NONLINEAR BOUSSINESQ CONVECTIVE MODEL
FOR LARGE SCALE SOLAR CIRCULATIONS

by

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We present extensive numerical calculations for a model of thermal convection of a Boussinesq fluid in an equatorial annulus of a rotating spherical shell. The convection induces and maintains differential rotation and meridian circulation. The model is solved for an effective Prandtl number \( P = 1 \), with effective Taylor number \( T \) in the range \( 10^2 < T < 10^6 \), and effective Rayleigh number \( R \) between the critical value for onset of convection, and a few times that value. With \( \Omega = 2.6 \times 10^{-6} \text{sec}^{-1} \), \( d = 1.4 \times 10^{10} \text{cm} \) (roughly the depth of the solar convection zone) the range of Taylor number is equivalent to kinematic viscosities between \( 10^{14} \) and \( 10^{12} \text{cm}^2/\text{sec} \), which encompasses eddy viscosities estimated from mixing length theory applied to the sun.

The convection does generally make equatorial regions rotate faster, the more so as \( T \) is increased, but local equatorial deceleration near the surface is also produced at intermediate \( T \) for large enough \( R \) above critical. The differential rotation is maintained primarily through momentum transport in the cells up the gradient, rather than by meridian circulation. Differential rotation energy increases relative to cell energy with increasing \( T \), surpassing it near \( T = 3 \times 10^5 \). The differential rotation tends to stretch out the convective cells, analogously to what is thought to happen to solar magnetic regions. Differential rotation and meridian circulations energies are nearly equal for \( T = 10^3 \), but the meridian circulation energy falls off relative to differential rotation like \( T^{-1} \) for larger \( T \). The meridian circulation is always toward the poles near the surface, contrary to models of Kippenhahn, Cocke, Köhler, and Durney and Roxburgh. The radial shear produced in the
differential rotation is almost always positive, as in the Köhler model, but contrary to the assumptions made by Leighton for his random walk solar cycle model.

Solutions in the neighborhood of $T = 3 \times 10^4$ seem to compare best with various solar observations including differential rotation amplitude, cell wavelength, tilted structure, horizontal momentum transport, and weak meridional circulation. The local equatorial deceleration (equatorward of $10 - 15^\circ$) has not been observed, although the techniques of data analysis may not have been sensitive to it. The most important deficiency of the model is that all the solutions with $T > 10^3$ show the vertical heat transport a rather strong function of latitude, with a maximum at the equator, no evidence of which is seen at the solar surface.
1. Introduction

There has been considerable theoretical effort devoted recently to the possible existence and implications of convective motions on the sun of much larger horizontal scale and longer duration than supergranules (Simon and Weiss, 1968; Kato, 1969; Kato and Nakagawa, 1969; Gilman, 1969a,b; Durney, 1970, 1971; Davies-Jones and Gilman, 1970, 1971; Busse, 1970a; Vickers, 1971; Yoshimura, 1971; Yoshimura and Kato, 1971). The motivation for most of these theoretical models has been some tantalizing but as yet not entirely conclusive observational evidence for large-scale motions. Ward (1964, 1965) has interpreted sunspot motion statistics as indication of such motions, but the reliability of spots as tracers is in some doubt. Plaskett (1966) has seen evidence for such motions in his doppler velocity measurements but his data sample is quite small. More recently, Howard and Harvey (1970) and Howard (1971) have reported first results from by far the most extensive doppler velocity measurements, which also show evidence of a very large scale system of horizontal motions. However, they have not been able yet to construct a very detailed picture of these motions. In addition, the sun's surface magnetic fields show very large scale structure in the form of unipolar and bipolar magnetic regions (Bumba and Howard, 1965) as well as the even larger scale sector structure (Wilcox, 1968), both of which may be related to the convection patterns (Starr and Gilman, 1965; Simon and Weiss, 1968).

One principal result of most of the theoretical papers cited above has been the indication that giant solar convection cells are likely to produce and maintain an equatorial acceleration. However, these calculations are only for the initial tendency to produce differential rotation; they do not really
allow for nonlinear effects. Consequently these models cannot calculate reliably convection or differential rotation amplitudes and some of their conclusions may be invalidated by the nonlinear effects they neglect. The model we present takes into account substantial nonlinearities, and surveys the parameter space much more extensively than previous models.

Our model represents a useful compromise between the Busse (1970a) and Durney (1970, 1971) models, and those of Davies-Jones and Gilman (1970, 1971) in that it takes into account in an approximate way the coriolis forces as they would act on convection in a spherical shell, while retaining the much simpler cartesian geometry. This is done by restricting consideration to an annular region of fluid, bounded by two latitude circles, which symmetrically straddles the equator. Eliminating spherical geometry effected a considerable saving of computing time and difficulty of programming for the NCAR computer.

The objects of this study and subsequent ones are several fold. We seek answers to the questions:

1. What kind and how much differential rotation can be induced by giant convective cells of a given intensity, as a function of the rotation rate of the fluid region, its depth, and the amount of smaller scale eddy diffusion present?

2. What are the details of the maintenance of the induced differential rotation and how do these compare to other models?

3. How do the induced differential rotation and meridian circulation modify the convection which induced them?

4. How do the resulting patterns of motion compare with solar observations that are available, and do they suggest new observations to be made?
5. Is the total heat flux and temperature structure in the model a function of latitude?

