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A Comparison of Shape Preserving Interpolators

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CONTENTS

Preface

A large number of interpolation schemes are evaluated in terms of their relative accuracy. The large number of schemes arises by considering combinations of interpolating forms (piecewise cubic polynomials, piecewise rational quadratic and cubic polynomials and piecewise quadratic Bernstein polynomials), derivative estimates (Akima, Hyman, arithmetic, geometric and harmonic means, and Fritsch-Butland), and modification of these estimates required to insure monotonicity and/or convexity upon the interpolant. *Shape preserving* methods maintain in the interpolant the monotonicity and/or convexity implied in the discrete data.

The schemes are compared by evaluating their ability to interpolate evenly spaced data drawn from three test shapes (Gaussian, cosine bell, and triangle) at two resolutions. Of the monotonic interpolants, the following are the most accurate: 1) The Hermite cubic interpolant with the derivative estimate of Hyman modified to produce monotonicity as suggested by de Boor and Swartz. 2) The second version of the rational cubic spline suggested by Delbourgo and Gregory, with the derivative estimate of Hyman modified to produce monotonicity. 3) The piecewise quadratic Bernstein polynomials suggested by McAllistor and Roulier with the derivative estimate of Hyman again modified. Imposing strict monotonicity at discrete extrema introduces significant errors. More accurate interpolations result if this requirement is relaxed at extrema. The Hermite cubic interpolant is improved by relaxing the strict monotonicity constraint to one suggested by Hyman at extrema. In a like manner, the accuracy of the rational and piecewise quadratic Bernstein polynomial interpolants can be improved by requiring only that convexity/concavity be satisfied rather than monotonicity.

1 Introduction

Shape preserving interpolation denotes a class of methods which maintain any monotonicity, and/or convexity suggested by data in the interpolant. These shape preserving properties provide a means of avoiding the oscillations often seen in polynomial interpolation. Many methods ([2], [9], [10], [11], [7], [3], [4], [8], [6], [5]) have been introduced in the past few years with shape preserving properties. They have usually been evaluated in terms of their "visually pleasing" nature, or via the error terms of the associated Taylor series. While these quantities are of great importance, many problems require the interpolation of data which are *not* strictly monotonic, or convex. Often the underlying form of the data has discontinuities in its derivatives, and the Taylor series error estimates are of limited utility. Thus, it is desirable to evaluate the accuracy of the interpolant in a more general context. In this report we objectively compare shape preserving interpolants with each other and with non-shape preserving forms in order to provide a sound basis for choosing one for a particular application.

This report provides a survey of a number of the currently accepted methods of shape preserving interpolation that have appeared in the literature. We present a review of the methods, and then compare the schemes qualitatively and quantitatively.

In the next section (2) a notation is established, and the various shape preserving interpolation schemes are reviewed. The interpolation schemes are compared in section 3 in terms of their accuracy in representing a variety of shapes which differ in their degree of continuity, monotonicity, and convexity. A summary of the interpolation comparison appears in section 4.

2 The Interpolation Problem

We begin by defining the grid $\{x_i\}_{i=1}^n$, $x_1 < x_2 < \cdots < x_n$, and the data values $\{f_i\}$, $f_i = f(x_i)$. It is also convenient to define the *discrete slopes*

$$\Delta_i = (f_{i+1} - f_i)/(x_{i+1} - x_i) \; . \tag{1}$$

The data are locally monotonic on the double grid interval $[x_{i-1}, x_{i+1}]$ if

$$\Delta_{i-1}\Delta_i > 0 \; , \tag{2}$$

and locally convex if

$$\Delta_{i-1} > \Delta_i \; . \tag{3}$$

For concave data, the previous inequality (3) is reversed. We note that with these definitions, some data may be interpreted as concave and convex on a single grid interval. We deal with this special case in section 3.2. We define the piecewise interpolant $p \in \mathcal{C}^K\,[x_1, x_n]$, with $K \geq 0$. On each subinterval $[x_i, x_{i+1}]$, let

$$\theta = (x - x_i)/h_i \qquad h_i = x_{i+1} - x_i \tag{4}$$

and

$$p(x) = p_i(\theta) \; . \tag{5}$$

The interpolant p has the following properties

$$p(x_i) = f_i, \qquad dp(x_i)/dx = d_i \; . \tag{6}$$

Here, d_i is some estimate of the derivative of f at the endpoints of the subinterval. The interpolant is specified on the subinterval in terms of the data f_i, and the derivative estimates d_i at the endpoints of the subinterval, that is

$$p_i(\theta) = p_i(\theta, f_i, f_{i+1}, d_i, d_{i+1}) \tag{7}$$

The interpolant thus adheres to the standard osculatory representation, although the functional form of p is not necessarily the usual Hermite form of the cubic polynomial. For the intercomparison, only interpolating forms which involve use of *local* information are included, *i.e.*, d_i is a function of a few surrounding values of f_i. In this fashion we have excluded from consideration many global schemes; for example, the classic C^2 cubic splines which minimize the integral of the curvature of the interpolant over the entire domain, exponential splines under tension [12], and global versions of the monotone, piecewise interpolants of [6] and [3]. These schemes require information over the entire domain. We chose to evaluate local methods because our major final goal was a local transport scheme. Local methods are also desirable because adding, changing or removing data in the domain will only change the interpolant in the vicinity of the change of data. Following this restriction, schemes which differ from each other in three major ways are considered:

- The method of estimating the derivative is varied according to algorithms that have appeared in the shape preserving literature.
- The type of interpolating function is varied to encompass cubic polynomials, rational functions, and quadratic Bernstein polynomials with extra knots.
- To guarantee monotonicity or concavity/convexity in the interpolating function, certain constraints are imposed on the derivative estimates. The appropriate constraint depends upon the interpolation form.

It is convenient to address these items in reverse order in the following subsections.

2.1 Constraints on the derivatives

Certain constraints must be imposed on the derivative estimates used in the interpolation schemes in order for the interpolants to maintain the properties of convexity/concavity or monotonicity present in the data. The constraints are reviewed in this section, proceeding from the least to the most restrictive form. The constraints can be written in terms of restrictions on the derivative estimates d at the endpoints of an interval, as a function of the discrete slope Δ within the interval. Because of this, the constraint on d_i based on Δ_{i-1} of the interval to the left may be different from that based on Δ_i of the interval to the right. One may choose to use a different derivative estimate at a point for interpolation over two adjacent intervals by constraining the estimate differently, in which case the interpolant is C^0, or insist that constraints associated with both intervals be satisfied simultaneously, in which case the same derivative estimate is used for the adjacent intervals, and the interpolant is C^1. When the constraint on d_i depends not only on the discrete slopes over the adjacent intervals Δ_{i-1} and Δ_i, but also the derivative estimate d_{i-1} or d_{i+1} at the other ends of the intervals, the C^1 interpolants become global. Such forms are not considered in this report.

The requirement that the continuous derivative estimates bound the discrete slope for a C^0 interpolant

$$(d_i - \Delta_i)(\Delta_i - d_{i+1}) > 0 \qquad \text{(NCC0)} \tag{8}$$

and lie between the adjacent discrete slopes for a C^1 interpolant

$$(d_i - \Delta_{i-1})(\Delta_i - d_i) > 0 \; . \qquad \text{(NCC1)} \tag{9}$$

must be true if the interpolant is to be convex/concave in the intervals $[x_i, x_{i+1}]$ and $[x_{i-1}, x_{i+1}]$, respectively. These requirements are identified as *Necessary Condition(s) for Convexity/Concavity*, C^0 and C^1 respectively.

In order that the interpolating function be monotonic on the interval $[x_i, x_{i+1}]$ and C^0, the derivatives must satisfy the *Necessary Condition for Monotonicity* C^0

$$\begin{array}{ll} \mathrm{sign}(d_i) = \mathrm{sign}(\Delta_i) = \mathrm{sign}(d_{i+1}) & \Delta_i \neq 0 \\ d_i = d_{i+1} = 0 & \Delta_i = 0 \end{array} \qquad \text{(NCM0)} \tag{10}$$

that is, the derivative estimate at the endpoints must have the same sign as the discrete slope on the interval. For a C^1 interpolant on the interval $[x_{i-1}, x_{i+1}]$

$$\begin{array}{ll} \mathrm{sign}(\Delta_{i-1}) = \mathrm{sign}(d_i) = \mathrm{sign}(\Delta_i) & \Delta_{i-1}\Delta_i > 0 \\ d_i = 0 & \Delta_{i-1}\Delta_i \leq 0. \end{array} \qquad \text{(NCM1)} \tag{11}$$

The derivative estimate at the datapoint must have the same sign as the discrete slopes surrounding it or be zero if the descrete datum is an extremum at this point. This condition is the *Necessary Condition for Monotonicity* C^1 (NCM1)

For the rational and piecewise quadratic Bernstein polynomial interpolation forms discussed below, the necessary conditions NCM0 and NCM1 are also sufficient conditions for monotonicity. Similarly, the NCC0 and NCC1 are sufficient conditions for convexity with these interpolants. On the other hand, for Hermite cubic interpolants NCM0 and NCM1 are necessary but not sufficient for monotonicity and must be augmented by additional constraints on the derivatives.

Fritsch and Carlson [6] have found both necessary and sufficient conditions for monotonicity of Hermite cubic interpolants. Let $\alpha = d_i/\Delta_i$, $\beta = d_{i+1}/\Delta_i$; then if $\Delta \neq 0$ the cubic interpolant will be monotonic if and only if (α, β) lies within the domain $\mathcal{M}_{ns}$ defined by the union of two domains

$$\mathcal{M}_{ns} = \mathcal{M}_e \cup \mathcal{M}_b \tag{12}$$

where

$$\mathcal{M}_e(\alpha, \beta) = \{\alpha, \beta : \phi(\alpha, \beta) \leq 0\}, \tag{13}$$

$$\mathcal{M}_b(\alpha,\beta) = \{\alpha,\beta : 0 \le \alpha \le 3, 0 \le \beta \le 3\} \tag{14}$$

and

$$\phi(\alpha,\beta) = (\alpha-1)^2 + (\alpha-1)(\beta-1) + (\beta-1)^2 - 3(\alpha+\beta-2)\ . \tag{15}$$

If $\Delta_i = 0$, then $d_i = d_{i+1} = 0$ and the necessary condition discussed earlier is also sufficient. Embedded in this domain $\mathcal{M}_{ns}$ is the region $\mathcal{M}_b$ recognized independently by de Boor and Swartz [2] which provides a sufficient condition for monotonicity for the Hermite cubic interpolant. This sufficient condition

$$0 \le \alpha \le 3, 0 \le \beta \le 3 \qquad \text{(SCM)} \tag{16}$$

is easier to apply than the more general necessary and sufficient condition ($\mathcal{M}_{ns}$) in which α and β may be dependent on each other. Throughout the remainder of this article, this simpler condition will be referred to as the *Sufficient Condition for Monotonicity* (SCM). As before, we define C^0 and C^1 forms depending on whether the derivatives d_i are bounded by just the Δ of the interval being interpolated or by the Δ of the two adjacent intervals simultaneously.

