A Non-staggered Block Jacobi Preconditioning Strategy in HOMME: Progress Report

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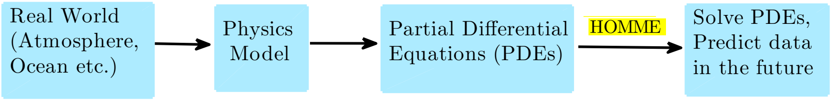
HOMME: High Order Method Modeling Environment



PDEs solved in **HOMME**:

- **Primitive Equations**
  Based on a more accurate model, more difficult to solve
- **Shallow Water Equations**
  Simplification of primitive equations, widely used for numerical test

My Project's Goal:
To **better** solve shallow water equations on **cubed sphere**

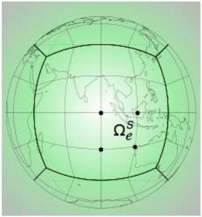
Shallow Water Equations (SWEQ)

$$
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + (f+\zeta)\mathbf{k}\times\mathbf{v} + \frac{1}{2}\bigtriangledown(\mathbf{v}\cdot\mathbf{v}) + \bigtriangledown\Phi &= 0,\\
\frac{\partial \Phi}{\partial t} + (\mathbf{v}\cdot\bigtriangledown)\Phi + (\Phi_0+\Phi)\bigtriangledown\cdot\mathbf{v} &= 0.
\end{aligned}
$$

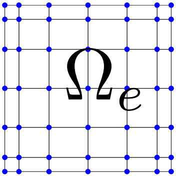
Two variables are to be solved:

- Velocity(2D): $\mathbf{v}(t,x,y)$
- Geopotential: $\phi(t,x,y)$

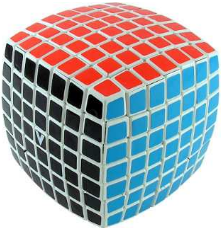
Computational Domain: cubed sphere



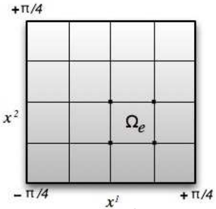
Sphere



Grids on One Element



Cubed Shpere



Elements on One Face

Discretization of shallow water equations:

Functions are continuous, but when you ask a computer to solve it, you have to discretize it!

Discretized Form of SWEQ

$$\begin{aligned} B_i \delta u - g^{ij} \Delta t \tilde{D}_j^T \delta\phi &= R_u^i \\ \tilde{B}\delta\phi + \Delta t \frac{\phi_0}{g} \tilde{D}_i g \delta u &= R_\phi, \end{aligned}$$

Variables to solve now: δu, $\delta\phi$.

Questions:

- What are B_i, $\tilde{B}$, g^{ij}, $\tilde{D}_i$, $\tilde{D}^T$, R_u^i, R_ϕ?
- Why δu, $\delta\phi$ instead of u, ϕ

Answer: They have something to do with how we discretize the equation.

Discretized form of SWEQ

$$B_i \delta u - g^{ij} \Delta t \tilde{D}_j^T \delta\phi = R_u^i \quad (1)$$

$$\tilde{B}\delta\phi + \Delta t \frac{\phi_0}{g} \tilde{D}_i g \delta u = R_\phi, \quad (2)$$

ϕ, **u**: functions of (t,x,y), on the cubed sphere.

- Time stepping:
  Semi-implicit Method: δu, $\delta\phi$, Δt
- Spacial discretization:
  Spectral Element Method: B_i, $\tilde{B}$, $\tilde{D}_i$, $\tilde{D}^T$, R_u^i, R_ϕ
- Coordinates transformation:
  Conformal Transformation: g^{ij}, g

Two equations, how to solve?

Compute δu^i from (1) explicitly using previous data of ϕ:

$$\delta u^i = B_i^{-1}(R_u^i + \Delta t g^{ij} \tilde{D}_j^T \delta\phi)$$

Substitute it back to (2):

The linear system we really need to solve

$$\underbrace{(g\tilde{B} + \Delta t^2 \phi_0 \tilde{D}_i g B_i^{-1} g^{ij} \tilde{D}_j^T)}_{H} \cdot \delta\phi = R_\phi,$$

H: Helmholtz operator, a symmetric positive definite (SPD) matrix

The problem now:

$$H \cdot \delta\phi = R_\phi$$

The same as to solve:

$$A \cdot x = b$$

A is huge and sparse, how to solve?

Iterative method: $x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_n \longrightarrow \cdots$
such that, x_n is closer and closer to the true solution x^*

In HOMME, the iterative method is: **conjugate gradient (CG)**

Back to **My Project's Goal:**
—"To **better** solve shallow water equation on **cubed sphere**"

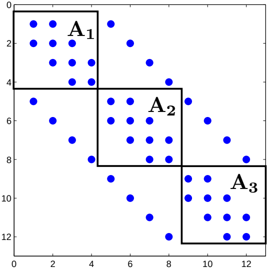
- **Basics:**
  Matrices from discretization are almost always "**bad**"
- **To make it "better":**

$$A \cdot x = b \;\Rightarrow P \cdot \; A \cdot x = P \cdot b, \text{ solve the latter instead}$$

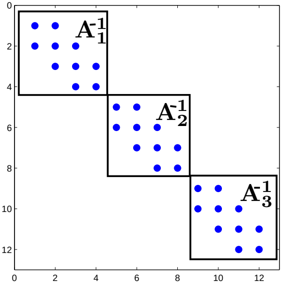
Same problem, "nicer" matrix PA, iteration steps will be largely reduced.
P is called **preconditioner** (usually an approximation of A^{-1})