6. How effective are the motions likely to be in maintaining a magnetic field?

2. Physical and geometrical assumptions

a) As in most other convection studies (including Busse 1970a,b; Durney 1970, 1971) we shall treat the fluid as Boussinesq. That is, we ignore density variations except when they are coupled with gravity. This is because the theory of compressible convection is presently in a relatively undeveloped state by comparison with the Boussinesq case. We also note that the bottom third or so of the convection zone can be treated as a nearly Boussinesq fluid. Further, in our opinion, coriolis forces are much more important than compressibility for determining differential rotation.

b) The depth of the annular region is taken to be \(0.2R_e\), with latitude extent of approximately 41.5°N to 41.5°S (see Figure 1). In the cartesian coordinate system we will be using, this corresponds to a width of eight times the depth. This region, then, would encompass the whole depth of the solar convection zone. In the sun it is possible that the giant convective cells would extend from the bottom almost to the top of the convective zone though they would be non-Boussinesq.

c) We choose simple boundary conditions, namely that the viscous stress vanishes at the top, bottom and sides, and that the top and bottom are perfect thermal conductors (temperature fixed) but the sides are thermal insulators. Thus radial but no latitudinal temperature gradients are imposed.
d) Since our model will represent explicitly only the very largest scales of convection on the sun, we must parameterize the turbulent transport effects of all the smaller scales. For the present, we shall simply assume that we may replace the molecular viscosity and radiative diffusivity of heat by a constant eddy viscosity $\nu$ and constant eddy heat diffusivity $\kappa$ respectively. In doing this, we are presuming rotational influences on granule and supergranule scale motions are insignificant by comparison with their influence on the giant cells. This seems reasonable since the characteristic times for the small scale motions are so short compared to the coriolis time scale. Unfortunately, the magnitudes of the eddy coefficients are not known very well; values ranging at least from $10^{12}$ to $10^{14}$ cm$^2$/sec for each are quoted various places in the literature. It would seem that $\nu$ and $\kappa$ should be about the same magnitude; there seems to be no strong argument for saying which is larger. For calculations presented below we shall therefore simply take the effective Prandtl number $P = \nu/\kappa = 1$.

e) In addition to not knowing either the viscosity or thermal diffusivity very well, we also do not know very accurately the temperature gradient the giant scale convection is likely to feel. Consequently it is very difficult to estimate the effective Rayleigh number for the giant cells. However, the mathematical limitations of our model are such that it can only treat convection that is at most "modestly nonlinear" (see section 3), so that we are effectively limited to Rayleigh numbers not more than a few times the critical value at which giant cell convection first occurs. Also, since we are considering only the largest scales in a spectrum of convection, a Rayleigh number based on molecular viscosity, which is very large for the sun, seems irrelevant.
f) The dimensionless parameter characterizing the rotation effects in the model is the effective Taylor number \( T \equiv 4\Omega^2d^4/\nu^2 \), in which \( \Omega \) is the sun's mean angular velocity (\( \approx 2.6 \times 10^{-6}\) sec\(^{-1} \) for the convection zone) and \( d \) is the depth of the convecting region. With the depth equal to \( 0.2R_\odot = 1.4 \times 10^{10} \text{cm} \), eddy viscosities \( \nu \) between \( 10^{12} \) and \( 10^{14} \) imply \( T \) varies by four orders of magnitude, between \( 10^6 \) and \( 10^2 \). Therefore we have solved the convection equations over this range.

We realize that the assumptions listed above make the model very simple by comparison with the real sun. On the other hand, the model we actually solve is probably more advanced than any previous model for solar differential rotation. It concentrates on those physical effects we believe to be the most important in determining differential rotation, namely the effects of coriolis forces. It is our belief that the differential rotation problem will only be solved by building a hierarchy of ever more complex, increasingly realistic models in which fewer approximations and parameterization of physical processes are made. At each step, understanding gained from the previous model would be built upon in the next model in which new elements of greater complexity are included, when computer size and development of solution techniques permit. To attempt now a nonlinear model which compressibility high Rayleigh numbers, spherical geometry, and the "proper" boundary conditions, (whatever they are) is in our view premature and very likely impossible to do successfully, because neither large enough computers nor good enough solution techniques exist.
3. Fundamental equations and energetics

a) Nonlinear equations

The approximate equations describing incompressible convection in the equatorial annulus are the standard Boussinesq equations for a rotating fluid, except that the geometric factors in the coriolis terms are expanded about the equator in the latitude coordinate, and only terms up to first order are retained (see Veronis 1963a,b). Otherwise, the equations are in cartesian form with \( x, y, z \) the coordinates in longitude, latitude and vertical height respectively. \( y = 0 \) at the equator, \( z = 0 \) at the bottom (see Figure 1). The equations are written relative to a uniformly rotating frame of reference, with the basic hydrostatic state of rest and the basic linear temperature profile determined by thermal conduction, subtracted out. The equations are in dimensionless form, with lengths scaled by the total depth \( d \) of the annulus, times by \( d^2/\kappa \), velocities by \( \kappa/d \), and temperature perturbations by \( \Delta \theta \), the magnitude of the imposed vertical temperature difference between bottom and top. The dimensionless velocities in the \( x, y, z \) directions are denoted respectively by \( u, v, w \); the time by \( t \); the temperature deviation from the linear profile by \( \theta \); the pressure deviation from hydrostatic, divided by the mean density, by \( \pi \). Dimensionless parameters that appear in the equations are the Rayleigh number

\[ R = g \alpha \Delta \theta d^3/\kappa \nu \]  

(in which \( g \) is gravity, and \( \alpha \) the coefficient of volume expansion), a length ratio \( \lambda = d/r_s \) (with \( r_s \) the sun's radius), and the Taylor number \( T \) and Prandtl number \( P \) defined earlier.

We are particularly interested in seeing what kind of longitudinally averaged circulations, i.e., differential rotation and meridian circulations, the convective cells force and what their reaction on the cells is. Therefore it is useful to separate each variable into a part averaged
over the total length of a latitude circle, denoted by a subscript \( \theta \), and a deviation from that average, denoted by a subscript \( \theta_1 \). The subscript \( \theta_1 \) variables, then, describe the forcing cells, and the subscript \( \theta \) variables the forced mean circulations. We further approximate the equations for the subscript \( \theta_1 \) variables by retaining only those nonlinear terms which represent interactions with the mean flows, and ignore self interactions which would produce other harmonics. The "mean field" equations then are

\[
\frac{\partial u_0}{\partial t} = -\frac{\partial}{\partial y} u_0 v_0 - \frac{\partial}{\partial z} u_0 w_0 + \lambda y PT \frac{k^2}{u_0} - PT \frac{k^2}{w_0} + \nabla^2 u_0 - \frac{\partial}{\partial y} (u_1 v_1) - \frac{\partial}{\partial z} (u_1 w_1)
\]  

\[\text{(1)}\]