Constraints on the derivative estimates are applied in the following fashion. The NCM constraints are imposed according to

$$d_i \leftarrow \begin{cases} d_i & d_i\Delta_j > 0 \\ 0 & d_i\Delta_j \le 0 \end{cases} \tag{17}$$

where $i = j, j+1$ for NCM0 interpolation on the interval $[x_j, x_{j+1}]$, and $j = i-1, i$ for NCM1 interpolation on the double interval $[x_{i-1}, x_{i+1}]$. Similarly, the SCM constraints use

$$d_i \leftarrow \mathrm{SIGN}(d_i)\min(|d_i|, |3\Delta_j|) \tag{18}$$

where $i = j, j+1$ for SCM0, and $j = i-1, i$ for SCM1, and are applied following the corresponding NCM constraint. Finally, we use the following algorithm to apply the NCC1 constraint.

$$d_i \leftarrow \begin{cases} d_i & (d_i - \Delta_{i-1})(\Delta_i - d_i) \geq 0 \\ d_{lim} & (d_i - \Delta_{i-1})(\Delta_i - d_i) < 0 \end{cases} \tag{19}$$

where

$$d_{lim} = \begin{cases} min(\Delta_{i-1}, \Delta_i) & d_i < \Delta_{i-1} \\ max(\Delta_{i-1}, \Delta_i) & d_i \geq \Delta_{i-1} \end{cases} \tag{20}$$

At an extremum where the data are not monotonic over the surrounding double interval, NCM1 limiting provides a severe restriction, as d_i is constrained to be zero there. The interpolant must put the extremum at the data point. For Hermit cubic interpolants Hyman [8] has relaxed the SCM1 limiting concept where the data reach a local extremum, and are not monotonic, in an attempt to mimic a convexity constraint. He proposed the following limit on the derivatives.

$$d_i \leftarrow \mathrm{SIGN}(d_i)\min(|d_i|,\ |3\Delta_{i-1}|,\ |3\Delta_i|) \tag{21}$$

This allows for overshoot on the interval next to local discrete extrema and thus is nonmonotonic, but does provide some control of the overshoot and, in particular, prevents oscillations at the edge of flat plateaus.

2.2 Interpolation forms

Three types of interpolating functions are considered — all have appeared in the recent literature regarding shape preserving interpolation;

- cubic polynomials ([2], [6], [8], [5])
- quadratic Bernstein polynomials with extra knots ([9], [10], [11])
- rational functions ([7], [3], [4])

The Hermite cubic and rational interpolating functions can be described using the formalism of Delbourgo and Gregory [3] . Consider the function

$$p_i = P_i(\theta)/Q_i(\theta) \tag{22}$$

on the interval $0 \leq \theta \leq 1$, equivalently $x_i \leq x \leq x_{i+1}$, where

$$P_i(\theta) = f_{i+1}\theta^3 + (r_i f_{i+1} - h_i d_{i+1})\theta^2(1-\theta) + (r_i f_i + h_i d_i)\theta(1-\theta)^2 + f_i(1-\theta)^3 \tag{23}$$

and

$$Q_i(\theta) = 1 + (r_i - 3)\theta(1-\theta) \tag{24}$$

We consider four choices of the parameter r_i.

- If $r_i = 3$, p_i reduces to the standard *Hermite cubic* polynomial interpolation form. Recall that the interpolant will be monotonic if the d_i lie within the domain M_b.

- If $r_i = 1 + (d_i + d_{i+1})/\Delta_i$, then P_i and Q_i reduce to quadratic polynomials, and p_i is identified as a rational quadratic interpolant. Delbourgo and Gregory [3] have shown that provided d_i and d_{i+1} satisfy the NCM0, p_i will be monotonic over the subinterval, otherwise this interpolant is not well defined. If NCM0 (10) is not satisfied we modify the derivatives to satisfy it in order to apply the rational quadratic via (17).

- If $r_i = 1 + \max(C_i/c_i, C_i/c_{i+1})$, where, $c_i = \Delta_i - d_i$, $c_{i+1} = d_{i+1} - \Delta_i$, $C_i = d_{i+1} - d_i$ then P_i is a cubic polynomial, and p_i is identified as the rational cubic interpolant version 1.

- If $r_i = 1 + c_{i+1}/c_i + c_i/c_{i+1}$, then P_i is again a cubic polynomial and p_i is identified as the rational cubic interpolant version 2. Delbourgo and Gregory [3] have shown that if the derivatives satisfy the convexity/concavity constraints NCC0 or NCC1 then both rational cubic versions will be convex/concave. If, in addition, the derivatives satisfy the monotonicity constraints NCM0 or NCM1 then both versions will be monotonic. Delbourgo and Gregory [3] have also shown that version 2 is in general more accurate than version 1. The derivative estimates must satisfy NCC0 for the two versions of the rational cubic interpolant to be well defined.

The quadratic Bernstein polynomials with extra knots cannot be described using the previous formalism. This interpolant is constructed by piecing together two quadratic Bernstein polynomials within each interval, with the point of intersection (the extra knot) determined by a rather complex algorithm which cannot be succinctly described with a few equations or figures. Because of this, the reader should refer to the descriptions found in the series of original articles ([9], [10], [11]). The characteristics of the Bernstein polynomials, together with the algorithms developed for constructing the knot, the value of the interpolant at the knot, and the interpolant derivative at the knot guarantee that the interpolant will be monotonic provided NCM is satisfied, and convex/concave provided NCC is satisfied.

2.3 Derivative estimation procedures

Table 1 lists the algorithms used in estimating derivatives at the nodes. Several of the algorithms suggested in the literature for shape preserving interpolation which differ for unequally spaced data reduce to a common form when the data become equally spaced. The tests which follow use only equally spaced data, and therefore common algorithms are grouped together. The table also includes an algorithm identified as Cubic, which does not usually appear as a derivative estimate. This scheme arises by computing a cubic interpolant through the four points surrounding the interval. The derivative estimates at the ends of the interval can then be written as a linear combination of the four surrounding data points.

Such a scheme results in an interpolant which is only C^0 continuous. It is included because this form of interpolation is often used in semi-Lagrangian problems. The Harmonic mean, Geometric mean and Fritsch-Butland derivative estimates automatically satisfy the NCM and NCC constraints. The others generally must be modified to satisfy them as described in section 2.1.

3 Intercomparisons of Interpolation Schemes

3.1 Test shapes and diagnostics

The accuracy of the interpolation schemes has been tested on a uniform grid using three shapes: a Gaussian, a cosine bell, and a triangle. For the cosine and triangle shapes, the nonzero portion is confined to the central half of the domain. These shapes were chosen because they have similar forms, but may be successively more difficult to approximate accurately. The Gaussian is C^∞, the cosine bell C^1 and the triangle C^0. The tests were made by embedding the shapes within the domain $[0, 2\pi]$. Tests using ten and forty intervals over the 2π domain were performed. The shapes were successively displaced 100 times, by 1/100 of the grid interval and measurements of the accuracy were made over the domain $[0, 2\pi]$. This was to establish the sensitivity of the representation to the relative position of the grid and test shape. The error varied by at least a factor of 5 over the 100 realizations. We compare the schemes using the error averaged over all realizations.

More precisely, define the data points in the evaluation domain $[0, 2\pi]$ to be $x_\ell = (\ell - 1)h$, where $h = 2\pi/N$, $\ell = 1, N+1$ and $N = 10$ or 40 for the two widths chosen. The test functions are given at these data points and the interpolation is evaluated over the set of points within $[0, 2\pi]$ given by $\tilde{x}_j = (j-1)h/13$, $j = 1$, $13N + 1$. Note, $x_\ell = \tilde{x}_{13(\ell-1)+1}$ and the data points where the interpolators fit exactly are included in the error measures. Extra data points were added outside the domain, when needed to compute the appropriate derivative estimates near 0 and 2π. The exact forms for the functions to be interpolated were

Gaussian:

$$F(x) = \exp\left[-2\left(\frac{(x - c_n)}{\delta}\right)^2\right] \tag{25}$$

Cosine Bell:

$$F(x) = \begin{cases} \frac{1}{2}\left\{1+\cos(\frac{\pi(x-c_n)}{\delta})\right\} & |x-c_n| < \delta \\ 0 & |x-c_n| \geq \delta \end{cases} \tag{26}$$

Triangle:

$$F(x) = \begin{cases} (1-\frac{|x-c_n|}{\delta}) & |x-c_n| < \delta \\ 0 & |x-c_n| \geq \delta \end{cases} \tag{27}$$

where $\delta = 10\pi/24$, specifies the width of the test shapes and $c_n = \pi - \frac{(n-1)h}{100} + \epsilon$ controls the offset of the test shape with respect to the grid. The additional small offset $\epsilon = 10^{-5}$ is included so that the maximum of the test functions never coincides with a sampled point. The error over the entire domain $[0, 2\pi]$, denoted the *total error*, is given by

$$E_T = \frac{1}{100}\sum_{n=1}^{100}\sum_{j=1}^{13N+1}\left[P(\tilde{x}_j) - F(\tilde{x}_j, c_n)\right]^2 h/13 \tag{28}$$

We also consider the error over the domain $[0, 2\pi]$ excluding the two grid intervals adjacent to the discrete maximum of the test shape. The test shapes are monotonic over this evaluation domain and we refer to this error as the *monotonic region error*

$$E_M = \frac{1}{100}\sum_{n=1}^{100}\sum_{j\epsilon\mathcal{M}}\left[P(\tilde{x}_j) - F(\tilde{x}_j, c_n)\right]^2 h/13 \tag{29}$$

where

$$\mathcal{M} = [j; j < j_L, j > j_R] \tag{30}$$

and the left and right bounds of the excluded grid points are determined from the grid intervals where the discrete slope of the test function changes sign, *i.e.*, given the data point ℓ^* at which

$$\Delta_{\ell^*-1}\Delta_{\ell^*} < 0\,, \tag{31}$$

the bounds are given by

$$j_L = 13(\ell^* - 2) + 1 \tag{32}$$

$$j_R = 13(\ell^*) + 1\,. \tag{33}$$

3.2 Description of interpolation schemes and notation

The error statistics are presented in the tables as a function of interpolation form, derivative approximation scheme, and derivative limiter. The least restrictive derivative limiter leading to a well posed interpolant is always included, even if it allows over/undershooting. Limiters are then included in the table which are expected to result in improvements in the interpolants, by reduction of the over/undershooting. The various forms of the interpolation are reviewed in the next few paragraphs and a naming convention for use in the tables is introduced.

