Effect of super-refraction and small-scale N-irregularities on RO inversions

(challenge of RO in the planetary boundary layer)

Sergey Sokolovskiy
UCAR – COSMIC Program
The structure of the moist PBL (in brief)

Super-refraction (SR) in geometric optics

Inapplicability of the Abel inversion in case of SR

Effect of the SR on RO inversions (negative N-bias)

Difference in frequency of occurrence of SR over different regions

Inverse problem in case of SR is ill-conditioned

SR in RH inversions

Effect of small-scale refractivity irregularities on RH inversions

Effect of truncation of RO signals on RH inversions

Parameters of RO signals and retrieved profiles that may indicate SR
Problems of radio occultation soundings in tropical lower troposphere: larger errors negative N-bias

Reasons:
(i) receiver tracking errors
(ii) super-refraction on top of moist planetary boundary layer (PBL) (errors of the Abel inversion)

Statistics of comparisons of RO refractivity with ECMWF (August 2002)
global

tropics
What is the super-refraction?

The super-refraction is the wave propagation condition when the radius of curvature of the ray at tangent point is smaller than the radius

\[ r_c = -n \left( \frac{dn}{dr} \right)^{-1} < r, \quad \frac{dN}{dr} < -157 N / km \quad \text{critical N gradient} \]

Since a ray is symmetric with respect to the tangent point (follows from Snell’s law), the ray which has tangent point in the super-refraction layer may not start and end outside the atmosphere. This is the internal (trapped) ray.

External rays may not have tangent points inside the SR layer. But external rays may have tangent points below the elevated SR layer, thus traversing the SR layer.
Meteorological conditions necessary for occurrence of the super-refraction gradient: (Kursinski et al., 2000)

(i) \( \frac{dT}{dz} > 140 \text{ K/km} \)

(ii) \( \frac{dP_w}{dz} < -34 \text{ mb/km} \)

Most important for RO:

Surface SR layer. Typically, \( \Delta z \) is in the range from several meters to several tens of meters. 
(i) warm air over cold surface ("superior mirage");
(ii) evaporation duct.

Elevated SR layer. Typically, \( \Delta z \) is in the range from several tens to several hundred meters. 
Top of moist planetary boundary layer (PBL). 
\( z_{top} \) is about 1-2 km.
N-gradient on top of PBL is especially large over tropical ocean. An example: $\Delta q \sim 6 \text{g} / \text{kg}, \quad \Delta e \sim 6 \text{mb}, \quad \Delta N \sim 25, \quad \Delta z \sim 40 \text{m}$

$dN / dz \sim -625 N / \text{km}$ (exceeds critical N-gradient 4 times)

Airborne lidar observations often show stability of top height of PBL (variations of tens of meters) over horizontal distances $\sim 100 \text{km}$ (over oceans).
Examples of SR observed on top of PBL over tropical ocean

Radiosonde data, St. Helena Island (16S, 5.7W). Typically, the SR layer on top of PBL is stable for days.
SR is a common phenomenon.

Max. height of SR reproduced by T106 between +/-45 deg lat. is ~600 m.

However, radiosondes often show SR at ~ 2km.

To reproduce the SR on top of marine PBL, an atmospheric models must have sufficiently dense vertical grid (with resolution of tens of meters) up to ~2 km.
An elevated super-refraction layer

\[ x = r \ n(r) \] is the refractional radius

Normally, refractional radius increases with height, \( \frac{dx}{dr} = n + r \frac{dn}{dr} > 0 \)
When \( \frac{dn}{dr} < -n/r \) \( (r_c < r) \) then \( \frac{dx}{dr} < 0 \) (super-refraction)

The critical N-gradient \( (\frac{dN}{dr} < -157N/km) \) is exceeded between \( r_2 \) and \( r_3 \)

\( r_1 \) is the radius at which \( x(r_1) = x(r_3) \)
An elevated super-refraction layer

The bending angle of rays with tangent points approaching \( r_1 \) from below and \( r_3 \) from above tends to infinity.

The rays with the tangent points between \( r_1 \) and \( r_3 \) are internal (trapped); they have infinitely large number of perigees (between \( r_1 \) and \( r_2 \)) and apogees (between \( r_2 \) and \( r_3 \)).