\[
\frac{\partial v_0}{\partial t} = -\frac{\partial}{\partial y} v_0^2 - \frac{\partial}{\partial z} v_0 w_0 - \lambda y PT \frac{k^2}{v_0} - \frac{\partial}{\partial y} (v_1^2) - \frac{\partial}{\partial z} (v_1 w_1)
\]  

\[\text{(2)}\]

\[
\frac{\partial w_0}{\partial t} = -\frac{\partial}{\partial y} v_0 w_0 - \frac{\partial}{\partial z} w_0^2 + PT \frac{k^2}{u_0} - \frac{\partial}{\partial y} v_1 w_1 - \frac{\partial}{\partial z} w_1^2
\]  

\[\text{(3)}\]

\[
\frac{\partial \theta_0}{\partial t} = -\frac{\partial}{\partial y} (v_0 \theta_0) - \frac{\partial}{\partial z} (w_0 \theta_0) + w_0 + \nabla^2 \theta_0 - \frac{\partial}{\partial y} (v_1 \theta_1) - \frac{\partial}{\partial z} (w_1 \theta_1)
\]  

\[\text{(4)}\]

\[
\frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = 0
\]  

\[\text{(5)}\]

The correlations of cell variables, or stresses, on the right-hand sides of (1) - (4) are seen to provide forcing functions for the mean field variables. The cell equations are

\[
\frac{\partial u_1}{\partial t} = -\frac{\partial}{\partial x} u_1 + \lambda y PT \frac{k^2}{u_1} - PT \frac{k^2}{w_1} + \nabla^2 u_1 - 2u \frac{\partial u}{\partial x} - \frac{\partial}{\partial y} (v_0 u_1 + u_0 v_1) - \frac{\partial}{\partial z} (w_0 u_1 + w_1 u_0)
\]  

\[\text{(6)}\]
\[ \frac{\partial v}{\partial t} = - \frac{\partial}{\partial y} \left( \lambda_y \frac{P}{T} u \right)_1 + \nabla^2 v \]
\[ - 2u \frac{\partial}{\partial x} - 2 \frac{\partial}{\partial y} (v \ n) - \frac{\partial}{\partial z} (w \ n \ w) \quad (7) \]
\[ \frac{\partial \omega}{\partial t} = - \frac{\partial}{\partial z} + \frac{P}{T} u + \frac{\partial \theta}{\partial z} + \nabla^2 \omega \]
\[ - 2u \frac{\partial}{\partial x} - \frac{\partial}{\partial y} (w \ w) - 2 \frac{\partial}{\partial z} (w \ w) \quad (8) \]
\[ \frac{\partial \theta}{\partial t} = + \omega + \nabla^2 \theta - 2u \frac{\partial}{\partial x} - \frac{\partial}{\partial y} (v \ n \omega) - \frac{\partial}{\partial z} (w \ n \ w) \quad (9) \]
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0 \quad (10) \]

As described in section 2d, the boundary conditions are
\[ \theta_0, \theta_1, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}, \frac{\partial v}{\partial z}, w_0, w_1 = 0 \text{ at } z = 0, 1 \quad (11) \]
\[ \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}, v_0, v_1, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial y} = 0 \text{ at } y = \pm 4 \quad (12) \]

We note that, with the above boundary conditions, the total momentum of the differential rotation \( u \) in the annulus is conserved. In all calculations which follow, we assume that momentum is initially zero with respect to the rotating frame. It therefore remains zero.

b) Differential rotation energy equation

It is very useful in interpreting the dynamics implied by the equations (1) - (12) to derive from them a kinetic energy equation for the differential rotation. It will tell us how, as functions of the parameters of the problem \( R, T, P \), the differential rotation is maintained.
by the convective cells. To obtain the kinetic energy equation for differential rotation (per unit mass, per unit length in x) we multiply (6) by \( u_0 \), integrate over the annulus cross section \( 0 \leq z \leq 1, -4 \leq y \leq +4 \), integrate by parts, using the continuity equation (5) and boundary conditions (11) and (12). This gives us

\[
\frac{\partial}{\partial t} \iint \frac{u^2}{2} \ dydz = \lambda PT \iint yu \ v \ dydz - PT \iint u \ w \ dydz
\]

\[
+ \iint (u \ v) \ \frac{\partial u}{\partial y} \ dydz + \iint (u \ w) \ \frac{\partial u}{\partial z} \ dydz
\]

\[
- P \iint \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \right] \ dydz
\]

(13)

From (13) we see that the differential rotation can be maintained against turbulent frictional dissipation (the last integral in (13)) in basically two ways. One is by coriolis forces acting on the meridian circulation \( v_0, w_0 \), represented by the first two integrals on the right. The second is by horizontal and vertical Reynolds stresses from the convective cells, \( (u \ v) \) and \( (u \ w) \) respectively. To make the kinetic energy of differential rotation increase, these stresses must selectively transport \( u \) momentum from regions of small (or negative) \( u_0 \) to regions of large \( u_0 \), that is, up the gradient of momentum. This is the opposite of a turbulent frictional decay process. Our results indicate that, for \( T > 10^3 \) and \( R \leq 3R^- \) critical for convection, these Reynolds stresses are more important in maintaining the differential rotation than is the mean meridional circulation. If the Reynolds stresses supply energy to the mean circulations, this energy must come, of course, from the kinetic energy of the cells themselves. The principal maintenance of the cells is of course by rising of warm fluid and sinking of cold in them. In all our calculations, only a rather small fraction of the energy
converted in this way is needed for the stresses which maintain the differential rotation. Most of it goes directly into dissipation.

4. Method of solution

Consistent with the mean field approximation described in section 3b, we represent the x variation of each of the primed variables associated with the convective cells by a single normalized fourier harmonic of longitudinal wavenumber $k$, whose phase may be a function of $y$ and $z$. The wavenumber $k$ is $k_c$, the critical value for the onset of convection according to a linear stability analysis of the conductive, no motion state (section 5).

We then approximate the equations for the $o,k$ amplitudes by finite difference equations in time and the spacial coordinates $y$ and $z$. The scheme we use is in most but not all respects similar to that laid out for a somewhat similar problem by Williams (1969), to which the reader is referred for details. For all calculations, the space increment $\Delta$ was taken to be 0.1, so that the fluid region was broken into 10 intervals in the vertical and 80 in latitude.