We have included seven types of derivative estimates in the intercomparison—the Akima estimate (AKI—[1], [6], [8]), Arithmetic mean (ARI—[6], [7], [3], [8]), Geometric mean (GEO—[3]), Harmonic mean estimate (HAR—[5], [7], [11]), Fritsch-Butland estimate (BUT—[5], [8]), the derivative estimate which gives rise to the simple piecewise cubic interpolant achieved by fitting the cubic through the data nearest the point of current interest (CUB), and the Hyman estimate (HYM—[8]).

The first spline form used is the *Hermite cubic interpolant.* Monotonic forms are constructed by applying the sufficient conditions (SCM0 and SCM1). Hyman's extension to the SCM1 limiter, HYM1, which allows limited overshooting in the vicinity of extrema, is included. As mentioned before, relaxation of the monotonicity condition at the discrete extrema is in the spirit of essentially nonoscillatory schemes which allow for an extremum to form between data points, but do not allow additional extrema to form. Hyman's version can be relaxed further by applying no limiting in the vicinity of extrema, that is, the sufficient condition for monotonicity is applied only in the vicinity of monotonic data (SCM0-EE and SCM1-EE). The EE notation is used to imply 'Except at Extrema.'

As discussed in section 2, the rational interpolant forms require less stringent derivative limiters. The *rational quadratic interpolant* is properly posed only when the derivative estimates satisfy NCM0. Like the Hermite cubic interpolant, the limiters are applied to constrain the derivatives in the whole domain, and except at extrema. For the NCM1-EE case, the NCM0 constraint is applied at the extrema to keep the scheme well defined.

As McAllister and Roulier [10] have pointed out, there are inevitable complications that arise when the data and corresponding derivative estimates switch between monotonic and concave/convex states. This change in the character of the data also requires a switching in the way the interpolants are constructed. McAllister and Roulier [10] and collaborators in [9], [11] have described a complete implementation for the piecewise Bernstein polynomials. The scheme is well posed for all data, but becomes monotonic, or convex/concave only if the derivative estimate satisfies the necessary conditions for monotonicity NCM or convexity/concavity NCC, respectively. Recall that the *Rational cubic interpolants* (versions 1 and 2) are properly posed only if the derivative estimates satisfy the convexity condition NCC0. If the data do not satisfy this condition (in the vicinity of an inflection point), some switching condition must be used. We have implemented this switching within the rational cubic interpolants in the following way.

- The derivative estimates are made,
- The derivative limiter (if any) is applied,
- The data and derivative estimate are used to see if NCC0 is satisfied, if so, the rational cubic interpolant is used, if not, the rational quadratic interpolant, as described above, is used there.

The tables of the errors provide a staggering amount of information, and the discussion in the following section justifying the conclusions about the interpolation from the tables is somewhat tedious. The results from the well resolved test shapes are summarized in section 3.6 and those from the poorly resolved shapes in section 3.10. The overall results are further summarized in section 4 and the important conclusions about the interpolation intercomparisons are highlighted there.

3.3 Well resolved cosine test shape

3.3.1 *Hermite cubic (Table 2)*

The errors associated with the Hermite cubic interpolation of the cosine bell shape over a domain resolved with 40 points appear in Table 2. Each column provides an indication of the error associated with the application of a different limiting form on the derivative. Each row describes the error associated with the use of a different derivative estimation scheme. The unbracketed number is the error calculated over the whole domain, the number in brackets is the subset of the errors calculated over the monotonic portion of the domain, *i.e.*, eliminating the two grid intervals adjacent to the discrete maximum in the test data.

When no limiters are applied (first column), the Hyman derivative estimation scheme has the least error, followed by the cubic then the arithmetic mean derivative estimates. In this case, the ordering agrees with the ordering by formal accuracy of the derivative estimates. The ordering by accuracy for the rest of the estimates becomes Akima, geometric mean, Fritsch-Butland, and harmonic mean. We have entered the BUT and HAR forms in the SCM1 column since they automatically satisfy that condition. When the error over only the monotonic part of the domain is considered, the geometric derivative estimate improves greatly in the ranking, with similar changes in the other derivative estimate schemes which automatically satisfy the NCM. Comparing the two types of error (bracketed and unbracketed) for estimates which automatically satisfy NCM with those which do not suggests that more than half of the error is concentrated near the extrema for the derivative estimates satisfying the NCM, with only a few percent of the error concentrated there for the other schemes.

Application of the SCM degrades the approximations over the whole interval (second or third column compared to first), more so for C^1 continuity (third column) than for C^0 (second column). The only exception is the arithmetic derivative estimate in the monotonic region. For all but the geometric derivative estimate, this degradation occurs in the vicinity of the extrema, since the error over the interval excluding the extrema actually decreases with the application of the SCM limiters, less so for C^1 continuity than for C^0.

In general, the application of the monotonicity condition to monotonic data improves the interpolation. In the vicinity of the extrema on the other hand, the error increases, because the monotonic derivative estimates must be zero there and the extrema must occur at a data point rather than in the interior of a subinterval. In the monotonic regions C^0 continuity provides a more accurate interpolation than C^1 continuity (second column vs. third) at the expense of smoothness of the interpolant. With the application of the SCM limiters, the advantage of the Hyman over the cubic and arithmetic derivative estimates decreases, although the Hyman remains superior. With the SCM1 limiter, the cubic derivative estimate falls behind the arithmetic derivative estimate, i.e., the C^1 condition compared to the C^0 condition does more harm to the cubic than to the arithmetic derivative estimates (column 3 vs. column 2).

As mentioned earlier, the Fritsch-Butland and harmonic derivative estimates automatically satisfy SCM1 and so are placed in the third column of Table 2 with the schemes to which they are comparable. Neither is as good as any of the better derivative approximations (Hyman, cubic or arithmetic) in either the monotonic region or the vicinity of the extrema. The geometric approximation to the derivatives is also not as good as these.

The monotonicity condition associated with the SCM1 requires the derivative estimate to be zero at extrema in the discrete data. The condition will naturally introduce errors when the function has a maximum between the discrete data points. This mismatch in structure is reflected in the error table by errors over the whole interval being larger than those over the monotonic region. The last three columns in Table 2 are schemes which allow for overshooting of a non-monotonic interpolant in one of the two intervals adjacent to the discrete extrema, the particular interval being the one in which the derivative estimates at the end have opposite sign. The amount of overshoot for the schemes labelled EE is not controlled except inherently by the derivative estimates themselves. The derivative limiter suggested by Hyman permits limited overshooting but eliminates it when the data imply an approach to a flat plateau structure.

The relaxation of the strict monotonicity condition at the extrema to allow overshooting improves the interpolant there with no effect in the strictly monotonic region. For example, the SCM0-EE limiter, (Column 5, Table 2), has less error over the entire interval than the SCM limiter applied over the entire interval (Column 2, Table 2). The errors over the monotonic interval remain the same. Thus the interpolant is improved for all derivative estimates by relaxing the monotonicity condition at extrema. Imposition of Hyman's limiter at the extremum does degrade the accuracy of the interpolant slightly (Column 4 vs. Column 6) in the region of the extrema such that it falls between the limited and non-limited cases.

To summarize, for the 40 point cosine bell with a Hermite cubic interpolant the Hyman derivative approximation always produces the best interpolation, followed in descending order by cubic, and arithmetic derivative estimates. Imposition of the monotonicity condition either implicitly as in the Fritsch-Butland and harmonic mean estimate, or explicitly with the SCM limiters, improves the interpolation of monotonic data, but can degrade the interpolation near the discrete extrema by suppressing overshooting when the underlying field actually overshoots the discrete values. SCM0 does less damage than SCM1. At the risk of over/undershooting, the limiters should not be applied at the extrema. The C^1 Hyman limiter will allow over/undershooting at isolated extrema but will prevent it on approaching flat plateaus.

3.3.2 *Rational quadratic (Table 3)*

As with the Hermite cubic interpolant, the best derivative estimate is the one proposed by Hyman, followed by the cubic derivative estimate and then the arithmetic (except the C^1 monotonic form where arithmetic is insignificantly better than cubic) estimate. The C^1 form of the approximation is better than the C^0 in the monotonic interval (bracketed terms of the second column vs. the first column), but not as good in the intervals near the extremum. Relaxing the necessary condition improves the interpolant there (third column vs. second column). The Akima, Fritsch-Butland, geometric and harmonic derivative estimates are not as good as the others.

Comparing Tables 2 and 3 shows that the rational quadratic is not as accurate as the Hermite cubic, *i.e.*, comparison of columns 1 and 2 of Table 3 with columns 2 and 3 of Table 2, respectively. The rational quadratic interpolant is not in general as accurate as the monotonic forms of the cubic interpolant. The only exception is the Hyman derivative estimate with C^1 continuity in the monotonic region. The interpolant can exceed the accuracy of the Hermite interpolant for monotonic data, with an accurate derivative estimate.