The Block-Jacobi Preconditioner:

A may look like:



Its block-Jacobi preconditioner *P*:



Construct the preconditioner matrix $P = (p_{ij})$

By **spectral element method**, P's entry

$$p_{ij} = \int_{\Omega} \phi_i \phi_j = \sum_k \int_{\Omega_k} \phi_i \phi_j,$$

- $\mathbf{\Omega}$: the whole cubed sphere;
- $\mathbf{\Omega_k}$: one element;
- $\phi_\mathbf{i}$: basis function, piecewise **polynomial**, constructed by interpolating on nodes *s.t.*

$$\phi_i = \begin{cases} 1 & \text{at Node i,} \\ 0 & \text{other nodes} \end{cases}$$

Staggered Grid *v.s* Non-staggered Grid on One Element

On One Element Ω_k:

Staggered Grid:

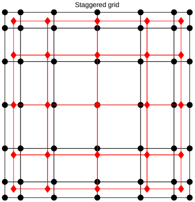


Figure: $np = nu - 2 = 4$

♦: *geopotential* ϕ, •: *velocity* $\mathbf{u}$

Non-staggered Grid:

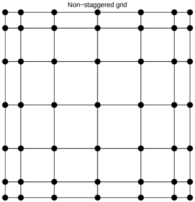
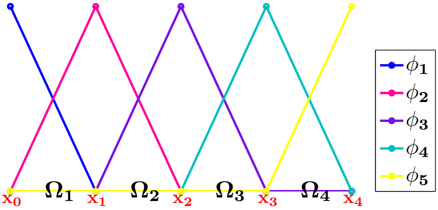


Figure: $np = nu = 6$

•: both *geopotential* ϕ and *velocity* $\mathbf{u}$



- **Staggered grid:**

$$p_{ij} = \sum_k \int_{\Omega_k} \phi_i \phi_j = \int_{\Omega_k} \phi_i \phi_j \quad \text{, for all Node i,j} \in \Omega_k$$

- **Non-staggered grid:**

$$p_{ij} = \sum_k \int_{\Omega_k} \phi_i \phi_j = \begin{cases} \int_{\Omega_k} \phi_i \phi_j, & \text{either Node i or j is inner node} \\ \int_{\Omega_k} \phi_i \phi_j + \int_{\Omega_{k+1}} \phi_i \phi_j + \cdots, & \text{Node i,j at edge or corner} \end{cases}$$

The Old Block-Jacobi preconditioner in HOMME:
Won't reduce the iteration number for Non-staggered Grids

Time_Step(*sec.*)	500	600	700
None	49	83	-
Block Jacobi	49	83	-

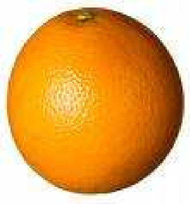
Table: CG Iteration Numbers with No Preconditioner or Block Jacobi in SWTC5

Note: In SWTC5, the whole simulation time is 15 days=1,296,000 sec., Time Step = 600 s. is pretty small

Chanllenges in Implementing Non-staggered Block Jacobi Preconditioner

Tough Part: Nodes on the Edge or Corner of a Face:

When construct P, still $p_{ij} = \int_{\Omega_k} \phi_i \phi_j + \int_{\Omega_{k+1}} \phi_i \phi_j + \cdots$?
– if so, you are adding apples with oranges.

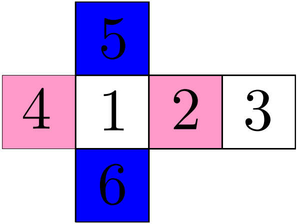
 $+$  $=$ **?**

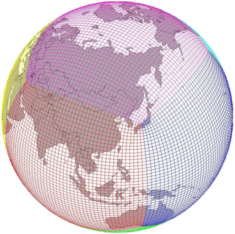
Why? —Coordinates (x, y) on different faces are NOT the same!

We know $\mathbf{v} = (\lambda, \theta) = (x, y)$, however,

- **Face 1:** $x = \tan\lambda$, $y = \frac{\tan\theta}{\cos\lambda}$
- **Face 5:** $x = \tan\theta\sin\lambda$, $y = -\tan\theta\cos\lambda$

So, (x, y) on Face 1 are different from on Face 5, thus the integration on them **scales differently**. Adding them together directly is just like adding apples with oranges





Problem shooting and results prediction

To solve the apple/orange problem:

- *Coordinates transformation:* Face A ⇒ Sphere ⇒ Face B
- *Edge rotation for vectors*

Results prediction after applying block-Jacobi preconditioner:
CG iteration numbers will be largely reduced.

Conclusion and Future Work

Overview

- Block Jacobi preconditioner for non-staggered grid entails communications with neighbors of each element;
- Extra efforts is needed due to coordinates transformation from the sphere to the cube.

Future Work

- Implement at least other two developed and tested preconditioners in HOMME, *i.e.*
  - 2-level additive Schwarz preconditioner with fast diagonalization method(FDM) as direct solver[7]
  - Non-overlapping optimized additive Schwarz preconditioner[5],[6]
- Test the performance using different preconditioners

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- [5] A. St-Cyr, M.J.Gander and S.J.Thomas, Optimized Multiplicative, Additive, and Restricted Additive Schwarz Preconditioning, *SIAM J. Sci. Comput. Vol. 29, No.6*
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- [7]S.J.Thomas *et al.*, A Schwarz Preconditioner for the Cubed-Sphere, *SIAM J. Sci. Comput. Vol 25, No.2*

Thank you ! & Questions?

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