar{\Lambda} \), as

\[
\text{var}(\bar{\mu}) = \left( \frac{d \mu}{d \eta} \right)^2 \left( \frac{\partial \eta}{\partial M_2} \right)^2 \text{var}(\bar{M}_2) + \left( \frac{\partial \eta}{\partial M_4} \right)^2 \text{var}(\bar{M}_4) + \left( \frac{\partial \eta}{\partial M_6} \right)^2 \text{var}(\bar{M}_6) + 2 \frac{\partial \eta}{\partial M_2} \frac{\partial \eta}{\partial M_4} \text{cov}(\bar{M}_2, \bar{M}_4) + 2 \frac{\partial \eta}{\partial M_2} \frac{\partial \eta}{\partial M_6} \text{cov}(\bar{M}_2, \bar{M}_6) + 2 \frac{\partial \eta}{\partial M_4} \frac{\partial \eta}{\partial M_6} \text{cov}(\bar{M}_4, \bar{M}_6),
\]
\[ (A9) \]

\[
\text{var}(\bar{\Lambda}) = \left( \frac{\partial \Lambda}{\partial \mu} \right)^2 \text{var}(\bar{\mu}) + \left( \frac{\partial \Lambda}{\partial M_2} \right)^2 \text{var}(\bar{M}_2) + \left( \frac{\partial \Lambda}{\partial M_4} \right)^2 \text{var}(\bar{M}_4) + \left( \frac{\partial \Lambda}{\partial M_6} \right)^2 \text{var}(\bar{M}_6) + 2 \frac{\partial \Lambda}{\partial M_2} \frac{\partial \Lambda}{\partial M_4} \text{cov}(\bar{M}_2, \bar{M}_4) + 2 \frac{\partial \Lambda}{\partial M_2} \frac{\partial \Lambda}{\partial M_6} \text{cov}(\bar{M}_2, \bar{M}_6) + 2 \frac{\partial \Lambda}{\partial M_4} \frac{\partial \Lambda}{\partial M_6} \text{cov}(\bar{M}_4, \bar{M}_6),
\]
\[ (A10) \]

\[
\text{cov}(\bar{\mu}, \bar{\Lambda}) = \frac{d \mu}{d \eta} \frac{\partial \eta}{\partial M_2} \left( \frac{\partial \Lambda}{\partial M_2} \right) \text{var}(\bar{M}_2) + \frac{d \mu}{d \eta} \frac{\partial \eta}{\partial M_4} \left( \frac{\partial \Lambda}{\partial M_4} \right) \text{var}(\bar{M}_4) + \frac{d \mu}{d \eta} \frac{\partial \eta}{\partial M_6} \left( \frac{\partial \Lambda}{\partial M_6} \right) \text{var}(\bar{M}_6) + \frac{d \mu}{d \eta} \frac{\partial \eta}{\partial M_2} \frac{\partial \eta}{\partial M_4} \text{cov}(\bar{M}_2, \bar{M}_4) + \frac{d \mu}{d \eta} \frac{\partial \eta}{\partial M_2} \frac{\partial \eta}{\partial M_6} \text{cov}(\bar{M}_2, \bar{M}_6) + \frac{d \mu}{d \eta} \frac{\partial \eta}{\partial M_4} \frac{\partial \eta}{\partial M_6} \text{cov}(\bar{M}_4, \bar{M}_6),
\]
\[ (A11) \]
Hence, (A9)–(A11) constitute the error analysis for the DSD estimated parameters $\hat{\mu}$ and $\hat{\Lambda}$. The derivative and partial derivatives in the above expressions are obtained from the functional relations (3) and (4) as well as the definition of the moment ratio $\eta$, given by

$$c = (\eta^2 + 14\eta + 1)^{1/2}, \quad (A13)$$

$$\eta = \frac{M_2}{M_2 M_6} = \frac{(\mu + 4)(\mu + 3)}{(\mu + 7)(\mu + 6)}, \quad (A14)$$

$$\frac{\partial \eta}{\partial M_2} = -\frac{M_3}{M_2 M_6} = -\frac{\eta}{M_2}, \quad (A15a)$$

$$\frac{\partial \eta}{\partial M_4} = \frac{2M_4}{M_2 M_6} = \frac{2\eta}{M_4}, \quad (A15b)$$

$$\frac{\partial \eta}{\partial M_6} = -\frac{M_3}{M_2 M_6} = -\frac{\eta}{M_6}, \quad (A15c)$$