3.3.3 *Rational cubic interpolation forms (Tables 4 and 5)*

Comparison of the two forms of the rational cubic in the tables entry by entry shows that the second version (Table 5) is consistently better than the first (Table 4) except for a few cases involving the Akima or Fritsch-Butland approximations. Since the Akima, Fritsch-Butland, geometric and harmonic approximations result in larger errors than the other schemes, we do not consider them further in conjunction with the rational cubic. Therefore, we consider only the second form of the rational cubic (Table 5) coupled with the Hyman, cubic and arithmetic derivative estimates.

For each derivative limiter (*i.e.*, each column) the Hyman derivative approximation continues to provide the best interpolant, followed by the cubic then arithmetic estimate. Again, they are ordered following their formal accuracy. As with the Hermite cubic, the modifications required for monotonicity degrade the solution at the extrema and improve it in the monotonic regions (columns 2 or 3 vs. column 1). Unlike the Hermite cubic, but like the rational quadratic interpolant, the rational cubic interpolant with C^1 continuity offers improvement over C^0 continuity (column 3 vs. column 2), except with the Hyman and cubic estimates measured over the entire domain. For practical purposes C^0 and C^1 produce the same average error with the Hyman estimate. When the strict monotonicity condition is relaxed at the extrema so that the rational cubic interpolant relies on its convexity properties, the error is reduced further (columns 4 and 5 vs. columns 2 and 3, respectively). It is also of interest to note that, in this case, the error over the entire domain is almost the same as that within the monotonic domain. (For example, compare bracketed with corresponding unbracketed terms in columns 4 and 5). The extrema are no longer responsible for a larger fraction of the error. Column 6, (labelled NCC1) represents implementation two of the interpolant as discussed in sections 2.2 and 3.2. We see that Implementation 1 and two result in identical accuracy for the arithmetic, and Akima derivative estimates, which automatically satisfy NCC1. Their accuracy is almost equivalent for the Cubic and Hyman derivative estimates, suggesting the particular implementation makes little difference.

3.3.4 *Bernstein quadratic (Table 6)*

Within any column, the relative standings of the derivative estimates are the same as with the rational cubic (Table 5) with the exception of a reversal of the Fritsch-Butland and geometric estimates. These two, along with the harmonic and Akima approximations, have the largest errors and will not be considered further here. The errors from the Bernstein quadratic scheme are consistently higher than the corresponding ones from the second version of the rational cubic (Table 6 compared to the corresponding entry in Table 5). The only exception is the cubic derivative estimate with no limiter, where the errors are identical to the accuracy shown in the table.

The Bernstein quadratic scheme tends to have slightly smaller errors than the Hermite cubic with the Hyman and cubic derivative approximations when no limiters are applied to the derivatives (column 1 of Table 6 vs. column 1 of Table 2) but the differences seem negligible. When the appropriate C^0 monotonic limiters are applied, the Hermite cubic tends to be the better of the two with these derivative estimates. The Bernstein quadratic interpolant tends to be better with the C^1 monotonic limiters (columns 2 and 3 of Table 6 vs. columns 2 and 3 of Table 2, respectively). In short, neither the Bernstein quadratic, or Hermite cubic forms are consistently more accurate than the other.

3.4 Well resolved Gaussian test shape

3.4.1 *Hermite cubic (Table 7)*

The relative ordering of the errors for the Hermite cubic can be compared with that for the well resolved cosine test shape in Table 2. An absolute comparison of the errors between the two tables is less relevant but possible since the shapes are not too unlike each other. When no limiter is applied to the derivatives, the ordering of the derivative estimates is the same for the Gaussian as for the cosine shape, *i.e.*, the best derivative approximation is that of Hyman, followed by the cubic followed by the arithmetic. The cubic and arithmetic derivative estimates show more separation with the Gaussian shape than they did with the cosine shape and the Hyman approximation is remarkably better in interpolating the Gaussian, probably due to the C^{∞} continuity of the shape. Application of the C^0 and C^1 SCM degrades the accuracy of the Hermite cubic interpolation with these three derivative estimates as it did with the well resolved cosine test shape, again more so for the C^1 than the C^0 continuity. When calculated over the entire domain, for all practical purposes, use of a limiter results in the same errors with any of the three better derivative estimates (*i.e.*, in columns 2 or 3 of Table 7, the unbracketed errors are insignificantly different for the arithmetic, cubic and Hyman differences). Examination of the errors calculated over the monotonic domain shows that the original relative ordering of the derivative approximations is preserved in the monotonic regions but that these errors are extremely small (by 1–3 orders of magnitude) compared with the errors near the extrema which dominate the total error. Since the monotonicity condition is the same for all three derivative approximations (*i.e.*, the derivative is zero at the extremum discrete point and possibly limited at the adjacent points), it is reasonable that they all have the same error there. The Akima, Fritsch-Butland, geometric and harmonic derivative approximations are not as accurate as the others.

3.4.2 *Rational quadratic (Table 8)*

The monotonic Hermite cubic interpolation remains slightly better than the corresponding rational quadratic interpolation (unbracketed terms in Table 8, column 1 vs. Table 7, column 2 and Table 8, column 2 vs. Table 7, column 3). For this test shape the error is dominated by the errors near the extrema. In the monotonic domain (the bracketed terms), the rational quadratic is often more accurate. The interpolation by either approach in these regions is so good that the difference becomes irrelevant. Near the extrema the error due to derivative modification to satisfy the NCM dominates the error as it did with the Hermite cubic. Unlike the cosine test shape, the NCM1 is not as accurate as NCM0 (column 2 vs. column 1), where additional error arises near the extrema. Relaxing the necessary condition at the extrema eliminates the error (column 3 equals column 1). Relaxing the monotonicity condition at the extremum shows that the additional error introduced by the C^1 continuity over that of the C^0 is all located near the extrema (column 3 compared to columns 2 and 1), as was observed above with the Hermite cubic.

3.4.3 *Rational cubic (Tables 9 and 10)*

Again, the second version is consistently better than the first with a few unimportant exceptions (*e.g.*, for the less accurate Akima and Fritsch-Butland derivative estimates). The Fritsch-Butland, geometric, harmonic and Akima derivative estimates produce larger errors than the other approximations modified to satisfy the NCM1 (Table 10, column 3). For this reason, we limit our attention here to the second version of the rational cubic with Hyman, cubic and arithmetic derivative estimates. The ordering of the more accurate derivative estimates (Hyman, then cubic, then arithmetic) is consistent with that found for the cosine shape. The NCM constraints degrade the interpolant at the extrema. In fact, for the well resolved Gaussian shape, evidently the only place the necessary condition is invoked is at the extrema since when the condition is not imposed at the extrema, the errors are the same as having no limiter (Table 10, column 5 or 6 vs. column 1). Again, the forms that rely on the convexity of the data near the extrema offer improvement over pure monotonicity, at the expense of possible overshooting. In the monotonic region, use of a convexity constraint does not always provide as accurate an interpolant as use of a monotonicity constraint. The additional error arises in the vicinity of the inflection point in the data.

3.4.4 *Bernstein quadratic (Table 11)*

In general, the relative results for the Bernstein quadratic scheme are the same as with the well resolved cosine test shape. The best derivative estimates are the Hyman followed by cubic and then arithmetic. The Bernstein quadratic scheme is consistently less accurate than the second version of the rational cubic. The unlimited and NCM0 versions are also consistently less accurate than the corresponding versions of Hermite cubic. However, it tends to be better than the Hermite cubic when C^1 monotonicity is required. The superiority of the Hermite cubic with C^0 continuity and the Bernstein quadratic with C^1 continuity is consistent with the results of interpolating the well resolved cosine test shape. The differences are not particularly large so these comparisons by themselves do not provide firm ground to choose one over the other.

3.5 Well resolved triangular test shape

3.5.1 *Hermite cubic (Table 12)*

As with the other test shapes, the Fritsch-Butland, geometric and harmonic derivative approximations have relatively large errors and we exclude them from further consideration.

When no limiter is applied, the Hyman derivative estimate is better than the cubic and arithmetic which are virtually the same. Unlike the previous test shapes, the Akima derivative estimate scores very well, having the least error. Examination of the form of the derivative estimates shows that with straight line segments, all the approximations give the same derivative when the support of the approximation does not exceed the straight line segment and the Hermite cubic interpolation collapses to the true linear straight line. Therefore, the differences between these Hermite cubic interpolations and the errors in interpolation occur only near the breaks in the line segments comprising the test shape. The Akima derivative estimate can be seen to maintain the linear structure closer to the break than the other schemes, thus it is the most accurate. Because of this property, the triangular test shape provides a less suitable basis for discriminating between combinations for more general use. Nevertheless, it is useful to include it as it provides an extreme test. When it is used to test the semi-Lagrangian advection, the successive interpolations as the feature moves across the grid destroy the straight line segments, making it a more relevant test shape.

Application of the C^0 and C^1 SCM degrades the interpolation near the apex of the triangle and improves it at the base where the shape changes to the flat region (*i.e.*, bracketed errors decrease). The C^0 condition tends to be somewhat more accurate than the C^1. The above properties hold for the Hyman, cubic and arithmetic derivative approximations and except for C^1 with the Akima. The Akima approximation remains the best with the monotonicity constraints. Relaxation of the monotonicity condition at the extrema improves the interpolation there at the expense of overshooting.

3.5.2 *Rational quadratic (Table 13)*

Again, far enough from the intersection of the straight line segments, the rational quadratic interpolation scheme also collapses to the true linear segment and thus the errors all occur near the apex and near the bottom. The Akima derivative estimate remains the most accurate, again because of the special nature of this test shape. A better indication of the significance of this is provided by the semi-Lagrangian advection tests in which the interpolation errors rapidly distort the linear segments. The C^0 monotonic Hermite cubic is consistently better than the rational quadratic. For the C^1 monotonic case, the Hermite cubic is better when the error is calculated over the entire domain, but the rational quadratic is better over the monotonic region, implying the Hermite cubic is better at the peak and not as good elsewhere.

