COMPARISON
OF
THE FINITE DIFFERENCE METHOD AND THE SPECTRAL METHOD
FOR
SOLVING THE ADVECTION EQUATION

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Summer Employment Program
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INTRODUCTION

The basic model equations for the atmosphere and oceans involve spatial derivatives. In order to solve these equations in time, values for the variables must first be known in space at some starting instant, the initial conditions. Once this set is known, the time change in the equations can be solved to yield a new set of values at some future time.

In this paper, we will solve the simple advection equation by two different numerical methods, finite difference and spectral methods, and the results from both methods will be compared. Figures are at the end of each part.

In Part 1, we present the solutions for the linear advection equation solved by the finite difference method with four schemes, central in time and central in space, forward in time and central in space, forward in time and forward in space, and forward in time and backward in space. We also discuss the stability conditions for each scheme and the error associated by using the finite difference method. In Part 2, solutions for the advection equation are presented and compared to the solutions in Part 1. In Part 3, the non-linear advection equation is solved by both methods. We also compare the results and discuss the stability conditions for both methods. In Part 4, we demonstrate the relaxation method, which uses the finite difference method to solve a two-dimensional equation. Two different methods are used, simultaneous relaxation and sequential relaxation, and the results are compared. We also discuss the stability condition for the relaxation method. A short note for the spectral method is attached at the end. Conclusions are stated in Part 5.
PART I

FINITE DIFFERENCE METHOD
FOR
LINEAR ADVECTION EQUATION
OBJECTIVE:

The most common numerical integration procedure for weather prediction is the finite-difference method in which the derivatives in the differential equations of motion are replaced by finite difference approximations at a discrete set of points in space and time. The resulting set of equations, with appropriate restrictions, can be solved by algebraic methods. Four types of difference schemes: central in time and central in space, forward in time and central in space, forward in time and forward in space, and finally forward in time and backward in space, will be examined for their stability when applied to the advective equation and the results will be compared to theoretical analysis.

THEORY AND METHODS:

The advection equation is considered to demonstrate the concepts involved in the finite-difference methods for solving partial differential equations because it is a first-order, linear equation and similar to a linearized version of the first two terms in the equation of motion, thermodynamic or vorticity. Since the numerical solutions for these finite-difference equations can be found, we can then compare our results with them and use the matrix method or Von Neumann method to make the stability analysis. Consider the advective equation:

$$\frac{\partial F}{\partial t} + c \frac{\partial F}{\partial x} = 0$$  \hspace{1cm} (1)

where \( c \) is a constant and may be interpreted generally as a phase velocity of gravity waves or wind velocity.

The analytic solution is assumed to be a simple wave:

$$F(x, t) = Ae^{i\mu(x - ct)}$$  \hspace{1cm} (2)

where \( \mu = \frac{2\pi}{L} \), \( A \) is the constant amplitude, \( L \) is the wave length, and \( c \) is the phase speed.

The numerical solution to this problem can be sought. First let

$$x_m = m\Delta x, \quad m = 0, \pm 1, \pm 2, \ldots, \quad \text{and} \quad t_n = n\Delta t, \quad n = 0, 1, 2, \ldots$$

which replaces the continuous \((x, t)\) space by a mesh or grid of discrete points in the space.

1. Central in time and central in space

Central difference approximation for both the space and time derivatives in (2) leads to

$$\frac{F_{m,n+1} - F_{m,n-1}}{2\Delta t} = -c \frac{F_{m+1,n} - F_{m-1,n}}{2\Delta x}$$  \hspace{1cm} (3)
so it is necessary to consider the case where \( \lambda > 1 \) for (6). The two roots will be imaginary numbers and one of these magnitudes will exceed unity. As a result the factor \((B^{\Delta t})^n\) will increase without bound with increasing time \((n \Delta t)\). Such amplification is referred to as computational instability and must be avoided. Consequently, it is apparent that a sufficient condition for the solution to remain computationally stable is that

\[
\left| \frac{c \Delta t}{\Delta x} \right| \leq 1.
\]