For most cases, symmetry conditions were imposed at the equator which reduced the calculations by half; $u,w,\theta,\tau$ are taken to be symmetric about the equator, and $v$ is taken antisymmetric. (We call these "symmetric modes"). It can easily be deduced from both the linear and nonlinear equations that if initial conditions with these symmetries are chosen, the evolving solutions retain them. Solutions with $u,w,\theta,\tau$ antisymmetric and $v$ symmetric (called "antisymmetric modes") about the equator also exist for the linear equations. As shown in section 5, these are less unstable, and therefore less important
for the nonlinear solutions near the critical Rayleigh number. In addition, this latter symmetry property does not hold for the nonlinear system, so that even if initial conditions contained it, the nonlinear terms would tend to produce solutions of the opposite symmetry.

5. Linear results

a) Stability properties

Figure 2 gives the critical $R_c$ (minimum $R$ for instability) as a function of $T$ for the onset of convection. We see that the symmetric modes are more unstable, increasingly so for large $T$. Consequently in the nonlinear calculations which follow we have concentrated on the symmetric modes. For comparison purposes, we have also included on Figure 2 the critical Rayleigh number for convection with the top and bottom boundaries when the rotation is parallel to gravity, corresponding to the poles on a sphere (from Chandrasekhar 1961, p. 95). Note that this type of convection is stabilized much more by rotation. Figure 3 gives the critical longitudinal wave number $k_c$ of the convection setting in at $R_c$. It increases with $T$ for both symmetries, but more slowly for the less inhibited symmetric modes. In general for $T > 0$, the unstable convective modes are not stationary, but propagate in $x$ (longitude) without change of shape. The phase velocity is slightly retrograde relative to the rotating reference frame for $T < \sim 5000$, and becomes increasingly prograde for larger $T$. 
b) Structure of unstable modes

The structure of the unstable convective modes changes rather markedly as the Taylor number $T$ is increased. In Figure 4, we see the horizontal velocity vectors (absolute magnitude arbitrary) of the symmetric modes at $z$ levels 10, 7, 3 (near the top, middle, and bottom, respectively), for $T = 10^2, 10^3, 10^4, 10^5$. All of these are for $k = k_c, R = 1.2R_c$ at the particular $T$, taken from Figures 2 and 3. At $T = 100$, the level 10 and 3 horizontal velocities show the convection is in the form of slightly distorted rolls with axis bent toward the positive $x$ direction at high latitudes in both hemispheres. The velocity vectors are turned by the modest coriolis force $-\lambda y PT u$ toward the right in the northern hemisphere and toward the left in the southern hemisphere, but the flow is still almost directly across the isobars from high to low pressure as in the nonrotating case. At larger $T$, the velocities at levels 7 and 10 are increasingly directed away from straight east-west flow of rolls, to a more rotary flow about centers at modest latitudes in each hemisphere. The flow at level three is largely damped out at high latitudes. (The patterns for $T = 10^6$ are very similar to those for $T = 10^5$ except the wavelength in longitude is considerably shorter.) The vertical velocities are maximum at the equator and, for the highest $T$, fall off fairly rapidly with latitude. The cell boundaries (defined by the nodes in vertical motions) now tilt in the opposite sense in latitude to what they do at low $T$. The flow is now nearly parallel to the isobars, with counterclockwise swirl about low pressure and clockwise swirl about high pressure in the northern hemisphere, and the opposite in the southern hemisphere. The swirl about high and low pressure is indicative of the flow becoming more and more heliostrophic (pressure gradients balancing coriolis forces) as $T$ is increased.
The xz profile of the convection (not shown) is complicated. Looking north, near the equator, the downward and upward flow vectors for $T = 10^3$, $10^4$ are tilted to their left. By $T = 10^5$ and $10^6$, their tilt is to the right. On the other hand, in middle latitudes, the upward and downward vectors are tilted increasingly to the right as $T$ increases. For a given $T$, this tilt is a smooth function of latitude, with higher latitude vectors tilted more to their right. In other words, the rolls have side boundaries which are tilted different amounts and different directions at different latitudes: the roll is twisted about its own axis. The temperature field, however, is almost unaffected by this twisting.

c. Interpretation of linear results

The convection adjusts in such a way as to minimize the constraint of rotation. In the present case it does this through the pressure gradient forces which tend to balance the coriolis forces giving rise to flow nearly parallel to the isobars. One consequence of the approach to heliostrophic balance is that the flow attempts to satisfy the so-called Taylor-Proudman theorem, which states that the flow should not vary along the axis of rotation. Consequently the convection increasingly takes the form of rolls with axis parallel to the local rotation vector. In the spherical shell, this direction is parallel to the upper and lower boundaries at the equator, but it cuts through the top of the shell at midlatitudes in each hemisphere (see dashed arrow in Figure 1). The swirling horizontal flow we see near the top in each hemisphere (Figure 4) then represents a diagonal cut across this roll.

The convection, however, is unable to be completely independent of the coordinate parallel to rotation, because of the upper and lower boundary conditions requiring the vertical motion to vanish. Consequently
the roll depth decreases with latitude, and heliostrophic balance in the vertical direction near the top is prevented and the Taylor-Proudman theorem violated. To a lesser extent, the free boundary conditions also prevent heliostrophic balance. Nevertheless, the symmetric modes are more nearly able to satisfy the Taylor-Proudman theorem and still convert potential energy than the antisymmetric ones, so they become unstable at a lower Rayleigh number (Figure 2).

The convective rolls have vorticity about the rotation axis. With large rotation, these can be treated as made up of nearly straight vortex tubes that are advected around in the roll. The length of tubes carried toward the surface must shorten; those carried away, lengthen. This leads to the prograde propagation in longitude of the whole convection pattern. This result is completely analogous to the spherical shell case examined by Busse (1970b). We were not, however able to confirm Busse's (1970a) result for small $T$ by our method, since the numerical integrations would have to have been much longer to pick up the small phase velocities.