3.5.3 *Rational cubic (Tables 14 and 15)*

The second version of the rational cubic remains consistently better than the first, except occasionally with the cubic derivative approximation. In these few cases the differences are very small, so we consider the first version no further. As with the other schemes, the rational cubic collapses to the appropriate straight line segment away from the vertices of the triangle where all the error occurs. The Akima derivative approximation again is the most accurate, followed by the Hyman approximation. The NCM0 and NCM1 monotonicity conditions degrade the interpolations near the pointed apex while improving it near the base as it approaches the flat region, except for the Akima derivative estimate where they have no affect away from the extrema. The degradation with the C^1 continuity near the vertex is such that the Hyman derivative approximation actually becomes better than the Akima. When the monotonicity condition is not applied at the extrema, the Akima approximation has the same errors as the unlimited version indicating that it was only required at these extrema and not at the base for strict monotonicity. This differs for the other approximations where application of the monotonicity condition except at the extrema improves the interpolation. The necessary condition improves the interpolation in the region approaching the flat base.

3.5.4 *Bernstein quadratic (Table 16)*

For the well resolved triangular test shape, the Bernstein quadratic scheme is consistently more accurate than the Hermite cubic and comparable to the second version of the rational cubic, neither being consistently better. The Akima derivative estimate, followed by the Hyman estimate is again the most accurate. The interpolation is degraded near the extrema and improved near the base when monotonicity constraints are applied. In the monotonic cases, the differences between the rational cubic and Bernstein quadratic seem small and do not provide firm ground for choosing between them.

3.6 Summary of tests of well resolved shapes

With the well resolved smooth test shapes (cosine and Gaussian), the Hyman derivative estimate is the best followed by the cubic which, in turn, is followed by the arithmetic—consistent with ordering by formal accuracy. The Akima approximation is usually not as good as any of these. However, when the data being interpolated consist of linked linear segments (the triangular test shape), the Akima derivative estimate becomes the best, followed by Hyman. For all cases, the geometric, harmonic and Fritsch-Butland approximations are consistently less accurate than other approximations. Thus one might choose the Akima approximation for data with special characteristics, but for the more general case the Hyman scheme seems desirable.

Application of monotonicity constraints degrades the interpolant near the extrema and tends to improve it (or at least not degrade it) in the strictly monotonic regions. In general, the C^1 constraints have larger errors than the C^0, again associated with the extrema, since the more severe constraint effects two adjacent intervals rather than just one.

The second version of the rational cubic interpolant is consistently more accurate than the first and is (with a few minor exceptions) more accurate than the Hermite cubic. Comparing the rational quadratic and the corresponding monotonic Hermite cubic interpolant provides a mixed signal; away from the extrema the monotonic Hermite cubic interpolant tends to be more accurate. The Bernstein quadratic interpolant is generally less accurate than the second version of the rational cubic except in some instances with the triangular test shape. It is also generally less accurate than the Hermite cubic when no monotonicity conditions are applied. The Bernstein quadratic scheme tends to be more accurate than the monotonic versions of the Hermite cubic interpolant when coupled to the higher order derivative approximations in the monotonic regions.

The best combination for general use, where strict monotonicity is not essential, seems to be the second version of the rational cubic with the Hyman derivative estimate constrained by the monotonicity condition where the data are strictly monotonic but relying on the convex/concave properties at the extrema where the data imply such structures. The choices between C^0 and C^1 seem less critical and could be chosen depending on the nature of the problem. If monotonicity is not imposed at the extrema then C^1 is probably preferable.

3.7 Poorly resolved cosine test shape

3.7.1 *Hermite cubic (Table 17)*

Like the well resolved cosine test shape, when no limiters are applied, the Hyman derivative estimate is more accurate than the cubic which in turn is better than the arithmetic. But now, the Akima derivative estimate falls in between the cubic and Hyman estimates. This suggests that the Akima derivative estimate is better with the more pointed shapes. As before, the Fritsch-Butland, geometric and harmonic derivative estimates are the least accurate and are considered no further.

Except with the Akima derivative estimate, application of the C^0 and C^1 SCM degrade the solution near the extrema and improve it in the monotonic regions. When the entire domain is considered, the relative ranking remains as with no limiters, except the Akima approximation falls behind cubic with C^0 continuity and behind arithmetic with C^1 continuity. Unlike the well resolved test case, the C^1 sufficient condition provides a better interpolation than the C^0. Relaxing the monotonicity condition at the extrema again gives a better interpolation at the expense of overshooting.

3.7.2 *Rational quadratic (Table 18)*

The relative ordering of the derivative estimates with the rational quadratic remain consistent with the Hermite cubic interpolant. Comparing the rational quadratic with the appropriate monotonic Hermite cubic gives somewhat mixed results for the derivative approximations. However, with the best derivative estimate (Hyman), the Hermite cubic is better than the rational quadratic.

3.7.3 *Rational cubic (Tables 19 and 20)*

The second version of the rational cubic remains more accurate than the first. But with the narrower shape, the Hermite cubic is also consistently better case for case than the second version of the rational cubic. The relative standings of the derivative estimates is essentially the same as with the Hermite cubic except the Akima derivative estimate is now not as good as the arithmetic estimate.

3.7.4 *Bernstein quadratic (Table 21)*

The Bernstein quadratic scheme is consistently worse than the second version of the rational cubic except with the Hyman derivative estimate and C^1 monotonicity when it is insignificantly better near the extrema.

3.8 Poorly resolved Gaussian test shape

3.8.1 *Hermite cubic (Table 22)*

When no modification is made to the derivative estimates with the Hermite cubic interpolant, the relative standings of the derivative estimates are similar to those of the poorly resolved cosine test shape, except the Hyman derivative estimate is the best by a larger margin and Akima derivative estimate moves below the cubic. The C^0 and C^1 sufficient conditions for monotonicity eliminate this margin although Hyman remains the best. The sufficient conditions degrade the interpolations near the extrema and unlike the earlier results, do not improve the interpolation in the monotonic region for the C^1 form. Unlike the poorly resolved cosine test shape, but like the well resolved cosine and Gaussian, the C^1 condition degrades the interpolation more than the C^0 does.

3.8.2 *Rational quadratic (Table 23)*

The rational quadratic and Hermite cubic show a nearly equal accuracy. In some regions one is better, in others the other is better. Conclusions regarding the relative accuracy of the derivative estimates used with the rational quadratic are similar to those using the Hermite cubic, except that the Akima estimate moving around somewhat but never becoming better than Hyman estimate.

3.8.3 *Rational cubic (Tables 24 and 25)*

The second version of the rational cubic remains more accurate than the first. The Hermite cubic is not consistently better than the second version of the rational cubic as it was with the poorly resolved cosine. With C^1 continuity, the rational cubic is the better of the two with the cubic and Hyman derivative estimate.

3.8.4 *Bernstein quadratic (Table 26)*

The Bernstein quadratic interpolations are not as good as the second version of the rational cubic except near the extrema for a few of the C^1 monotonic cases, namely the Akima and Hyman, where the improvement is fairly small.

3.9 Poorly resolved triangle test shape

3.9.1 *Hermite cubic (Table 27)*

The derivative estimates follow the same order seen with the well resolved triangle shape, Akima being the best, followed by Hyman, again illustrating that the Akima approximation is accurate with shapes defined by piecewise continuous linear segments. Monotonicity constraints degrade the solution near the peak but improve it near the base where the test function changes from the flat base to the side of the triangle. The improvement near the base is not as obvious for the C^1 monotonicity because our error integrals do not make a clear domain separation. In the C^1 case the error integral excluding the extrema includes intervals which are affected by the condition applied on the intervals next to the extrema. Thus it does not represent error at the base only. The interpolations are improved by relaxing the monotonicity condition at the extrema, at the expense of overshooting.

3.9.2 *Rational quadratic (Table 28)*

Comparing the rational quadratic with the appropriate monotonic Hermite cubic gives a mixed signal; with the better derivative estimates (Akima and Hyman) the Hermite cubic is better everywhere with C^0 continuity and near the extrema with C^1 continuity — but, the rational quadratic is better over the monotonic domain with C^1 continuity.

3.9.3 *Rational cubic (Tables 29 and 30)*

Again, the second version of the rational cubic is better than the first version. With the second version, the Akima and Hyman derivative estimates remain the most accurate. The Hyman estimate is better in the monotonic region, the Akima is better near the extrema. This advantage of the Akima approximation is lost when the necessary condition for monotonicity is applied and, of course, regained when the condition is relaxed at the extrema.

For the unlimited case and for NCM0 and NCM1 limiters an examination of tables 27 and 29 reveals the following: (1) the second version of the rational cubic is more accurate than the Hermite cubic with the Hyman and cubic derivative estimates, and (2) The Hermite cubic and rational interpolants when coupled with the Akima estimate give mixed signals about which is more accurate, depending upon whether one looks at the monotonic or whole domain, and whether one applies NCM constraints at the extrema or not.

3.9.4 *Bernstein quadratic (Table 31)*

The Bernstein quadratic scheme is more accurate than the second version of the rational cubic with the Hyman derivative estimate but less accurate with the Akima estimate. The differences tend to be small however, and by themselves do not provide a firm basis for choosing one over the other.

3.10 Summary of tests of poorly resolved shapes

With the poorly resolved test shapes, the Hyman and Akima derivative estimates consistently rank as the most accurate. For shapes which are strongly peaked, the Akima slope is more accurate than the Hyman estimate. The geometric, harmonic and Fritsch-Butland approximations are consistently less accurate than the other approximations.

The monotonicity constraints degrade the interpolation near the extrema and usually improve the solution in the monotonic region. The C^1 condition does not always degrade the interpolation more than the C^0.

The second version of the rational cubic is consistently better than the first. The Hermite cubic is better than the rational cubic when evaluated using the cosine shape but not always with the Gaussian or triangular shapes. The rational quadratic compared to the corresponding monotonic Hermite cubic gives a mixed signal. The Bernstein quadratic interpolant (with some few exceptions) tends to be less accurate than the second version of the rational cubic interpolant.

4 Conclusions

In the previous section we have compared the various schemes in interpolating well-resolved cosine bell, Gaussian and triangular test shapes, and the corresponding poorly resolved test shapes. Out of the mass of numbers considered in these comparisons there are logical inferences to be drawn relating the various schemes to each other. These conclusions may not be universal, as definite known properties of particular fields might be used to advantage in the interpolation scheme. Minor exceptions can be found in our tables that might imply some other scheme is ideal for such specific applications. It is also possible that our conclusions would change with a different error measure.

We begin by itemizing our conclusions regarding the interpolating functions.