2. Forward in time and central in space

Consider the following difference equation as an approximation to (1) and assume the same initial condition:

\[
F_{m,n+1} = F_{m,n} - \frac{\lambda}{2}(F_{m+1,n} - F_{m-1,n}).
\]  

(9)

Again assume a solution of the form (5) and substitute it into (9). After simplifying, the equation reduces to

\[
B^{\Delta t} = 1 - \lambda i \sin \mu \Delta x
\]

Placing the complex number on the right in polar form gives

\[
B^{\Delta t} = [1 + \left(\frac{c \Delta t}{\Delta x}\right)^2 \sin^2 \mu \Delta x]^{1/2} e^{-i \theta}
\]  

(10)

where

\[
\theta = \arctan \frac{c \Delta t}{\Delta x} \sin \mu \Delta x.
\]

Hence the finite difference solution for \( F \) may be expressed as

\[
F_{m,n} = A[1 + \left(\frac{c \Delta t}{\Delta x}\right)^2 \sin^2 \mu \Delta x]^{n/2} e^{i \mu(m \Delta x - n \theta / n)}.
\]  

(11)

It is apparent from (11) that the difference scheme which is forward in time and central in space is always computationally unstable for all wavelengths (except when \( \mu \Delta x \) is a multiple of \( \pi \)) because the amplitude of the finite difference wave grows continuously with time.

3. Forward in time and forward in space

Consider the next scheme that is forward both in time and space,

\[
F_{m,n+1} = F_{m,n} - \lambda(F_{m+1,n} - F_{m,n})
\]  

(12)
RESULT AND STABILITY ANALYSIS:

1. Central in time and central in space

While keeping $|\frac{c\Delta t}{\Delta x}| < 1$, from Figs. 1A and Fig. 1B which are central both in time and space but with c greater than and less than zero, we can see that the solutions are always computationally stable (neutral) as expected from the theory.

Using the Von Neumann method of stability analysis to check our results, we first introduce a new variable, $G_{m,n} = F_{m,n-1}$, which will permit a formal vector representation in terms of only two levels as follows:

$$F_{m,n+1} = G_{m,n} - \frac{c\Delta t}{\Delta x} (F_{m+1,n} - F_{m-1,n})$$

$$G_{m,n+1} = F_{m,n}$$

Now since the equations are linear, separate solutions are additive; thus only a single Fourier term can represent F and G, instead of a Fourier series. Hence let

$$F_{m,n} = \sum B_n e^{i\mu \Delta x}, \quad G_{m,n} = \sum B_n e^{i\mu \Delta x}$$

where the leading subscript indicates the column number. Substituting (16) into the system (15) and simplifying leads to

$$\begin{pmatrix} B_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{-2ic\Delta t}{\Delta x} \sin \mu \Delta x & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B_n \\ B_n \end{pmatrix}$$

or

$$B_{n+1} = AB_n = A^2 B_{n-1} = .......$$

Thus, the eigenvalues of A must be examined to find the stability of the scheme. They are the roots of the equation:

$$\lambda^2 + \lambda\left(\frac{2ic\Delta t}{\Delta x} \sin \mu \Delta x\right) - 1 = 0$$

giving

$$\lambda = -i \frac{c\Delta t}{\Delta x} \sin \mu \Delta x \pm [1 - (\frac{c\Delta t}{\Delta x})^2 \sin^2 \mu \Delta x]^{1/2}$$

Clearly both eigenvalues have a magnitude of unity provided that

$$\frac{c\Delta t}{\Delta x} \leq 1$$

which is the stability condition derived earlier for the leapfrog scheme.

