Space Discretized Partial Differential Equations Viewed as Systems of Ordinary Differential Equations: Method of Lines

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1. Introduction

Scientists and engineers in many fields use differential equations to describe a vast number of their physical problems. Unfortunately, a large number of the differential equations which result from scientific problems have no analytic solution. However, the differential equations can be solved by numerical methods. Therefore, an understanding of the methods used to obtain numerical solutions of differential equations is helpful. The computer is an essential tool used in the numerical solution of differential equations.

In this paper we will describe a method used to transform systems of partial differential equations into systems of ordinary differential equations. We show that such ordinary differential equations can be viewed as scalar differential equations. We also discuss difference approximations used in the numerical solution of ordinary differential equations.
2. Continuum Theory for Ordinary Differential Equations

We first considered the canonical constant coefficient ordinary differential equation

\[ y_t = \lambda y, \quad y = y(t) \]  \hspace{1cm} (2.1)

with the initial condition

\[ y(0) = y_0. \]

The solution to the differential equation is

\[ y = y_0 e^{\lambda t}. \]

Four cases of the analytical solution were considered.

Case 1: \( \lambda < 0 \)

The solution is exponentially decaying.

Case 2: \( \lambda > 0 \)

The solution is exponentially growing.

Case 3: \( \lambda \) purely imaginary, i.e. \( \lambda = ib \) with \( b \) real.

Then the analytical solution is

\[ y = y_0 e^{ibt} = y_0 [\cos(bt) + i\sin(bt)], \]

and the solution is purely oscillatory.

In this case the differential equation can be written as

\[ y_{rt} + iy_{it} = i[y_r(t) + iy_i(t)] = iy_r - y_i \]  \hspace{1cm} (2.2)

where \( y_r \) and \( y_i \) denote the real and imaginary parts of the solution \( y \). Equating the real and imaginary parts we can rewrite (2.2) as

\[
\begin{bmatrix}
y_r \\
y_i
\end{bmatrix}_t =
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
y_r \\
y_i
\end{bmatrix}.
\]

Case 4: \( \lambda \) complex
Let \( \lambda = a + ib \) with \( a \) and \( b \) real.

Then

\[
y = y_0 e^{(a + ib)t} = y_0 e^{at} [\cos(bt) + i\sin(bt)].
\]

This solution is a product of the solution of either Case 1 or Case 2 multiplied by the solution of Case 3. Thus it is simply damped or growing oscillation.

We now show how systems of constant coefficient ordinary differential equations can be transformed into uncoupled scalar differential equations in canonical form.

Let us consider the ordinary differential equation system

\[
y_t = Ay,
\]

where \( A \) is a real constant matrix. Assume \( A \) has a complete set of eigenvectors. Let \( P \) denote the matrix of eigenvectors and \( \Lambda \) denote the diagonal matrix of eigenvalues of \( A \), i.e.

\[
P^{-1}AP = \Lambda.
\]

Applying \( P^{-1} \) to (2.3)

\[
P^{-1}y_t = P^{-1}APP^{-1}y
\]

\[
(P^{-1}y)_t = \Lambda (P^{-1}y)
\]

We now make the change of dependent variables

\[
v = P^{-1}y.
\]

The system can be written in terms of \( v \) as

\[
v_t = \Lambda v.
\]  

(2.4)

The \( i \)th component of equation (2.4) can now be written

\[
v_{it} = \Lambda_i v_i.
\]  

(2.5)
with \( \lambda \) belonging to one of the cases of the previous section. Hence, any system of constant coefficient differential equations can be transformed into an uncoupled system with each equation being in canonical form.

We now show how a system of linear variable coefficient differential equations can be transformed analogous to the constant coefficient case. Consider the following ordinary differential equation system.

\[
y_t = B(t)y
\]

where \( y \) is a \( n \)-dimensional vector and \( B \) is a \( n \times n \) matrix with time dependent coefficients. Assume \( B \) has a complete set of eigenvectors. Let \( P \) denote the matrix of eigenvectors and \( \Lambda \) denote the diagonal matrix of eigenvalues of \( B \), i.e.

\[
P^{-1}BP = \Lambda
\]

Applying \( P^{-1} \) to (2.6)

\[
P^{-1}y_t = p^{-1}By
\]

\[
p^{-1}y_t = P^{-1}BPp^{-1}y
\]

Since \( (P^{-1}y)_t = P^{-1}y_t + (P^{-1})_t y \)

\[
(P^{-1}y)_t - (P^{-1})_t y = \Lambda(P^{-1}y)
\]

\[
(P^{-1}y)_t - (P^{-1})_t P(P^{-1}y) = \Lambda(P^{-1}y)
\]

\[
(P^{-1}y)_t = \Lambda(P^{-1}y) + (P^{-1})_t P(P^{-1}y)
\]

\[
(P^{-1}y)_t = \Lambda(P^{-1}y) - p^{-1}P_t(P^{-1})y
\]
Again we can make the change of variables, \( v = P^{-1} y \).