The tilts in the velocity vectors in $xz$ and $xy$ planes produced by the Coriolis forces are crucial to the question of what kind of differential rotation is induced, because they determine the direction of transport of $u$ momentum. That is, rising fluid with a component of $u$ in the negative $x$ direction, and sinking fluid with positive $u$, as we have found near the equator for $T < 10^5$, gives a momentum transport downward, or negative stress $(u \, w)_{110}$, while the opposite sense of tilt we found in middle latitudes implies upward transport (positive $(u \, w)_{110}$). At higher $T$, the transport is upward at all latitudes. The horizontal velocity vector tilts (Figure 4) indicate a
predominantly equatorward transport of momentum (negative \( u v \) in northern hemisphere, positive in southern) from middle latitudes in each hemisphere. That is, the fluid particles moving fastest in the positive x direction also are moving toward the equator, while the slower moving particles are moving toward the pole. This transport is concentrated more in upper levels as \( T \) increases. These deductions are confirmed by the nonlinear calculations.

6. Nonlinear results

In all our nonlinear calculations, the net \( u_0 \) momentum integrated over the entire cross section \(-4 \leq y \leq +4, 0 \leq z \leq 1\) is always zero relative to the rotating frame of reference. It is clear that if we are to achieve an equatorial acceleration at the surface as is observed, the equatorward transport of momentum by the convection must overpower the downward momentum transport near the equator, as well as the tendency of the meridian circulation \( v_0 \) and upward momentum flux to speed up the upper levels of the convection zone at higher latitudes. If the vertical momentum transport and meridian circulation dominate, we will get an equatorial deceleration.

In choosing which nonlinear solutions to emphasize, we have been guided somewhat by the observations of Howard and Harvey (1970) and Howard (1971). From these, we can obtain probably the most reliable mean profiles of the differential rotation. They were unable to find any systematic meridional circulation, with an upper limit of 30 m/sec line of sight velocity, while the differential rotation they found in the latitudes our model covers is of magnitude 150 m/sec. This implies the energy in the differential rotation in our model should be many times that of the meridian circulation.
to be relevant to the solar case. In addition, the large scale horizontal velocity fields they see in addition to the differential rotation are probably at most roughly the same amplitude as the observed shears in the differential rotation and perhaps less. Therefore the convection in our model should not be large in amplitude compared to the differential rotation it produces.

We have done nonlinear calculations for $T$ from $10^2$ all the way to $10^6$. For $T < 10^3$, we find the solutions have a number of properties which are contrary to our observational guidelines including large amplitude oscillations between equatorial acceleration and deceleration, as well as large cell and meridian circulation amplitudes, so we omit detailed discussion of them here. What we present below are the results from a large number of computer runs for different values of $R$ with $T$ in the range $10^3$ to $10^6$.

a) Mean circulation profiles

In Figure 5 we give the form of the induced differential rotation $u_0$ and meridian circulation $v_0, w_0$ for $T = 10^3, 10^4, 10^5, 10^6$ with $R = 1.2 R_c$. Two features of the circulations are immediately evident. First, for all $T$, equatorial regions rotate faster than higher latitudes, and surface layers rotate faster than the deep part of the convection zone. Second, the meridian circulation is in all cases a single dominate cell in each hemisphere, with poleward flow near the upper boundary and rising flow centered on the equator. Thus, the convective cells do produce equatorial acceleration. We do note, however, some slight dips in the rotation right near the equator at the surface (most evident in the $T = 10^4$ plot). These dips can, depending on $T$, amplify considerably as we push $R$ further above critical. We can show that as $T$ increases, the differential rotation contours become more and more parallel to the local rotation vector in the $yz$ plane, resulting
in the concave upward patterns seen. This differential rotation itself is
increasingly heliostrophic as \( T \) increases, and therefore it more nearly satisfies the Taylor-Proudman theorem, as do the convective cells.

Clearly, at \( R = 1.2 R_c \) for \( T = 10^3 \) and \( 10^4 \) the equatorward transport of momentum is enough to accelerate surface layers relative to higher latitudes and lower levels, against the decelerating effects of downward momentum flux \( (u \ w) \) near the equator and the coriolis forces acting on the meridian circulation. For \( T = 10^5 \) and particularly \( T = 10^6 \), the upward momentum flux near the equator actually reinforces the equatorial acceleration, and the dip in the surface rotation near the equator is very small.

The meridian circulation is essentially unchanged in structure as \( R \) increases, except for some tendency for the horizontal flow to concentrate near the top and bottom boundaries (the magnitude of the circulation of course increases). The differential rotation, on the other hand, shows increased equatorial deceleration near the surface. For a given \( R/R_c \), this effect is less pronounced for \( T = 10^5, 10^6 \) than for \( T = 10^4, 3 \times 10^4 \). For these latter values, we actually get two "jets" forming 10-15° on either side of the equator (see Figure 10 and section 7b). The reason is that while strong equatorward transport of momentum is retained the downward transport near the equator by the cells increases in latitude extent and intensity compared to the upward transport at higher latitudes. This is due to the fact that flow is becoming less heliostrophic as \( R \) increases. Rising and sinking fluid particles in the cells now have more tendency to conserve their momentum, and therefore be deflected to their left in the xz plane, leading to more downward momentum flux.

b) Mean circulation amplitudes

Figure 6 gives a measure of how large a differential rotation we get compared to the convective cells which force it. The plot is of the ratio of the total kinetic energy \( U^2 \) in the differential rotation, to the kinetic
energy $U^2_1$ of the same component of motion in the cells, as a function of $R/R_c$ for several Taylor numbers. We computed solutions out to the points where we were no longer getting approximately steady solutions, but instead began to get energy fluctuations in the solutions of 10 or 20%. We see in Figure 6 that for $T = 10^3$, the differential rotation energy is never more than about 6% of the energy of the cells, whereas by $T = 3 \times 10^4$, we get an energy ratio of roughly unity. With still higher $T$, the steepness of the curves suggest that in a model with greater resolution we can get much larger differential rotation compared to the cells.