2. Forward in time and central in space
In (21), the matrix A is a tridiagonal matrix and the eigenvalues of a tridiagonal matrix, with elements a,b,c in that order, are

\[ \lambda_k = b + 2(\sqrt{ac}) \cos \left( \frac{k\pi}{J} \right), \quad k = 1, \ldots, J - 1. \]

Therefore, the eigenvalues of A are

\[ \lambda_k = 1 + \frac{c \Delta t}{\Delta x} i \cos \left( \frac{k\pi}{J} \right), \quad k = 1, 2, \ldots, J - 1. \]  \hspace{1cm} (25)

From (25), it is apparent that the magnitude of all the eigenvalues are greater than unity. This implies that the forward in time and central in space difference scheme is computationally unstable for all values of c and this agrees with our earlier discussion.

3. Forward in time and forward in space

Figure 3A shows that when c is positive (downstream), the difference scheme of forward in time and forward in space is computationally unstable because it amplifies and Fig. 3B shows that when c is negative (upstream) the system damps which means it is computationally stable.

The matrix method is again used for stability analysis:

\[ F_{m,n+1} = F_{m,n} - \lambda (F_{m+1,n} - F_{m,n}) \]

or

\[ F_{m,n+1} = (1 + \lambda) F_{m,n} - \lambda F_{m+1,n}. \]  \hspace{1cm} (26)

Following the same steps as in the previous section, the matrix A can be found and is as follows:

\[
\begin{pmatrix}
F_{1,n+1} \\
F_{2,n+1} \\
F_{3,n+1} \\
\vdots \\
F_{j-1,n+1}
\end{pmatrix} =
\begin{pmatrix}
1 + \lambda & -\lambda & 0 & \cdots & \cdots & 0 \\
0 & 1 + \lambda & -\lambda & \cdots & \cdots & 0 \\
0 & 0 & 1 + \lambda & -\lambda & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 1 + \lambda
\end{pmatrix}
\begin{pmatrix}
F_{1,n} \\
F_{2,n} \\
F_{3,n} \\
\vdots \\
F_{j-1,n}
\end{pmatrix}
\]

or

\[ A = I + \lambda T \]

where

\[
T = \begin{pmatrix}
1 & -1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & -1 & \cdots & \cdots & 0 \\
0 & 0 & 1 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & 1
\end{pmatrix}
\]

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Therefore, the eigenvalues of $A$ are:

$$\lambda_k = 1 - \lambda = 1 - \frac{c\Delta t}{\Delta x}. \quad (29)$$

We can see that from (29) the results for the forward in time and backward in space difference scheme are the same as for the forward in time and forward in space difference scheme, namely the system is stable with upstream ($c > 0$ in this case) difference and the system is computationally unstable with downstream ($c < 0$ in this case) difference.

Again, the results from our figures agree with the stability analysis using the matrix method and our earlier theoretical analysis.
Fig. 1A, Finite difference method: centered in time and central in space with $c < 0$. (Neutrals)
FIG. 24. Finite difference method: forward in time and central in space with \( c < 0 \).
Fig. 3A. Finite difference method: forward in time and forward in space with $c < 0$. 

\[ x \]

\[ y \]

\[ \text{advection} \]
Fig. 4A. Finite difference method: forward in time and backward in space with $c > 0$. (Damped)
PART II

SPECTRAL METHOD

FOR

LINEAR ADVECTION EQUATION
OBJECTIVE:

Solutions to the advection equation will be examined using the spectral method, we compare the results with those from the finite-difference method.

THEORY AND METHODS:

Consider the advection equation:

\[ \frac{\partial F}{\partial t} = -c \frac{\partial F}{\partial x} \]  \hspace{1cm} (1)

where \( F \) is a function of both \( x \) and \( t \), and \( c \) is the wind velocity or phase velocity. We will discuss the case where \( c \) is a constant.

There are three steps required to evaluate the advection equation by the spectral method. Details will be discussed in the next section, but here is a brief description of the steps: (1) Forward transform, where the coefficients of the Fourier series for the original function \( F \) are determined (2) Inverse transform, where the function \( F \) is reconstructed from the Fourier series with the coefficients from (1) and (3) Time integrate, where the future positions of \( F \) are calculated.