Then in terms of \( v \) the system becomes

\[
v_t = \Lambda v - P^{-1} P_y v.
\]  

(2.7)

For \( B \) independent of \( t \), \( P_t = 0 \) and (2.7) reduces to the constant coefficient case. If \( P_t \) is small, the last term in eq. (2.7) is small and will not significantly effect the solution of eq. (2.7). Thus it can be ignored, and we again have an uncoupled system. If \( P_t \) is large, a change of independent variables can be made that will decrease it. Hence, this case can also be reduced to scalar equations.

By perturbation methods, a nonlinear system can be studied as a linear variable coefficient system.

Consider the system

\[
y_t = F(y, t).
\]  

(2.8)

Assume

\[
y = \overline{y} + y',
\]

where

\[
y = \text{true solution},
\]

\[
\overline{y} = \text{known function},
\]

\[
y' = \text{deviation from true solution}.
\]

Substituting this expression into (2.8) we have

\[
\overline{y}_t + y'_t = F(\overline{y} + y', t)
\]

\[
\overline{y}_t + y'_t = F(\overline{y}, t) + F_y(\overline{y}, t)y' + ...
\]  

(2.9)
where we have expanded $F$ in a Taylor series about $\bar{y}$.

For small perturbations $y'$ the higher order terms in equation (2.9) can be ignored. We are interested in the solution of the homogeneous equation. Hence we will ignore the forcing terms $\bar{y}_t$ and $F(\bar{y}, t)$ and the system becomes

$$y'_t = F_y(\bar{y}, t)y'.$$

System (2.8) has now been reduced to a system of linear variable coefficient ordinary differential equations which can be handled by the methods described above.
3. Finite Difference Approximations of Ordinary Differential Equations

Numerical solutions of systems of ordinary differential equations are obtained by replacing the time derivatives by difference approximations. A difference approximation must be accurate and stable. The truncation error of a finite difference approximation of a derivative is the bound on the difference of the derivative and the finite difference approximation of the solution of the differential equation. It is obtained by expanding the difference approximation via Taylor series. Two difference approximations were examined for their truncation errors and stability.

Euler's difference approximation is given by

$$y(t) = \frac{y(t + \Delta t) - y(t)}{\Delta t} \quad (3.1)$$

Expanding $y(t + \Delta t)$ around $y(t)$ by Taylor's theorem, we find that

$$y(t + \Delta t) = y(t) + y_t(t)\Delta t + \frac{y_{tt}(t)\Delta t^2}{2} + ...$$

Rewriting this becomes

$$\frac{y(t + \Delta t) - y(t)}{\Delta t} - y_t(t) = \frac{y_{tt}(t)\Delta t}{2} + ...$$

Neglecting higher order terms we have

$$\left| \frac{y(t + \Delta t) - y(t)}{\Delta t} - y(t) \right| \leq \alpha \Delta t \quad \text{where} \ \alpha \ \text{is a constant.}$$

Because $\Delta t$ is raised to the first power, Euler's method is called a first order method.

Notice $\epsilon(\Delta t) \leq \alpha \Delta t$

$$e(\Delta t) = \alpha(\Delta t)$$
and the error drops off linearly as the mesh is refined.

Fig. 1 shows the error in Euler's method when $\Delta t = .1$.

Fig. 2 shows the error in Euler's method when $\Delta t = .01$.

Notice that the error at any time with $\Delta t = .01$ is one-tenth of the error at that same time with $\Delta t = .1$.

The Leapfrog difference approximation is

$$y_t(t) = \frac{y(t + \Delta t) - y(t - \Delta t)}{2\Delta t}$$  \hspace{1cm} (3.2)

We expand $y(t + \Delta t)$ and $y(t - \Delta t)$ in a Taylor series about $y(t)$:

$$y(t + \Delta t) = y(t) + y_t(t)\Delta t + \frac{y_{tt}(t)\Delta t^2}{2} + \frac{y_{ttt}(t)\Delta t^3}{6} + \ldots$$

$$y(t - \Delta t) = y(t) - y_t(t)\Delta t + \frac{y_{tt}(t)\Delta t^2}{2} - \frac{y_{ttt}(t)\Delta t^3}{6} + \ldots$$