The bending angle as the function of impact parameter ( = refractional radius at tangent point) has a singularity at \( x_1=x_3 \) and no gap.
RO signal for a simple model of elevated SR layer (calculated by MPS method)

An elevated SR layer does not bend external rays to surface. The external rays are “sliding” on top of the SR layer.
Abel inversion

The bending angle:

\[ \alpha(r_0) = -2r_0 n(r_0) \int_{r_0}^{\infty} \frac{dn/dr}{\sqrt{r^2 n^2(r) - r_0^2 n^2(r_0)}} \, dr \]

where \( r_0 \) is the radius to TP

to be inverted, must be subject to transform of variables:

\[ x = rn(r), \quad a = r_0 n(r) \]

\[ \alpha(a) = -2a \int_{a}^{\infty} \frac{d \ln n/\,dx}{\sqrt{x^2 - a^2}} \, dx \]

\[ \ln[n(x)] = \frac{1}{\pi} \int_{x}^{\infty} \frac{\alpha(a)}{\sqrt{a^2 - x^2}} \, da \]

In the presence of SR, 
\( r(x) \) and \( n(x) \) are multi-valued functions,
thus the Abel inversion is invalid below \( r_3 \)
What happens when the Abel inversion is applied below the top of SR layer? (the singularity of the bending angle as function of impact parameter is integrable)

Since the retrieved function $n(x)$ will be single-valued, it will not reproduce absolute refractivity gradient larger than the critical ($dN/dz \sim -157N/km$)

Thus the retrieved $N(r)$ will be negatively biased below the top of the SR layer. This bias is related to inapplicability of the Abel inversion, not to the integration through the singularity in bending angle (which can be as accurate as necessary)
The frequency of occurrence of SR on top of PBL is different over different ocean regions.

Simulation study was based on radiosonde data during winter 2001-2002, for two sites: (1) St. Helena Is. (Atlantic Ocean); (2) At. Kwajalein (Pacific Ocean). For each N profile, bending angles were calculated and subject to Abel inversion.

The top of PBL is more sharp and errors of the Abel inversion much larger for site (1) than for site (2).

A more detailed study with the use of global RAOBS data base would be useful for aiding assimilation of RO data in PBL (when OL tracking is implemented).
Important: since the refractive index $n_{ret}(x)$ retrieved from the $\alpha_{true}(a)$ in case of SR, is the single-valued function, inverting it back to bending angle will yield: $\alpha_{ret}(a) = \alpha_{true}(a)$

The true N profile in case of SR and the Abel-retrieved N profile correspond to strictly the same bending angle profile (in geometric optics).

The difference in RO signals is due to only diffraction effects (that are small).

The RO signals simulated by MPS method

The RO signal simulated from true N-profile with SR.

The RO signal simulated from the Abel-retrieved N-profile.
Generalized Abel inversion

Commonly, Abel inversion is applied for the rays that traverse the atmosphere from space to space.

The generalized Abel inversion is applied for rays below $r_1$

The bending angle below $r_1$:

$$\alpha_1(a) = -2a \int_a^0 \frac{d \ln n / d x}{\sqrt{x^2 - a^2}} dx, \quad a_1 = r_1 n(r_1)$$

After the substitution of variables $x^2 = u$, $a^2 = v$ and the Abel transform:

$$\ln[n(x)] = \ln[n(x_1)] + \frac{1}{\pi} \int_x^{x_1} \frac{\alpha_1(a)}{\sqrt{a^2 - x^2}} da, \quad x_1 = r_1 n(r_1)$$
The generalized Abel inversion can be applied, with additional constraints, below the SR layer (the bending angle is defined below $r_1$)

\[
\ln[ n(x) ] = \ln[ n(x_1) ] + \frac{1}{\pi} \int_{x}^{x_1} \alpha_1(a) \frac{d}{\sqrt{a^2 - x^2}} \, da
\]

\[
\alpha_1(a) = \alpha(a) + 2a \left[ \int_{x_1}^{x_2} + \int_{x_3}^{x_2} + \int_{x_3}^{\infty} \right] \frac{d \ln n / dx}{\sqrt{x^2 - a^2}} \, dx
\]

- known from RO observations;
- calculated from Abel-retrieved refractivity above the SR layer;
- must be taken from some ancillary data
Solving the inverse problem in case of super-refraction

The true and the Abel-retrieved N-profiles, in case of the SR layer, correspond to strictly the same bending angle as function of impact parameter.