1. Forward Transform:

We will assume the solution \( F(x,t) \) can be expressed as a Fourier series:

\[ F(x,t) = \sum_{j=-\infty}^{j=+\infty} \lambda_j(t) e^{iu_j x} \]  \hspace{1cm} (2)

where \( u_j \) is the wave number.

The coefficients \( \lambda_j(t) \) for the Fourier series can be determined by first multiplying both sides of (2) by \( e^{-iu_k x} \), the complex conjugate of the basis function, and integrating both sides from 0 to \( 2\pi \), the domain of definition:

\[ \int_0^{2\pi} F(x,t)e^{-iu_k x} \, dx = \sum_{j=-\infty}^{j=+\infty} \int_0^{2\pi} e^{i(u_j-u_k) x} \, dx. \]  \hspace{1cm} (3)

To find the integral of \( \int_0^{2\pi} e^{i(u_j-u_k) x} \, dx \), let us consider two cases:
(a) when \( j=k \), the integral is

\[ \int_0^{2\pi} dx = 2\pi. \]  \hspace{1cm} (4)

(b) when \( j \neq k \), the integral is

\[ \int_0^{2\pi} e^{i(u_j-u_k) x} \, dx = \int_0^{2\pi} \cos(u_j - u_k) x \, dx + i \int_0^{2\pi} \sin(u_j - u_k) x \, dx. \]  \hspace{1cm} (5)
RESULTS AND DISCUSSION:

Figures 1 and 2 are the results of using the spectral method with centered in time and forward in time schemes respectively to approximate the time the time derivative. Comparing Figs. 1 and 2 with Figs. 1A and 2A from the finite difference method using the same schemes, we can see that the results are identical, as expected.
Fig. 2. Spectral method: forward in time
PART III

SPECTRAL METHOD
AND
FINITE DIFFERENCE METHOD
FOR
LINEAR ADEQUATION EQUATION
OBJECTIVE:

The nonlinear advection equation, in which \( c \) is a function of \( x \) and \( t \), will be examined using both the finite difference and the spectral methods. The results from the two methods are also compared.

THEORY AND METHOD:

(a) Finite Difference Method:

The theory for the finite difference method is the same as in Part I of this paper where the advection equation is linear. Here we use the central in time and central in space difference scheme as an example.

(b) Spectral Method:

The theory for the spectral method nonlinear equations is quite different from that for the linear equation but the components are the same as in Part II, namely (1) Forward transform, (2) Inverse transform and (3) Time integration. Consider the advection equation:

\[
\frac{\partial F}{\partial t} = -F(x) \frac{\partial F}{\partial x}.
\]  

(1)

1. Forward Transform:

We will assume the solution \( F(x,t) \) can again be expressed as a Fourier series:

\[
F(x,t) = \sum_{j=-\infty}^{j=+\infty} \lambda_j(t) e^{iujx}.
\]  

(2)

To make the forward transformation, we want to find a single Fourier series to express the right-hand side of (1). In order to do that, we have to find the coefficients for the function \( F \) as in Part II and then we can approximate the partial derivative of \( F \) with respect to \( x, \frac{\partial F}{\partial x} \), with the coefficients of \( F \). After we have both approximations for \( F \) and \( \frac{\partial F}{\partial x} \), multiply them together and then apply the orthogonality condition as before to obtain the coefficients for the product and this gives the forward transformation.

2. Time Integration:

After the coefficients are found from step (1), \( F(x,t) \frac{\partial F}{\partial x}(x,t) \) can be written:

\[
F(x,t) \frac{\partial F}{\partial x}(x,t) = \sum_{j=-k}^{j=+k} T_j(t) e^{iujx}.
\]  

(3)
RESULT AND DISCUSSION:

(a) Finite Difference Method:

Figures 1, 2, 3 and 4 are the results from using the finite different method. I have set the prediction range the same for all the figures, which is equal to the product of the number of time steps, \( m \), and the time step, \( \Delta t \). The range of the function is set from -2.5 to +2.5 to make it easier for us to compare the results.