Substituting

$$\frac{y(t + \Delta t) - y(t - \Delta t)}{2\Delta t} - y_t(t) = \frac{y_{ttt}(t)\Delta t^2}{6} + \ldots$$

$$\left|\frac{y(t + \Delta t) - y(t - \Delta t)}{2\Delta t} - y_t(t)\right| \leq c\Delta t^2.$$

Since $e(\Delta t) \leq c\Delta t^2$

then $e(\frac{\Delta t}{n}) = c(\frac{\Delta t}{n})^2$,

and leapfrog difference approximation is a second order method. The error drops off quadratically as the mesh is refined. We can now conclude that a Leapfrog method is more accurate than Euler's method.

Let us now consider the general form of a difference equation.
Assume \( y^n = r^n \)

\[ y^{n+m} = \sum_{i=0}^{m-1} a_i y^{n+i}. \]

Substituting this form of solution into the difference equation, we obtain the so called characteristic polynomial

\[ r^{n+m} = \sum_{i=0}^{m-1} a_i r^{n+i}. \]

Dividing by \( r^n \)

\[ r^m - \sum_{i=0}^{m-1} a_i r^i = 0. \]

This is a polynomial of degree \( m \), and we assume there are \( m \) distinct roots \( r_i, i = 1, 2, \ldots, m. \)

Then the general solution of the finite difference scheme is of the form

\[ y^n = \sum_{i=1}^{m} c_i r_i^n. \]

Now, we examine the stability of the difference approximations considered above for the three cases of section 1.

Case 1: \( \lambda < 0 \)

Euler's method is given by

\[ \frac{y(t + \Delta t) - y(t)}{\Delta t} = -y(t) \quad (3.3) \]

Assume

\[ y^n \approx y(n\Delta t) \]

Then

\[ y^{n+1} = (1 - \Delta t)y^n \]
Let

\[ y^n = r^n \]

Then the characteristic polynomial is

\[ r^{n+1} = (1 - \Delta t) r^n \]

with solution

\[ r = (1 - \Delta t). \]

Since \(|r| < 1\), the method is stable and Fig. 3 shows the computed solution.

The leapfrog method is given by

\[ \frac{y^{n+1} - y^{n-1}}{2\Delta t} = -y^n \]  \hspace{1cm} (3.4)

\[ y^{n+1} + 2\Delta t y^n - y^{n-1} = 0 \]

The characteristic polynomial is

\[ r^{n+1} + 2\Delta t r^n - r^{n-1} = 0 \]  \hspace{1cm} (3.5)

Dividing (3.5) by \( r^n - 1 \)

\[ r^2 + 2\Delta t r + 1 = 0 \]

So

\[ r = \frac{-2\Delta t \pm \sqrt{4\Delta t^2 + 4}}{2} \]

\[ r = -\Delta t \pm \sqrt{1 + \Delta t^2} \]

\[ r \approx -\Delta t \pm (1 + \frac{1}{2}\Delta t^2) \]

\[ r_1 \approx -(1 + \Delta t + \frac{1}{2}\Delta t^2) \]

\[ |r_1| > 1 \]

\[ r_2 \approx 1 - \Delta t + \frac{1}{2}\Delta t^2 \]
\[ |r_2| < 1 \]

For \( \lambda < 0 \), the true solution of (3.3) decays with time.

Since the solution \( |r_1| \) grows in time, the Leapfrog method is unsuitable for this case. See Fig. 4.

Case 2: \( \lambda > 0 \)

Euler's method is given by

\[
\frac{y^{n+1} - y^n}{\Delta t} = y^n \tag{3.6}
\]

\[ y^{n+1} = (1 + \Delta t)y^n. \]

The characteristic polynomial is

\[ r^{n+1} = (1 + \Delta t)r^n \]

\[ r = 1 + \Delta t. \]

Now \( |r| > 1 \) but the true solution grows exponentially so Euler's method is stable for this case. Fig. 5 shows Euler's method for this case.

The leapfrog method is given by

\[
\frac{y^{n+1} - y^{n-1}}{2\Delta t} = y^n \tag{3.7}
\]

\[ y^{n+1} + 2\Delta t y^n - y^{n-1} = 0 \]

then

\[ r^{n+1} - 2\Delta tr^n - r^{n-1} = 0 \]

\[ r^2 - 2\Delta tr + 1 = 0 \]

\[ r = \Delta t \pm \sqrt{\frac{4\Delta t^2}{2} + 4} \]

\[ r = \Delta t \pm \sqrt{1 + \Delta t^2} \]

\[ r \approx \Delta t + (1 + 1/2\Delta t^2) \]
\[ r_1 = \Delta t - 1 - 1/2\Delta t^2 \]

\[ |r_1| < 1 \]

\[ r_2 = \Delta t + 1 + 1/2\Delta t^2 \]

\[ |r_2| > 1 \]

The true solution of (3.2) grows in time, therefore the Leapfrog method is stable. See Fig. 6.