Thus the inverse problem is ill-conditioned and needs to be constrained.

Theoretically, the inverse problem can be solved as the least-square problem with inequality condition, by satisfying the observation bending angle, but by forcing N to be larger than the $N_{\text{AR}}$ below the top of the SR layer. The top of SR layer is traced by one point of critical N-gradient in the $N_{\text{AR}}(r)$.

$$n(r) = n_{AR}(r), \quad r \geq r_3$$

$$n(r) = n_{AR}(r) + v^2(r), \quad r \leq r_3$$

$$I[v(r)] = \|\alpha_{\text{true}}(a) - \alpha_{\text{cnst}}(a)\| = \min$$

where:

$$\alpha_{\text{cnst}}(a) = -2a \left[ \int_{x_3}^{x_2} + \int_{x_3}^{x_2} + \int_{x_3}^{x_2} \right] \frac{d \ln n(x)}{dx} \frac{dx}{\sqrt{x^2 - a^2}}$$

\( \alpha_{\text{cnst}}(a) \) depends only on the $N_{\text{AR}}$ above $r_3$

\( \alpha_{\text{cnst}}(a) \) depends on the $N_{\text{AR}}$ and on the $v(r)$
At first, it looks like the solution is found (but only in theory!)

In practice: there are subsets of N profiles between $N_{\text{TRUE}}$ and $N_{\text{AR}}$ (like those shown in Fig.A) that result in very small difference in the bending angles.

It is not clear, how well the second minimum in the cost function $I[v]$ (which corresponds to $N_{\text{TRUE}}$) is distinguishable from the main minimum $I[v]=0$ (which corresponds to $N_{\text{AR}}, v(r)=0$ ) on the background of the effect of noise and non-spherically-symmetric N-irregularities.
Direct variational assimilation of the bending angles into NWP models might help to improve the solution of the problem.

The variational assimilation consists of minimizing the cost function:

\[
J = \|\alpha_{true}(a) - \alpha_{cnst}(a)\| + \|\text{other observations}\| + \|\text{1}^{\text{st}}\text{ guess}\| = \text{min}
\]

The additional terms in the cost function, related to other assimilated observations and the NWP model 1\textsuperscript{st} guess, may put additional constraint on N(r) by nudging it to positive values from the N_{AR}(r).

This can be the subject of the future research.

A necessary (but not sufficient) condition for assimilating RO data affected by SR: the NWP model must be capable of reproducing real SR layers, i.e. it must have sufficiently dense vertical grid (tens of meters) up to the height of marine PBL ~2 km.
Radio-holographic inversions in case of super-refraction

RO signals simulated by MPS method from radiosonde N-profiles
St. Helena Is., (A) Jan. 22, 2002; (B) Jan. 24, 2002
The bending angles were calculated:
(i) in geometric optics (GO) directly from N-profiles;
(ii) by canonical transform (CT) method from complex simulated RO signals.
Refractivity was retrieved from both bending angles by Abel inversion.

Conclusion: RH methods, in case of SR, do not introduce errors additional to those existing in GO (related to inapplicability of the Abel inversion).
Effect of small-scale N-irregularities on RH inversions

Vertical spectra of small-scale N-irregularities in the troposphere were estimated from hires tropical radiosondes. Relative RMS fluctuation of N in (1/1000-1/50) m\(^{-1}\) spatial frequency interval is 1.5%. The spectra were averaged, approximated, and used for generation of random 2-D N-fields with different horizontal-to-vertical aspect ratio.
The generated 2D random N-fields were used for simulation of RO signals by MPS method.
(A) isotropic random N-field (rms=1.5%)
(B) spherically-symmetric random N-field (rms=1.5%)
The RO signal (A) ends at higher altitude and does not show the frequent fluctuation of amplitude typical for multi-path.
The bending angles were calculated from the simulated RO signals by canonical transform (CT) method. A – isotropic N-field; B – spherically-symmetric N-field.

In case of 2-D (isotropic) N-irregularities, the fluctuation of the bending angle is not large. The fluctuation of the transformed amplitude is large and the transition to zero is smeared (does not allow to accurately determine the cut-off).