Figure 7, which gives the ratio of meridian circulation energy $V^2_0 + W^2_0$ to the differential rotation energy as a function of $T$, for several different $R$ values, shows that meridian circulation gets smaller and smaller by comparison as $T$ increases, roughly like $T^{-1}$. This is a natural consequence of approaching heliostrophic balance, since the Coriolis forces associated with the differential rotation can be balanced by pressure gradients and buoyancy, while those linked with the meridian circulation, being in the $x$ direction, have no axisymmetric pressure gradients in that direction to balance against.

c) Kinetic energy balance

Figure 8 gives an analysis of the kinetic energy equation (13) for the maintenance of the induced differential rotation $u_0$. The various curves correspond to integrals on the right-hand side of (13) and are defined as follows:

- **HCF** - work done by Coriolis forces in $xy$ plane (first integral)
- **VCF** - work done by Coriolis forces in $xz$ plane (second integral)
- **MMT** - work done by the meridional Reynolds stress $(u_1 v_1)$ in transporting $u$ momentum against the momentum gradient $\partial u_0 / \partial y$ (third integral)
- **VMT** - work done by vertical Reynolds stress $(u_1 w_1)$ in transporting $u$ momentum against the momentum gradient $\partial u_0 / \partial z$ (fourth integral)
All of these integrals are normalized with respect to the dissipation of differential rotation kinetic energy by smaller scale turbulence, the last integral in (13). Thus positive values in Figure 6 represent work done against dissipation, and negative values represent work done which adds to the depletion of kinetic energy of differential rotation. The sum of the individual curves in Figure 6 adds to unity since a steady state is reached.

The important point of Figure 6 is that for all $T$, the maintenance of the differential rotation is by the transport of $u$ momentum up the gradient of momentum by the convective cells which counterbalance the downgradient viscous diffusion of momentum. For the lower values of $T$, this maintenance is principally by the equatorial transport of momentum in the horizontal Reynolds stress, while for higher $T$ the upward transport of momentum at all latitudes significantly. The Coriolis forces acting on meridian circulation act either to destroy differential rotation (for $10^3 < T < \sim 3 \times 10^4$) or contribute essentially nothing to its maintenance. These conclusions basically hold for $R > 1.2 R_c$ up to the largest $R$ values we have studied, except that near $T = 10^3$, at which the meridian circulation does become more important in maintaining differential rotation.

We see that the destructive effect of $xz$ plane Coriolis forces outweighs the positive effect of $xy$ plane Coriolis forces, except for large $T$, for which they almost exactly cancel. It should be pointed out that although the meridian circulation does not contribute to the maintenance of the differential rotation as a whole, it is of course locally important in determining details in its profile.
d) Heat transports

The total vertical heat transport, that is transport by the sum of the thermal conduction \( (w_1 \theta)_0 \) and \( w_0 \theta \) by the induced meridian circulation, shows in all cases a pronounced maximum at the equator, varying from roughly 50% to 20% greater at the equator than at high latitudes as \( T \) varies from \( 10^3 \) to \( 10^6 \) for \( R = 1.2R_c \). For \( R > 1.2R_c \), the differences in heat flux become still greater. For example with \( R = 2.8R_c \) and \( T = 3 \times 10^4 \), almost two and one half times as much heat emerges at the equator than at higher latitudes. This property represents the most serious difficulty in applying the model to the sun, because such heat flux differences are not observed. The temperature contours show a general warming of equatorial latitudes at a given level relative to higher latitudes. This is produced by a modest equatorward heat flux from the cells.

e) Distortions of convective cells by mean circulations

The structure of the convective cells is significantly distorted by the induced mean circulations as \( R \) is increased. Figure 9 gives the vertical and horizontal motion fields of the cells for \( T = 3 \times 10^4 \), and we note that the patterns are increasingly "strung out" by the differential rotation giving a tilt with latitudes of roughly half a cell wavelength by \( R = 2.8R_c \). Similar stretching occurs in the temperature and pressure fields. At the bottom of Figure 9, we see the total horizontal motions \( (u_0 + u_1) \hat{x} + (v_0 + v_1) \hat{y} \) at \( z \) level 10 (near the top) showing we get a pattern of tilted swirls, each of the same sense of swirl within a single hemisphere. This pattern is quite characteristic of all the high Taylor number solutions in which the differential rotation energy is comparable to
the cell energy. The predominance of positive differential rotation at the
equator can be seen quite clearly in the $R = 2.0, 2.8 R_c$ cases.

At higher $T$ than $3 \times 10^4$, the distortions are less, even though the
mean circulation amplitudes relative to the cells are larger for a given
ratio $R/R_c$. This is because the rotational constraint is stronger, and is
not broken as easily by the effects of shear on the cells.

7. Comparisons with Observations

If we compare our model solutions to those solar observations that are
available, which we do in a) - f) below, we find that solutions in the
neighborhood of $T = 3 \times 10^4$ tend to agree best in several respects, although,
since our model is highly idealized, such agreements should be treated with
some caution.

a) Cell structure

Howard (1971) reports that his doppler data gives evidence for largely
east-west horizontal motions with a wavelength of roughly $25^\circ$ in longitude.
In our model, the wave number $k$ of these motions would be about 2.9 which
corresponds to the most unstable mode at onset of convection with $T \approx 2 \times 10^4$
(from Figure 3). Lower $T$ would give cells somewhat too large, larger $T$ quite
a bit too small. The cell structure in the nonlinear case should look similar
to Figure 9, including the shearing due to differential rotation. It seems
possible that the tilts with latitude of the cell boundaries could be related
to solar bipolar magnetic regions, and that the tilts in the horizontal velocity
vectors might be related to the tilt in the axis of sunspot groups, if such
groups tend to line up with their axis parallel to the flow.
b) Mean motions

We found earlier (Figure 6) that for $T < 3 \times 10^4$, the energy in the differential rotation is significantly less than that in the corresponding component of motion in the cells. From Howard (1971) this does not appear to be the case on the sun. On the other hand our solutions for $T = 10^5$ and $10^6$ for the Rayleigh numbers we have studied give much too small absolute magnitude of differential rotation, which can be seen if we put back in the dimensional scale factors. Around $T = 3 \times 10^4$, the differential rotation approaches the correct magnitude within a factor of two by $R = 2.8 R_c$, with comparable energies in the cells. Figure 10 gives a plot of the solutions for several $R$ values at $T = 3 \times 10^4$ compared to the mean profile from all data of Howard and Harvey (1970) both plotted in a rotating frame with $\Omega = 2.6 \times 10^{-6}$ sec$^{-1}$. We see that a significant local equatorial deceleration is present (also discussed in Section 6) as a deficit between two "jets", one in each hemisphere. At higher $T$, the profile is much smoother, with little dip at the equator, but much too small in amplitude for the Rayleigh numbers studied. At lower $T$, we also get some dip, or a broad flat nose, but here the cell energies are too big.