- The Hermite cubic and the second version of the rational cubic interpolant appear to be the most useful interpolation formulas. The first version of the rational cubic interpolant is consistently inferior to the second.

- The Bernstein quadratic interpolant is generally of comparable accuracy to the rational form mentioned above. We found it to be somewhat more difficult to program for the various special cases, which results in a corresponding increase in the complexity of computer code and execution time.

- The rational quadratic interpolant is of comparable accuracy to the SCM limited Hermite cubic for monotonic data, but it does not allow the flexibility of the Hermite cubic near extrema, or allow for the concave/convex structure provided by versions of the rational cubic interpolant. For data which have an extremum, this scheme is not recommended, because there is no alternative to assuming the slope goes to zero at a discrete extremum. This results in much larger errors in the vicinity of the extremum, than the cubic, rational cubic and piecewise quadratic spline forms.

Conclusions regarding the derivative estimates follow:

- The geometric mean, harmonic mean and Fritsch-Butland derivative estimates are consistently less accurate than the others. Their virtue is their simplicity. While they may result in visually pleasing interpolants they are generally of insufficient accuracy for many applications. The rational-linear derivative estimate [7], equivalent to that suggested by McAllistor and Roulier [11], and to the harmonic mean estimate suggested by Frisch and Butland [5] for equally spaced data, is the least accurate of all the derivative estimates. The Fritsch-Butland derivative estimate is always more accurate than the rational linear estimate.

- The Akima approximation performs extremely well for data with small scale features, but less well for the broader, more rounded shapes. Careful examination of the results suggests the Akima scheme is actually quite accurate in the vicinity of the extrema, and much less accurate over the rest of the domain.

- Except for the intersection of straight lines such as triangular peaks where the Akima estimate shines, the Hyman derivative estimate is the most accurate, followed generally by the cubic, then arithmetic. The disadvantage of the cubic derivative approximation is that it does not provide for a C^1 continuous interpolant while the others do.

- Monotonicity constraints generally improve the interpolation of monotonic data and data approaching a flat plateau. These constraints degrade the interpolation near extrema by not allowing any overshoot that might be implied in the underlying data.

The derivative estimate is constrained to be zero in the vicinity of an extrema with the C^1 form. The C^0 continuity constraint is less serious in this than the C^1.

- Where strict monotonicity is not required, relaxation of the monotonicity condition at any extremum seems desirable to allow the interpolant to form an extremum somewhere other than at a data point. Application of Hyman's limiter for the Hermite cubic seems desirable to prevent overshooting in the approach to a flat or nearly flat plateau.

In general the accuracy of the interpolation does not vary by more than a factor of two or three between schemes. The exception to this statement occurs when the shape to be interpolated is analytic and resolution is high, in which case the high formal accuracy of the cubic interpolant, and the Hyman fourth order derivative estimate result in substantial increases in accuracy over other schemes. We saw this demonstrated in the well resolved Gaussian test shape. points.

We mention in passing that we also tested other C^0 sufficient conditions for monotonicity with the Hermite cubic interpolation form which modify the derivative estimates to lie on the elliptical boundary of $\mathcal{M}_{ns}$ (12) rather the to the more restrictive boundary of $\mathcal{M}_b$. These limiting forms involve extra calculations and result in minute but discernible improvements in the accuracy of the representation.

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TABLE 1.

Identifier	Algorithm
Akima ([1], [6], [8])	$d_i = \begin{cases} \frac{\alpha\Delta_{i-1}+\beta\Delta_i}{\alpha+\beta} & \alpha+\beta \neq 0 \\ \frac{(\Delta_{i-1}+\Delta_i)}{2} & \alpha+\beta = 0 \end{cases}$ $\alpha = \lvert\Delta_{i+1}-\Delta_i\rvert,\ \beta = \lvert\Delta_{i-1}-\Delta_{i-2}\rvert$
Arithmetic Mean ([6], [7], [3], [8] *Deficient Spline*	$d_i = \frac{(\Delta_{i-1}+\Delta_i)}{2}$
Geometric Mean ([3])	$d_i = \begin{cases} \text{sign}(\Delta_i)\sqrt{\Delta_{i-1}\Delta_i} & \Delta_{i-1}\Delta_i \geq 0 \\ 0 & \Delta_{i-1}\Delta_i < 0 \end{cases}$
Harmonic Mean ([5]) *Rational Linear* ([7]) *McAllister-Roulier* ([11])	$d_i = \begin{cases} \frac{2\Delta_{i-1}\Delta_i}{(\Delta_{i-1}+\Delta_i)} & \Delta_{i-1}\Delta_i \geq 0 \\ 0 & \Delta_{i-1}\Delta_i < 0 \end{cases}$
Fritsch-Butland ([5], [8])	$d_i = \begin{cases} \frac{3\lvert\Delta_{i-1}\rvert\,\lvert\Delta_i\rvert}{\max(\Delta_{i-1},\Delta_i)+2\min(\Delta_{i-1},\Delta_i)} & \Delta_{i-1}\Delta_i \geq 0 \\ 0 & \Delta_{i-1}\Delta_i < 0 \end{cases}$
Cubic	$d_i = \begin{cases} \frac{(2\Delta_{i-1}+5\Delta_i-\Delta_{i+1})}{6} & x \in (x_i, x_{i+1}) \\ \frac{(-\Delta_{i-2}+5\Delta_{i-1}+2\Delta_i)}{6} & x \in (x_{i-1}, x_i) \end{cases}$
Hyman ([8])	$d_i = \frac{\Delta_{i-2}-7\Delta_{i-1}+7\Delta_i-\Delta_{i+1}}{12}$

Algorithms for derivative estimates as they simplify for evenly spaced data.

Slope	No limiter	SCM0	SCM1	HYM1	SCM0-EE	SCM1-EE
Ari	1.25(-6)	2.43(-6)	2.71(-6)	1.54(-6)	1.22(-6)	9.66(-7)
	[1.24(-6)]	[8.85(-7)]	[6.32(-7)]	[6.32(-7)]	[8.85(-7)]	[6.32(-7)]
Cub	1.07(-6)	2.22(-6)	2.80(-6)	1.60(-6)	1.01(-6)	1.02(-6)
	[1.06(-6)]	[6.78(-7)]	[6.80(-7)]	[6.80(-7)]	[6.78(-7)]	[6.80(-7)]
Aki	2.55(-6)	4.04(-6)	5.27(-6)	3.93(-6)	2.83(-6)	3.25(-6)
	[2.52(-6)]	[2.52(-6)]	[2.94(-6)]	[2.94(-6)]	[2.52(-6)]	[2.94(-6)]
But			4.01(-6)			
			[1.46(-6)]			
Geo	3.22(-6)	3.38(-6)	3.44(-6)			
	[1.05(-6)]	[1.07(-6)]	[1.14(-6)]			
Har			5.70(-6)			
			[2.65(-6)]			
Hym	7.05(-7)	1.96(-6)	2.75(-6)	1.52(-6)	7.50(-7)	9.17(-7)
	[7.05(-7)]	[4.36(-7)]	[6.03(-7)]	[6.03(-7)]	[4.36(-7)]	[6.03(-7)]

Table 2. Error measures for the Hermite interpolant, cosine bell shape, using 40-point resolution. Unbracketed numbers represent the ensemble average of the error integral associated with the 100 realizations of the shape. The numbers within square brackets represent the error excluding the intervals adjacent to the extrema.

Slope	NCM0	NCM1	NCM1-EE
Ari	2.89(-6)	3.35(-6)	2.28(-6)
	[1.24(-6)]	[6.39(-7)]	[6.39(-7)]
Cub	2.47(-6)	3.36(-6)	2.25(-6)
	[8.32(-7)]	[6.10(-7)]	[6.10(-7)]
Aki	4.38(-6)	5.69(-6)	4.38(-6)
	[2.78(-6)]	[2.78(-6)]	[2.78(-6)]
But		5.19(-6)	
		[2.04(-6)]	
Geo		4.36(-6)	
		[1.45(-6)]	
Har		6.80(-6)	
		[3.52(-6)]	
Hym	2.28(-6)	3.16(-6)	2.02(-6)
	[6.60(-7)]	[3.96(-7)]	[3.96(-7)]

Table 3. Error measure for the rational quadratic interpolant, cosine bell shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	2.37(-6)	4.00(-6)	3.08(-6)	2.37(-6)	1.33(-6)	2.37(-6)
	[2.35(-6)]	[2.35(-6)]	[1.31(-6)]	[2.35(-6)]	[1.31(-6)]	[2.35(-6)]
Cub	1.19(-6)	2.62(-6)	2.54(-6)	9.85(-7)	7.62(-7)	1.06(-6)
	[1.18(-6)]	[9.72(-7)]	[7.50(-7)]	[9.72(-7)]	[7.50(-7)]	[1.05(-6)]
Aki	2.26(-6)	3.84(-6)	4.19(-6)	2.26(-6)	2.26(-6)	2.26(-6)
	[2.24(-6)]	[2.24(-6)]	[2.24(-6)]	[2.24(-6)]	[2.24(-6)]	[2.24(-6)]
But			3.40(-6)			
			[1.11(-6)]			
Geo			3.88(-6)			
			[1.64(-6)]			
Har			6.61(-6)			
			[3.17(-6)]			
Hym	8.50(-7)	2.40(-6)	2.25(-6)	7.78(-7)	4.55(-7)	8.49(-7)
	[8.49(-7)]	[7.77(-7)]	[4.54(-7)]	[7.77(-7)]	[4.54(-7)]	[8.48(-7)]

Table 4. Error measure for the rational cubic interpolant version 1, cosine bell shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	1.73(-6)	3.20(-6)	2.56(-6)	1.73(-6)	8.27(-7)	1.73(-6)
	[1.72(-6)]	[1.72(-6)]	[8.16(-7)]	[1.72(-6)]	[8.16(-7)]	[1.72(-6)]
Cub	1.05(-6)	2.20(-6)	2.34(-6)	7.40(-7)	5.64(-7)	8.10(-7)
	[1.04(-6)]	[7.32(-7)]	[5.55(-7)]	[7.32(-7)]	[5.55(-7)]	[8.02(-7)]
Aki	2.40(-6)	3.80(-6)	4.39(-6)	2.40(-6)	2.40(-6)	2.40(-6)
	[2.37(-6)]	[2.37(-6)]	[2.37(-6)]	[2.37(-6)]	[2.37(-6)]	[2.37(-6)]
But			3.58(-6)			
			[1.26(-6)]			
Geo			3.25(-6)			
			[1.15(-6)]			
Har			5.50(-6)			
			[2.39(-6)]			
Hym	6.42(-7)	1.99(-6)	2.10(-6)	5.36(-7)	2.93(-7)	6.03(-7)
	[6.42(-7)]	[5.35(-7)]	[2.93(-7)]	[5.35(-7)]	[2.93(-7)]	[6.03(-7)]