In case of 1-D (spherically-symmetric) N-irregularities, the fluctuation of the bending angle is large. The large positive spikes (and the corresponding spikes in the transformed amplitude) trace critical and super-critical N-gradients. The transition of the transformed amplitude to zero allows to accurately determine the cutoff.
Abel inversion of the bending angles

A – isotropic N-field; B – spherically-symmetric N-field.
RO signals were used down to $-150\text{km}$ height of straight line transmitter-receiver. Small random errors and no N-bias in 2-D case (isotropic N-irregularities (A)) Negative N-bias below $\sim3\text{ km}$ (where random N-gradient exceeds critical) in 1-D case (spherically-symmetric N-irregularities (B))

Again, inversions of the GO and CT bending angles are in excellent agreement. RH methods, in case of SR, do not introduce errors additional to those in GO.
A – isotropic N-field; B – spherically-symmetric N-field.
RO signals were used down to −50km height of straight line transmitter-receiver.
Cut-off (∼3.8-3.9km) determined from the CT amplitude.
No difference in 2-D case (isotropic N-irregularities (A)).
Additional significant negative N-bias above the cut-off in 1-D case (spherically-symmetric N-irregularities (B)).

Diffraction by small-scale N-irregularities results in that information about N at a certain z spreads over a large section of complex RO signal (radio-hologram).
Truncation of the hologram results in lower quality of reconstruction of the image.
An example of inversion of the truncated RO signal (GPS/MET)
Effect of super-refraction and small-scale $N$ irregularities on detection of the reflected signals
Parameters of RO signal and retrieved profiles that may indicate SR

(i) large positive spike in the bending angle
(ii) gradient, close to critical, in the Abel-retrieved N-profile
(iii) deep fading of amplitude (not in all cases, depends on complexity of N-profile)
Summary

Super-refraction is the wave propagation condition when radius of curvature of rays is smaller than the distance to the center of sphericity.

This results in that external rays may not have tangent points in a certain height interval (inside and below the SR layer).

An elevated SR layer does not bent rays to surface.

In the presence of SR, the refractive index is a multi-valued function of the refractional radius. This makes the Abel inversion inapplicable.

Application of the Abel inversion in case of the SR results in negative N-bias below the top of the SR layer. This bias is not related to inaccurate integration through singularity in bending angle.

The SR in RF domain occurs, most commonly, on top of PBL (over tropical and sub-tropical oceans).

The occurrence of SR on top of PBL, statistically, can be very different in different geographic regions over tropical and sub-tropical oceans.
Summary

Inverse problem in case of SR is ill-conditioned.

The true and the Abel-retrieved N-profiles correspond to strictly the same bending angle profile, i.e. they may not be distinguished in GO (the difference in real RO signals is due to only diffraction effects that are small).

Theoretically, the inverse problem may consist in finding the refractivity profile with larger values than the Abel-retrieved profile below the point of critical refraction, by satisfying the observation bending angle profile (a least square problem with inequality conditions).

In practice, it is not clear how stable this solution is to non-spherically symmetric N-irregularities and observation errors. This can be the subject of the future study.

An additional constraint on the solution can be put by an NWP model when directly assimilating the bending angle. It is necessary that the NWP model must have dense enough vertical grid in order to be able to reproduce the SR at an arbitrary height <2km.
Summary

Before reliable solution is found and tested, it is better to discard RO data, thought to be affected by the SR below the top of PBL.

Parameters of RO signal which may indicate the SR:

(i) strong positive spike in the bending angle profile at the expected top height of PBL;

(ii) the corresponding N-gradient, close to critical, in the Abel-retrieved N-profile;

(iii) deep fading of the amplitude. But, if N-profile is enough complicated above and below the SR layer, the amplitude may be not subject to deep fade.

Analysis of global radiosonde data and outlining regions with high frequency of occurrence SR on top of PBL may help assimilating or discarding RO profiles. This can be the subject of the future research.
Summary

Application of RH methods (CT, FSI) for processing tropospheric RO signals, in case of SR, does not introduce errors additional to those in GO (related to inapplicability of the Abel inversion).

N-irregularities with small horizontal-to-vertical aspect ratio ("turbulence") do not result in significant errors of reconstruction of N by RH methods. But they result in strong fluctuation of the transformed (CT) amplitude and smear its transition to zero thus degrading the cut-off accuracy.

Horizontally-elongated N-irregularities may result in larger errors and introduce negative N-bias (when vertical N-gradient exceed critical value).

It is important to acquire and to use RO signal for RH inversion down to sufficiently low height. Truncation of tropical RO signals may introduce additional errors above the cutoff.