The local equatorial deceleration has not been reported in any of the observational data, but then it has not really been looked for, either. It seems doubtful that the procedure used by Howard and Harvey (1970), which was to fit the doppler shift data to an angular velocity formula of the form $A + B \sin^2 \phi + C \sin^4 \phi$, in which $\phi$ is latitude, would pick up a detail like this, since the functions $\sin^2 \phi$, $\sin^4 \phi$ weight higher latitude data much more heavily. It would be quite useful if that data could be reanalyzed in equatorial regions at least, or using spherical harmonics.
Howard and Harvey were not able to detect any systematic meridian circulation. This tends to favor our solutions with $T$ as large as $3 \times 10^4$ or larger, for which the mean poleward motion is no more than 15 m/sec in the Rayleigh number range we studied, which is below their resolution limits.

c) Meridional momentum transports

Ward (1965) has inferred from sunspot motion statistics that there is a systematic equatorward transport of momentum $(u_1 v_1)$ by large scale eddy motions. In our model equatorward transport by the convective cells is also a very persistent feature and is responsible for what equatorial acceleration we do get. If we compare our computed momentum transports near the top with those of Ward (1965) for the case $T = 3 \times 10^4$ (Figure 11), we see that the magnitude is roughly correct. Ward's values peak at a higher latitude, but this could easily be due to the fact that our model contains side walls at roughly 40°N and S latitude. For $T < 3 \times 10^4$, we have found the equatorward transport in the model to be much larger than given by Ward's data (for example, with $T = 10^3$, $R = 2.0R_c$, they are ten times too big) while for $T > 3 \times 10^4$, the values become very weak by comparison (as does the calculated differential rotation). Thus the momentum transport statistics tend also to favor solutions in the neighborhood of $T = 3 \times 10^4$.

d) R.M.S. cell velocities

As seen earlier, the cell amplitudes decrease relative to the differential rotation as $T$ increases. Howard (1971) reports he sees mostly east-west motions but no north-south velocities with an upper limit of 30 m/sec line of sights. We typically find that our north-south and east-west velocities are comparable. For $T = 10^3$, $R = 2.4R_c$, we get rms velocity in $v$ of about 200 m/sec near 20° latitude, which Howard should have found if it were there. On the other hand, by $T = 3 \times 10^4$
this rms velocity is down to 80 m/sec, which for low latitudes is near his resolution limit. At high T, these values are still lower, but then the model produces too small a differential rotation.

e) Other comparisons

We note in addition that with T as large as $3 \times 10^4$, modes symmetric about the equator are rather strongly favored over the antisymmetric modes, which then favors a differential rotation symmetric about the equator. Howard and Harvey's (1970) observations indicate the observed differential rotation is predominantly, but not entirely, symmetric. Finally, we note that, since we approach the correct magnitude of differential rotation at $T = 3 \times 10^4$ with $R$ only a few times critical, we are probably not far from the correct effective Rayleigh number at this $T$ for large scale convection. Presumably, much larger $R$ for which we would need more modes and resolution at this $T$ would give too large a differential rotation. Larger $T$ would require larger $R$ to bring the differential rotation up to the right magnitude.

f) The heat transport problem

While the velocity field and structure comparisons are encouraging, there is a serious problem with the heat transports. That is, we persistently find for all the Taylor numbers studied the heat flux is a maximum at the equator and may be twice as large or more than at higher latitudes, which is certainly not observed at the solar surface. (Because our model is Boussinesq it is very hard to compare the absolute magnitude of the heat flux it produces with the real sun). Our hope is that with more degrees of freedom, we will excite sufficient amplitude in modes with other $k$ values that give more transport at higher latitudes while still giving us an equatorial acceleration. This is
certainly not guaranteed, however. It is known that shear flow without rotation tends to favor convection with patterns elongated in the direction of the mean flow (e.g., Davies-Jones, 1971), and references cited therein), and these should give more heat transport at high latitudes than low. In addition, at higher $R > R_c$ and with more degrees of freedom we may get turbulent convection. This, together with more sophisticated representation of the small scale turbulence, may help reduce the temperature and heat flux difference. We could have, of course, imposed constant heat flux difference. We could have, of course, imposed constant heat flux rather than constant temperature boundary conditions, but this would simply have led to large surface temperature differences, which are also not observed.

8. Comparisons with Other Models

Differential rotation models by Kippenhahn (1963), Cocke (1967), and Köhler (1971) rely on anisotropic eddy viscosity to produce differential rotation. The momentum transports by Reynolds stresses of all scales of convective motions are therefore represented only in a parametric way. In the present model, on the other hand, we instead calculate explicitly the Reynolds stresses for the largest scale of convection, leaving the smaller scales parameterized. The Kippenhahn-Cocke-Köhler model sidesteps all problems of heat transports because it contains no thermodynamics. But surely the difficulty is there. The model assumes the anisotropy is produced by the presence of a preferred direction, determined by gravity. But in the rotating case the large spacial scale long time scale convection feels two directions determined by gravity and rotation, whose difference is a function of latitude. Consequently the resulting eddy viscosity should be more complicated, as should the "eddy thermal diffusivity", giving the
possibility of latitude dependent heat flux. Only if we postulate that all solar convective elements are sufficiently small scale and short lived that rotation is unimportant will only the one direction be felt.

A very persistent difference between our results and those of Kippenhahn, Cocke and Köhler and also Osaki (1970), Durney and Roxburgh (1971) is that our meridian circulation near the surface is toward the poles rather than toward the equator. This circulation might therefore contribute toward the transports of magnetic flux poleward, and together with the large scale cell motions compete with the Leighton (1964, 1969) random walk transport by the smaller scale convection. With regard to the Leighton model for the solar cycle, we should point out also that in no case do we find differential rotation with radial shear much stronger than latitudinal shear. The rotation generally increases outwards, rather than inwards, as Leighton requires. The increase outwards is required if there is to be an equatorial acceleration and the Taylor-Proudman theorem is to hold in the interior of the convecting region.