Table 5. Error measure for the rational cubic interpolant version 2, cosine bell shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	2.18(-6)	3.79(-6)	2.98(-6)	2.18(-6)	1.10(-6)	2.18(-6)
	[2.16(-6)]	[2.16(-6)]	[1.09(-6)]	[2.16(-6)]	[1.09(-6)]	[2.16(-6)]
Cub	1.05(-6)	2.40(-6)	2.52(-6)	7.86(-7)	6.01(-7)	1.00(-6)
	[1.04(-6)]	[7.77(-7)]	[5.92(-7)]	[7.77(-7)]	[5.92(-7)]	[9.94(-7)]
Aki	2.51(-6)	4.08(-6)	4.65(-6)	2.51(-6)	2.51(-6)	2.51(-6)
	[2.47(-6)]	[2.47(-6)]	[2.47(-6)]	[2.47(-6)]	[2.47(-6)]	[2.47(-6)]
But			3.67(-6)			
			[1.34(-6)]			
Geo			3.75(-6)			
			[1.56(-6)]			
Har			6.46(-6)			
			[3.16(-6)]			
Hym	7.05(-7)	2.23(-6)	2.30(-6)	6.26(-7)	3.50(-7)	8.42(-7)
	[7.04(-7)]	[6.26(-7)]	[3.49(-7)]	[6.26(-7)]	[3.49(-7)]	[8.41(-7)]

Table 6. Error measure for the piecewise quadratic Bernstein polynomial interpolant, cosine bell shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	SCM0	SCM1	Hyman	SCM0-EE	SCM1-EE
Ari	7.39(-8)	1.08(-6)	1.59(-6)	8.25(-7)	2.88(-7)	4.53(-7)
	[6.51(-8)]	[6.51(-8)]	[2.30(-7)]	[2.30(-7)]	[6.51(-8)]	[2.30(-7)]
Cub	2.28(-8)	1.03(-6)	1.59(-6)	8.08(-7)	2.40(-7)	4.28(-7)
	[1.51(-8)]	[1.51(-8)]	[2.03(-7)]	[2.03(-7)]	[1.51(-8)]	[2.03(-7)]
Aki	4.02(-7)	1.37(-6)	2.20(-6)	1.31(-6)	5.78(-7)	8.61(-7)
	[3.72(-7)]	[3.72(-7)]	[6.55(-7)]	[6.55(-7)]	[3.72(-7)]	[6.55(-7)]
But			2.09(-6)			
			[4.21(-7)]			
Geo	1.68(-6)	1.77(-6)	1.80(-6)			
	[2.51(-7)]	[2.51(-7)]	[2.88(-7)]			
Har			2.56(-6)			
			[5.57(-7)]			
Hym	5.83(-10)	1.00(-6)	1.60(-6)	8.00(-7)	2.07(-7)	4.04(-7)
	[4.10(-10)]	[4.10(-10)]	[1.98(-7)]	[1.98(-7)]	[4.10(-10)]	[1.98(-7)]

Table 7. Error measures for the Hermite interpolant, Gaussian bell shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	NCM0	NCM1	NCM1-EE
Ari	1.14(-6)	1.84(-6)	1.14(-6)
	[6.15(-8)]	[6.15(-8)]	[6.15(-8)]
Cub	1.09(-6)	1.82(-6)	1.09(-6)
	[1.28(-8)]	[1.28(-8)]	[1.28(-8)]
Aki	1.41(-6)	2.28(-6)	1.41(-6)
	[3.62(-7)]	[3.62(-7)]	[3.62(-7)]
But		2.70(-6)	
		[6.33(-7)]	
Geo		2.31(-6)	
		[4.04(-7)]	
Har		3.11(-6)	
		[9.52(-7)]	
Hym	1.07(-6)	1.81(-6)	1.07(-6)
	[4.37(-10)]	[4.37(-10)]	[4.37(-10)]

Table 8. Error measure for the rational quadratic interpolant, Gaussian bell shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	3.34(-7)	1.41(-6)	1.47(-6)	3.34(-7)	3.34(-7)	3.34(-7)
	[3.12(-7)]	[3.12(-7)]	[3.12(-7)]	[3.12(-7)]	[3.12(-7)]	[3.12(-7)]
Cub	6.35(-8)	1.13(-6)	1.22(-6)	6.35(-8)	6.35(-8)	9.17(-8)
	[5.24(-8)]	[5.24(-8)]	[5.24(-8)]	[5.24(-8)]	[5.24(-8)]	[8.07(-8)]
Aki	4.31(-7)	1.46(-6)	1.71(-6)	4.31(-7)	4.31(-7)	4.31(-7)
	[4.14(-7)]	[4.14(-7)]	[4.14(-7)]	[4.14(-7)]	[4.14(-7)]	[4.14(-7)]
But			1.80(-6)			
			[3.01(-7)]			
Geo			1.84(-6)			
			[3.68(-7)]			
Har			2.77(-6)			
			[5.06(-7)]			
Hym	2.93(-8)	1.09(-6)	1.20(-6)	2.93(-8)	2.93(-8)	5.68(-8)
	[2.81(-8)]	[2.81(-8)]	[2.81(-8)]	[2.81(-8)]	[2.81(-8)]	[5.56(-8)]

Table 9. Error measure for the rational cubic interpolant version 1, Gaussian bell shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	C' NC EE	NCM1-EE	NCC1
Ari	1.86(-7)	1.15(-6)	1.31(-6)	1.86(-7)	1.86(-7)	1.86(-7)
	[1.76(-7)]	[1.76(-7)]	[1.76(-7)]	[1.76(-7)]	[1.76(-7)]	[1.76(-7)]
Cub	3.04(-8)	9.91(-7)	1.19(-6)	3.04(-8)	3.04(-8)	5.73(-8)
	[2.26(-8)]	[2.26(-8)]	[2.26(-8)]	[2.26(-8)]	[2.26(-8)]	[4.95(-8)]
Aki	4.00(-7)	1.31(-6)	1.72(-6)	4.00(-7)	4.00(-7)	4.00(-7)
	[3.73(-7)]	[3.73(-7)]	[3.73(-7)]	[3.73(-7)]	[3.73(-7)]	[3.73(-7)]
But			1.83(-6)			
			[3.19(-7)]			
Geo			1.62(-6)			
			[2.43(-7)]			
Har			2.38(-6)			
			[3.40(-7)]			
Hym	7.05(-9)	9.61(-7)	1.19(-6)	7.05(-9)	7.05(-9)	3.33(-8)
	[6.85(-9)]	[6.85(-9)]	[6.85(-9)]	[6.85(-9)]	[6.85(-9)]	[3.31(-8)]

Table 10. Error measure for the rational cubic interpolant version 2, Gaussian bell shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	2.74(-7)	1.34(-6)	1.49(-6)	2.74(-7)	2.74(-7)	2.74(-7)
	[2.62(-7)]	[2.62(-7)]	[2.62(-7)]	[2.62(-7)]	[2.62(-7)]	[2.62(-7)]
Cub	3.46(-8)	1.09(-6)	1.29(-6)	3.46(-8)	3.46(-8)	1.18(-7)
	[2.68(-8)]	[2.68(-8)]	[2.68(-8)]	[2.68(-8)]	[2.68(-8)]	[1.10(-7)]
Aki	4.55(-7)	1.48(-6)	1.87(-6)	4.55(-7)	4.55(-7)	4.55(-7)
	[4.25(-7)]	[4.25(-7)]	[4.25(-7)]	[4.25(-7)]	[4.25(-7)]	[4.25(-7)]
But	1.89(-6)					
	[3.69(-7)]					
Geo	1.82(-6)					
	[3.79(-7)]					
Har	2.71(-6)					
	[5.47(-7)]					
Hym	1.22(-8)	1.07(-6)	1.28(-6)	1.22(-8)	1.22(-8)	9.54(-8)
	[1.19(-8)]	[1.19(-8)]	[1.19(-8)]	[1.19(-8)]	[1.19(-8)]	[9.51(-8)]

Table 11. Error measure for the piecewise quadratic Bernstein polynomial interpolant, Gaussian bell shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	SCM0	SCM1	Hyman	SCM0-EE	SCM1-EE
Ari	8.35(-5)	1.03(-4)	1.09(-4)	8.65(-5)	8.42(-5)	8.19(-5)
	[2.96(-5)]	[2.61(-5)]	[2.38(-5)]	[2.38(-5)]	[2.61(-5)]	[2.38(-5)]
Cub	8.35(-5)	9.84(-5)	1.21(-4)	9.20(-5)	8.50(-5)	8.66(-5)
	[2.94(-5)]	[2.61(-5)]	[2.77(-5)]	[2.77(-5)]	[2.61(-5)]	[2.77(-5)]
Aki	3.23(-5)	6.80(-5)	9.68(-5)	6.20(-5)	3.94(-5)	4.66(-5)
	[1.08(-5)]	[9.16(-6)]	[1.64(-5)]	[1.64(-5)]	[9.16(-6)]	[1.64(-5)]
But			1.15(-4)			
			[2.71(-5)]			
Geo	1.15(-4)	1.15(-4)	1.16(-4)			
	[2.71(-5)]	[2.72(-5)]	[2.78(-5)]			
Har			1.25(-4)			
			[3.35(-5)]			
Hym	7.85(-5)	1.00(-4)	1.12(-4)	8.68(-5)	7.97(-5)	8.07(-5)
	[2.78(-5)]	[2.36(-5)]	[2.46(-5)]	[2.46(-5)]	[2.36(-5)]	[2.46(-5)]