Case 3: purely imaginary

Euler's method is given by

\[ \frac{y^{n+1} - y^n}{\Delta t} = iy^n \]

\[ y^{n+1} = (1 + i\Delta t)y^n. \]

The characteristic polynomial is

\[ r^{n+1} = (1 + i\Delta t)r^n \]

with the root

\[ r = 1 + i\Delta t \]

Notice \(|r| > 1\) so Euler's method is unstable.

Leapfrog method is given by

\[ \frac{y^{n+1} - y^{n-1}}{2\Delta t} = iy^n \]

\[ y^{n+1} - i2\Delta t y^n - y^{n-1} = 0 \]

The characteristic polynomial is

\[ r^{n+1} - i2\Delta tr^n - r^{n-1} = 0 \]

\[ r^2 - i2\Delta tr - 1 = 0 \]

\[ r = i2\Delta t \pm \sqrt{4\Delta t^2 + 4} \]
\[ r = i\Delta t + \sqrt{1 - \Delta t^2} \]

Assume \( 1 - \Delta t > 0 \), \( (\Delta t < 1) \), then \( |r_1| = |r_2| = 1 \) For \( \Delta t < 1 \) the method is stable, Fig. 6 shows the computed solution.
4. Partial Differential Equations: Method of Lines

In this section, we will show how the wave equation and subsequently any system of nonlinear partial differential equations can be transformed into a nonlinear system of ordinary differential equations using the method of lines. Hence, the numerical solution of a system of nonlinear partial differential can be analyzed using the methods described in the previous sections.

Consider the wave equation

\[ u_{tt} = u_{xx} \]  \hspace{1cm} (4.1)

where

- \( u \) represents displacement,
- \( t \) represents time,
- \( x \) represents space

with the initial conditions \( u(x,0), u_t(x,0) \) given.

Let

\[ v = u_t, \quad w = u_x \]

Writing equation (4.1) in first order form

\[ \frac{\partial}{\partial t} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} w_x \\ v_x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}. \]

Approximating \( \frac{\partial}{\partial x} \) by \( D_{ox} \), i.e.

\[ \frac{\partial v}{\partial x} \approx D_{ox} v = \frac{v_{j+1} - v_{j-1}}{2\Delta x} \]

then

\[ \frac{\partial}{\partial t} \begin{bmatrix} v \\ w \end{bmatrix} j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} D_{ox} \begin{bmatrix} v \\ w \end{bmatrix} j. \]
Given the boundary conditions \( u(0,t) = u(1,t) = 0 \),
we can now represent (4.1) in matrix form as

\[
\begin{bmatrix}
  v_1 \\
v_2 \\
  \vdots \\
v_{n-1} \\
w_0 \\
w_1 \\
  \vdots \\
w_n
\end{bmatrix}
= \frac{1}{2\Delta x}
\begin{bmatrix}
w_2 - w_0 \\
w_3 - w_1 \\
  \vdots \\
w_n - w_{n-2} \\
2(v_1 - v_0) \\
v_2 - v_0 \\
  \vdots \\
2(v_n - v_{n-1})
\end{bmatrix}
= \begin{bmatrix}
  D_0x v_1 \\
  D_0x v_2 \\
  \vdots \\
  D_0x v_{n-1} \\
  D_0x w_0 \\
  D_0x w_1 \\
  \vdots \\
  D_0x w_n
\end{bmatrix}
\]

System (4.1) has now been transformed into a system of nonlinear ordinary differential equations. Fig. 9 and Fig. 10 show the numerical solution of (4.1) at different times.

In general the method of lines can be used on most systems of partial differential equations. Consider the system

\[
u_t = F(x,t,u,u_x).
\]

Discretizing in space only

\[
u_x \approx D_0x u = \frac{u_{j+1} - u_{j-1}}{2\Delta x}.
\]

Let \( v \) be a discrete approximation (in \( x \)) of \( u \)

\[
\frac{d}{dt}v_j(t) = F(x_j,t,v_j,D_0x v_j).
\]

We now have a nonlinear system of ordinary differential equations. The methods which were explained in earlier sections can now be used to analyze this system of nonlinear ordinary differential equations which in turn provides an approximate understanding of the system of
nonlinear partial differential equations.
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