The models of Busse (1970a) and Durney (1970, 1971) for differential rotation induced by giant scale convection are for small Taylor number and do not allow for a reaction of the induced differential rotation back on the convection. Our calculations show that the small T nonlinear solutions are deficient in several respects in that we get large amplitude probably permanent oscillations involving large equatorial decelerations, as well as too much energy in the meridian circulations and in the convective cells compared to the differential rotation.
9. Concluding Remarks

Clearly, we need to move toward a model with more degrees of freedom, and a more sophisticated parameterization of the small scales of motion we cannot explicitly calculate. This will probably involve a non-linear form of the turbulent viscosity, such as that discussed by Lilly (1967) and others. We should also look at shallower convecting layers that would apply more directly to the nearly Boussinesq bottom levels of the convection zone and we should look at a spherical model. Presuming that such calculations yield more reasonable heat transport near the surface that is not a strong function of latitude, we then should add magnetic fields to the model to assess the dynamo action of the motions. The motions generated in the current model do appear to have the right ingredients of swirl and rising motion (helicity) as well as differential rotation. Such a model could supplement or provide an alternative to the random walk model of Leighton (1964, 1969) or the "\(\alpha\) effect" dynamos of Krause, Rädler and Steinbeck (see Weiss 1971, Roberts and Stix, 1971). In our opinion, it will be considerably harder to do a full compressible convection model. Much work should be done first on the compressible convection problem without the complications of rotation, spherical geometry, magnetic fields, etc.

The heat transport problem we have encountered in this study points to a rather fundamental dilemma in solar and, indeed, stellar physics: if the solar convection zone is as deep as 20% of the radius, by any reasonable argument, such as Simon and Weiss (1968), we must expect very large scale convection, with a long circulation time, to exist in the convection zone, since there is nothing to prevent it, and since it would.
be more efficient than smaller scales in converting potential to kinetic energy. But the long circulation time (say 1 month or more) allows coriolis forces to significantly influence the heat transport, in different amounts at different latitudes. However, no large differences in heat flux or temperature are seen at the surface. It is difficult to see how the smaller scale granules and supergranules near the surface could smooth this out without showing, for example, a systematic variation in their scale or intensity with latitude, which is apparently also not observed. Is it possible that the giant convection cells and their induced meridian circulation and differential rotation on the sun are organized in just such a way as to minimize the differences in heat flux? If so, why? As another possibility, could the effective Prandtl number for the large scale motions be in fact much less than unity? If so, for a given Rayleigh number, the effective Nusselt number would be closer to unity, so that the percentage difference in heat flux with latitude would be less. In other words, the smaller the Prandtl number, the more the heat transport would be by the small scale parameterized convection, and the less by the large scale, explicitly calculated convection.
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References


Figure Legends

Figure 1: Schematic drawing of the equatorial annulus (shaded cross-section).

Figure 2: Critical Rayleigh number $R_c$ at which convection sets in as a function of Taylor number $T$, for symmetric and antisymmetric modes in the equatorial annulus, and for the case of rotation parallel to gravity.

Figure 3: Critical wavenumber $k_c$ at the onset of convection.

Figure 4: Horizontal velocity vectors in linear convective modes at $z$ levels 3, 7, 10 (near bottom, middle, top) for increasing Taylor number (computer drawn figures).

Figure 5: Induced mean differential rotation and meridian circulation in the northern hemisphere (computer drawn) for $R = 1.2 R_c$, $T = 10^3, 10^4, 10^5, 10^6$. Equator is at the left-hand edge of each section. Differential rotation is symmetric about the equator, meridian circulation a mirror reflection. To obtain dimensionless velocity amplitudes for differential rotation, multiply by $10^{-3}$ for $T = 10^3, 10^4$, and by $10^{-2}$ for $T = 10^5, 10^6$.

Figure 6: Kinetic energy conversion rates for maintenance of the differential rotation, normalized with respect to the dissipation of kinetic energy, for $T = 10^3$ through $10^6$ with $R = 1.2 R_c$. Labelling of curves defined in the text.

Figure 7: Ratio of energy $U_2^2$ of the differential rotation to energy $U_1^2$ of the same component of motion in the cells, as a function of $R/R_c$ for various $T$ values.
Figure 8: Radio of energy $V^2 + W^2$ of meridian circulation to the energy $U_0^2$ of differential rotation, as a function of $T$ for various $R$ values.

Figure 9: Plots of cell vertical velocities, horizontal velocities, and total horizontal velocities near the top (z level 10) for increasing $R$ at $T = 3 \times 10^4$, which show distortion of cells by induced differential rotation. For correct magnitudes of vertical velocities, multiply by $10^{-2}$.

Figure 10: Dimensional differential rotation near the top (z level 10) produced from the model for several Rayleigh numbers at Taylor number, $T = 3 \times 10^4$, compared to the grand average differential rotation profile from Howard and Harvey (1970) plotted in the same rotating frame. Scale at top is latitude, at bottom, $y$ of model.

Figure 11: Dimensional meridional momentum transport $(u\,v)$ for $T = 3 \times 10^4$ compared with Wards sunspot data.
Figure 4

CELL HORIZONTAL VELOCITIES

T = 10²   T = 10³   T = 10⁴   T = 10⁵
Figure 5

DIFFERENTIAL ROTATION  \( R = 1.2 R_c \)  MERIDIAN CIRCULATION
Figure 6

\[ \frac{U_0^2}{U_i^2} \] vs. \( R/R_c \)

- \( T = 10^6 \)
- \( T = 10^5 \)
- \( T = 3 \times 10^4 \)
- \( T = 10^4 \)
- \( T = 10^3 \)
Figure 7

\( \frac{V_0^2 + W_0^2}{U_0^2} \)
Figure 8

Normalized energy conversion rates vs. T

- HCF
- MMT
- VMT
- HCF + VCF
- MMT + VMT

T - 10^3, 10^4, 10^5, 10^6
Figure 9
Figure 10

DIFFERENTIAL ROTATION $U_0$ (m/sec)

$T = 3 \times 10^4$

Howard + Harvey

$4.4R_c$

$2.8R_c$

$2.0R_c$

$1.2R_c$
Figure 11