From Fig. 1, we can see there is a shock near the stationary point at the end of the forecast period. This is because the curves are moving tilted to the right in the upper half and to the left in the lower half. This causes the slope near the stationary point, \( x = \pi \), at end of the forecast period to approach infinity. When we apply the grid point method to approximate the slope, the result will approach infinity also.

Figure 2 shows the result as we increase the horizontal resolution by increasing the number of the grid points. We see that the shock is confined more locally as compared to Fig. 1 but there seems to be a greater error near the stationary point. This is because with smaller grid size, the approximation of the slope sees the function as a vertical line and the slope for a vertical line is infinity, therefore, the shock in Fig. 2 has much greater value as compared to Fig. 1. By examining the nonlinear advection equation carefully, we realize that the derivative of the function is singular at the stationary point, but in the mean time the function itself is zero. If the rate of the function goes to zero is greater or equal to the rate the derivative goes to infinity, then the shock is not existed.

There is temporal truncation error besides the spatial truncation error. By comparing Fig. 3 and Fig. 4, we can see the results are quite different. We increase the temporal resolution in Fig. 4, by decreasing the time step from 0.3 to 0.025. The result seems to look better when the time step is larger, but that is because with large time step, it tends to smooth out the shock that occurs at the stationary point. As we increase the temporal resolution, the approximation for the function is more accurate, we can the shock is produced again.

Comparing Fig. 2 and Fig. 4 where they either have high horizontal or temporal resolutions, we see the similar results between them.

(b) Spectral Method:

The same result is shown in using the spectral method in Figs 1A, 2A, 3A and 4A. Again, I have set the forecast period and the range of the function the same for all the figures as in the finite difference method.

From Fig. 1A, we can see that there is no shock, however there is error in the smallest wavelength throughout the domain of the function. That is because in the spectral method the transformation is done globally, so a local error will influence to the entire function. On the other hand, the approximation for the function is done locally in the finite difference method, so a local error has only a local influence, as seen in the figures.

Increasing the horizontal resolution is equivalent to increasing the truncation limit in the spectral method. In Fig. 2A, we increase the truncation limit from 10 to 50 we see that the curve at the end of the forecast period is much smoother because the truncation
Fig. 1. Finite difference method: central in time and central in space
Fig. 3. Finite difference method: central in time and central in space.
Fig. 1A Spectral method: central in time

k=10, m=24, dt=0.05

advection eqn
Fig. 3: Spectral method: central in time.
PART IV

RELAXATION METHOD
OBJECTIVE:

Given the Laplacian operator $\nabla^2$ and the function $\zeta$, we use relaxation methods to find the finite difference solutions for $\psi$ such that $\nabla^2\psi = \zeta$. The stability for this type of finite difference scheme is also discussed.

THEORY AND METHODS:

Let the function $\psi$ be given by:

$$
\psi = \Psi - a \ U \cos \theta + A \sin^3 \theta \cos \theta \cos 3\lambda
$$

where we set $\Psi$, $a$, and $U$ all equal to zero and $A$ equals to 0.5, $\theta$ ranges between $-\pi$ and $0$ and $\lambda$ ranges between $0$ and $2\pi$. So the equation can be simplified to:

$$
\psi = 0.5 \sin \theta^3 \cos \theta \cos 3\lambda.
$$

Figure 1 is the plot of $\psi$ which is called the stream function.