$$\frac{\partial \Lambda}{\partial \mu} = \frac{1}{2} \left[ \frac{M_2(2\mu + 7)^2}{M_2 M_4} \right]^{1/2} = \frac{\Lambda(\mu + 3.5)}{(\mu + 4)(\mu + 3)}, \quad (A16a)$$

$$\frac{\partial \Lambda}{\partial M_2} = \frac{1}{2} \left[ \frac{(\mu + 4)(\mu + 3)}{M_2 M_4} \right]^{1/2} = \frac{\Lambda}{2M_2}, \quad (A16b)$$

$$\frac{\partial \Lambda}{\partial M_4} = -\frac{1}{2} \left[ \frac{M_2(\mu + 4)(\mu + 3)}{M_4} \right]^{1/2} = -\frac{\Lambda}{2M_4}. \quad (A16c)$$

**APPENDIX B**

**Derivation of Variances of Estimated Rain Physical Parameters due to Errors in DSD Parameters**

Rain physical parameters, rain rate ($R$), and median volume diameter ($D_0$) are related to gamma rain DSD parameters by (9) and (10). The analysis of error propagation gives relations between the errors of the estimated rain parameters and those of DSD parameters. We have the estimate error in rain rate as expressed by that of $\mu$ and $\Lambda$ as

$$\delta R = \frac{\partial R}{\partial \mu} \delta \mu + \frac{\partial R}{\partial \Lambda} \delta \Lambda. \quad (B1)$$

After performing the partial derivatives of rain rate (12) and substituting them in (B1), we have

$$\delta R = \frac{R}{N_0} \delta N_0 + \left[ \frac{\Gamma'(\mu + 4.67)}{\Gamma(\mu + 4.67)} - \ln(\Lambda) \right] R \delta \mu - \frac{(\mu + 4.67)}{\Lambda} R \delta \Lambda. \quad (B2)$$

The variance of rain estimates is an ensemble average of the square of the estimation error as

$$\text{var}(\hat{R}) = \left( \frac{R}{N_0} \delta N_0 + \left[ \frac{\Gamma'(\mu + 4.67)}{\Gamma(\mu + 4.67)} - \ln(\Lambda) \right] R \delta \mu - \frac{(\mu + 4.67)}{\Lambda} R \delta \Lambda \right)^2$$

$$= \frac{R^2}{N_0^2} \text{var}(\hat{N}_0) + \left[ \frac{\Gamma'(\mu + 4.67)}{\Gamma(\mu + 4.67)} - \ln(\Lambda) \right]^2 R^2 \text{var}(\hat{\mu}) + \frac{(\mu + 4.67)^2}{\Lambda^2} R^2 \text{var}(\hat{\Lambda})$$
\[\begin{align*}
+ 2 \frac{R^2}{N_0} \left[ \Gamma'(\mu + 4.67) - \ln(\Lambda) \right] \text{cov}(N_0, \hat{\mu}) - 2 \frac{R^2(\mu + 4.67)}{N_0\Lambda} \text{cov}(N_0, \hat{\Lambda}) \\
- 2 \frac{R^2(\mu + 4.67)}{\Lambda} \left[ \Gamma'(\mu + 4.67) - \ln(\Lambda) \right] \text{cov}(\hat{\mu}, \hat{\Lambda}).
\end{align*}\]

(B3)

For the error in estimated median volume diameter (13), we perform the analysis of error propagation and have
\[
\delta D_h = \frac{\partial D_h}{\partial \mu} \delta \mu + \frac{\partial D_h}{\partial \Lambda} \delta \Lambda = \frac{1}{\Lambda} \delta \mu - D_h \frac{\delta \Lambda}{\Lambda}.
\]

(B4)

Hence, its variance is
\[
\text{var}(\hat{D}_h) = (\delta D_h)^2 = \left( \frac{1}{\Lambda} \delta \mu - D_h \frac{\delta \Lambda}{\Lambda} \right)^2 = \frac{1}{\Lambda^2} \text{var}(\hat{\mu}) + \frac{D_h^2}{\Lambda^2} \text{var}(\hat{\Lambda}) - 2 \frac{D_h}{\Lambda^2} \text{cov}(\hat{\mu}, \hat{\Lambda}).
\]

(B5)

Equations (B3) and (B5) constitute the error analysis for rain rate and median volume diameter estimates due to the errors in estimated gamma DSD parameters.

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