The vorticity $\zeta$ is defined as:

$$
\nabla^2\psi = \zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}.
$$

Therefore, we can find the vorticity from $\psi$ by taking the second order partial derivatives of (1) with respect to $\theta$ and $\lambda$:

$$
\zeta = a \ U \cos \theta + 6 \ A \sin \theta \cos^3 \theta \cos 3\lambda - 19 \ A \sin^3 \theta \cos \theta \cos 3\lambda
$$

with the values we have above for $a$, $U$, and $A$. Figure 6 is the plot of $\zeta$.

Figures 2, 3, 4, and 5, depict the first and second order partial derivatives of $\psi$ with respect to $\lambda$ and $\theta$, are the intermediate steps for finding the vorticity. Next, we use both the simultaneous and sequential relaxation methods to find the numerical solutions for $\psi$ and compare the numerical results with that in Fig.1 for Eq(1).

1. Simultaneous Relaxation:

We first define the error of the Nth guess at point $(m,n)$ as the residual $R_{m,n}^N$.

$$
\nabla^2\psi_{m,n}^N - \zeta_{m,n} = R_{m,n}^N
$$

where $\psi_{m,n}^N$ is the Nth guess of the stream function at point $(m,n)$.

Clearly, if the residuals were zero at every point of the grid, the numerical solution would be $\psi_{m,n}^N$. In general it is unlikely that the error will be zero everywhere. Thus the
RESULTS AND DISCUSSION:

Figures 7 and 8 are the results of numerical $\psi$ from the two different relaxation methods, simultaneous and sequential, respectively, when we set the boundary conditions the same as our first guess. Comparing these two figures with Fig. 1, analytical $\psi$, we see that the results are very close to the expected results, except at the boundary.

Figures 9 and 10 have the same conditions as Fig. 7 and 8, except the boundary conditions are assigned to be equal to the boundary conditions of the analytical equation, (2). The results are very satisfactory because they are almost identical to Fig. 1, the actual $\psi$ field.

By decreasing the error tolerance, $E$, the results should be more accurate as shown in Figs. 11 and 12. The error tolerance is decreased from $10^{-3}$ to $10^{-5}$ and the resulting figures show that the numerical solutions are even closer to the analytical solution.

STABILITY ANALYSIS:

We can examine the stability for the difference schemes by finding the solutions of the following equation and checking if the magnitudes are less than unity, the stability criterion.

$$\frac{\partial \psi}{\partial t} = k \nabla^2 \psi$$  \hspace{1cm} (9)

where $k$ is a constant and equal to unity in our case. Assume an exponential solution of (9)

$$\psi_{m,n,t} = B^t e^{i(m\mu + n\lambda) \Delta x}.$$ \hspace{1cm} (10)

Let $\Delta x = \Delta y$ and $\sigma = \frac{k \Delta t}{\Delta x^2}$, the fourier number. Substituting (10) into (9) and simplifing, we get

$$B^t = (1 - 4\sigma) + 4\sigma \cos(\mu \Delta x).$$

We can see from some calculation that, for the magnitude of $B^t$ to be less than unity, $\sigma$ must be between 0 and 0.25. That is $0 < \frac{k \Delta t}{\Delta x^2} < 0.25$. 

3
Fig. 1. Vorticity: Analytic solution
Fig. 7. Finite difference method: Simultaneous Relaxation

without boundary conditions
Fig. 9. Finite difference method: Simultaneous Relaxation

with boundary condition, error constant = 0.001
PART V

CONCLUSIONS
CONCLUSIONS

The results for the linear advection equation from the two numerical methods are similar.

There are differences in the results for the nonlinear advection equation from the two numerical methods due to the horizontal resolution (spatial truncation) and temporal resolution.

We used the finite difference method, also known as the relaxation method, to solve a two-dimensional second order equation, first with the boundary condition is fixed at the first guess and then with the boundary condition is the analytical boundary condition. The solutions with the analytical boundary conditions are similar to the analytical solution. We see how the boundary condition affects the solutions by comparing Fig. 7 and Fig. 9 in Part VI.

After becoming acquainted with the numerical methods outlined in this paper, we can go on to solve more complicated mathematical models.