Table 12. Error measures for the Hermite interpolant, triangle shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	NCM0	NCM1	NCM1-EE
Ari	1.10(-4)	1.20(-4)	1.04(-4)
	[2.76(-5)]	[2.14(-5)]	[2.14(-5)]
Cub	1.01(-4)	1.30(-4)	9.67(-5)
	[2.66(-5)]	[2.19(-5)]	[2.19(-5)]
Aki	7.22(-5)	1.07(-4)	7.22(-5)
	[1.16(-5)]	[1.16(-5)]	[1.16(-5)]
But		1.33(-4)	
		[3.19(-5)]	
Geo		1.32(-4)	
		[3.09(-5)]	
Har		1.42(-4)	
		[3.87(-5)]	
Hym	1.08(-4)	1.24(-4)	1.05(-4)
	[2.47(-5)]	[2.16(-5)]	[2.16(-5)]

Table 13. Error measure for the rational quadratic interpolant, triangle shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	9.86(-5)	1.17(-4)	1.27(-4)	9.86(-5)	9.22(-5)	9.86(-5)
	[3.10(-5)]	[3.10(-5)]	[2.46(-5)]	[3.10(-5)]	[2.46(-5)]	[3.10(-5)]
Cub	7.96(-5)	9.56(-5)	9.50(-5)	7.73(-5)	7.25(-5)	7.60(-5)
	[2.76(-5)]	[2.54(-5)]	[2.05(-5)]	[2.54(-5)]	[2.05(-5)]	[2.47(-5)]
Aki	2.54(-5)	6.80(-5)	1.03(-4)	2.54(-5)	2.54(-5)	2.54(-5)
	[8.47(-6)]	[8.47(-6)]	[8.47(-6)]	[8.47(-6)]	[8.47(-6)]	[8.47(-6)]
But			1.39(-4)			
			[3.45(-5)]			
Geo			1.42(-4)			
			[3.57(-5)]			
Har			1.57(-4)			
			[4.58(-5)]			
Hym	6.61(-5)	8.83(-5)	8.43(-5)	6.50(-5)	6.03(-5)	8.00(-5)
	[2.28(-5)]	[2.17(-5)]	[1.71(-5)]	[2.17(-5)]	[1.71(-5)]	[2.43(-5)]

Table 14. Error measure for the rational cubic interpolant version 1, triangle shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	9.41(-5)	1.13(-4)	1.23(-4)	9.41(-5)	8.74(-5)	9.41(-5)
	[2.95(-5)]	[2.95(-5)]	[2.28(-5)]	[2.95(-5)]	[2.28(-5)]	[2.95(-5)]
Cub	8.03(-5)	9.54(-5)	9.51(-5)	7.75(-5)	7.17(-5)	7.35(-5)
	[2.81(-5)]	[2.53(-5)]	[1.95(-5)]	[2.53(-5)]	[1.95(-5)]	[2.39(-5)]
Aki	2.06(-5)	6.40(-5)	9.87(-5)	2.06(-5)	2.06(-5)	2.06(-5)
	[6.88(-6)]	[6.88(-6)]	[6.88(-6)]	[6.88(-6)]	[6.88(-6)]	[6.88(-6)]
But			1.35(-4)			
			[3.27(-5)]			
Geo			1.38(-4)			
			[3.39(-5)]			
Har			1.53(-4)			
			[4.39(-5)]			
Hym	6.33(-5)	8.60(-5)	8.17(-5)	6.19(-5)	5.69(-5)	7.58(-5)
	[2.20(-5)]	[2.06(-5)]	[1.56(-5)]	[2.06(-5)]	[1.56(-5)]	[2.29(-5)]

Table 15. Error measure for the rational cubic interpolant version 2, triangle shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	7.16(-5)	9.09(-5)	8.49(-5)	7.16(-5)	6.56(-5)	7.16(-5)
	[2.39(-5)]	[2.39(-5)]	[1.79(-5)]	[2.39(-5)]	[1.79(-5)]	[2.39(-5)]
Cub	7.92(-5)	9.55(-5)	9.39(-5)	7.66(-5)	7.15(-5)	6.37(-5)
	[2.76(-5)]	[2.50(-5)]	[2.00(-5)]	[2.50(-5)]	[2.00(-5)]	[2.12(-5)]
Aki	2.26(-5)	6.67(-5)	6.67(-5)	2.26(-5)	2.26(-5)	2.26(-5)
	[7.54(-6)]	[7.54(-6)]	[7.54(-6)]	[7.54(-6)]	[7.54(-6)]	[7.54(-6)]
But			8.71(-5)			
			[1.84(-5)]			
Geo			9.17(-5)			
			[2.13(-5)]			
Har			1.01(-4)			
			[2.55(-5)]			
Hym	6.14(-5)	8.51(-5)	8.10(-5)	6.04(-5)	5.60(-5)	5.53(-5)
	[2.09(-5)]	[1.99(-5)]	[1.56(-5)]	[1.99(-5)]	[1.56(-5)]	[1.84(-5)]

Table 16. Error measure for the piecewise quadratic interpolant, triangle shape, using 40-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	SCM0	SCM1	Hyman	SCM0-EE	SCM1-EE
Ari	3.05(-3)	4.12(-3)	3.34(-3)	2.44(-3)	3.06(-3)	2.13(-3)
	[2.13(-3)]	[1.88(-3)]	[9.55(-4)]	[9.55(-4)]	[1.88(-3)]	[9.55(-4)]
Cub	2.93(-3)	3.78(-3)	3.12(-3)	2.39(-3)	2.97(-3)	2.02(-3)
	[2.09(-3)]	[1.85(-3)]	[9.02(-4)]	[9.02(-4)]	[1.85(-3)]	[9.02(-4)]
Aki	2.39(-3)	3.94(-3)	3.59(-3)	3.00(-3)	2.66(-3)	2.68(-3)
	[1.33(-3)]	[1.32(-3)]	[1.34(-3)]	[1.34(-3)]	[1.32(-3)]	[1.34(-3)]
But			3.63(-3)			
			[1.09(-3)]			
Geo	3.66(-3)	3.72(-3)	3.71(-3)			
	[1.15(-3)]	[1.16(-3)]	[1.15(-3)]			
Har			4.30(-3)			
			[1.44(-3)]			
Hym	1.65(-3)	2.77(-3)	2.59(-3)	1.59(-3)	1.62(-3)	1.15(-3)
	[1.38(-3)]	[1.04(-3)]	[5.72(-4)]	[5.72(-4)]	[1.04(-3)]	[5.72(-4)]

Table 17. Error measures for the Hermite interpolant, cosine bell shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	NCM0	NCM1	NCM1-EE
Ari	3.96(-3)	3.79(-3)	3.16(-3)
	[1.89(-3)]	[1.09(-3)]	[1.09(-3)]
Cub	3.75(-3)	3.98(-3)	3.02(-3)
	[1.85(-3)]	[1.11(-3)]	[1.11(-3)]
Aki	3.68(-3)	3.71(-3)	3.68(-3)
	[1.16(-3)]	[1.16(-3)]	[1.16(-3)]
But		3.93(-3)	
		[1.10(-3)]	
Geo		3.89(-3)	
		[1.06(-3)]	
Har		4.43(-3)	
		[1.44(-3)]	
Hym	2.78(-3)	3.14(-3)	2.28(-3)
	[1.09(-3)]	[5.88(-4)]	[5.88(-4)]

Table 18. Error measure for the rational quadratic interpolant, cosine bell shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	4.06(-3)	5.17(-3)	5.27(-3)	4.06(-3)	3.53(-3)	4.06(-3)
	[2.31(-3)]	[2.31(-3)]	[1.78(-3)]	[2.31(-3)]	[1.78(-3)]	[2.31(-3)]
Cub	3.45(-3)	4.40(-3)	4.62(-3)	3.30(-3)	2.97(-3)	3.43(-3)
	[2.25(-3)]	[2.10(-3)]	[1.77(-3)]	[2.10(-3)]	[1.77(-3)]	[2.08(-3)]
Aki	4.08(-3)	5.43(-3)	5.57(-3)	4.08(-3)	4.08(-3)	4.08(-3)
	[2.35(-3)]	[2.35(-3)]	[2.35(-3)]	[2.35(-3)]	[2.35(-3)]	[2.35(-3)]
But			5.44(-3)			
			[1.85(-3)]			
Geo			5.66(-3)			
			[1.94(-3)]			
Har			6.60(-3)			
			[2.51(-3)]			
Hym	2.30(-3)	3.63(-3)	4.04(-3)	2.22(-3)	1.83(-3)	2.23(-3)
	[1.50(-3)]	[1.43(-3)]	[1.04(-3)]	[1.43(-3)]	[1.04(-3)]	[1.45(-3)]

Table 19. Error measure for the rational cubic interpolant version 1, cosine bell shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	3.76(-3)	4.89(-3)	4.99(-3)	3.76(-3)	3.23(-3)	3.76(-3)
	[2.19(-3)]	[2.19(-3)]	[1.65(-3)]	[2.19(-3)]	[1.65(-3)]	[2.19(-3)]
Cub	3.23(-3)	4.09(-3)	4.25(-3)	3.03(-3)	2.64(-3)	3.24(-3)
	[2.19(-3)]	[1.99(-3)]	[1.60(-3)]	[1.99(-3)]	[1.60(-3)]	[2.00(-3)]
Aki	3.80(-3)	5.14(-3)	5.28(-3)	3.80(-3)	3.80(-3)	3.80(-3)
	[2.23(-3)]	[2.23(-3)]	[2.23(-3)]	[2.23(-3)]	[2.23(-3)]	[2.23(-3)]
But			5.20(-3)			
			[1.73(-3)]			
Geo			5.40(-3)			
			[1.82(-3)]			
Har			6.32(-3)			
			[2.37(-3)]			
Hym	2.01(-3)	3.28(-3)	3.70(-3)	1.91(-3)	1.53(-3)	1.98(-3)
	[1.36(-3)]	[1.27(-3)]	[8.86(-4)]	[1.27(-3)]	[8.86(-4)]	[1.32(-3)]

Table 20. Error measure for the rational cubic interpolant version 2, cosine bell shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	4.23(-3)	5.37(-3)	5.12(-3)	4.23(-3)	3.71(-3)	4.23(-3)
	[2.39(-3)]	[2.39(-3)]	[1.87(-3)]	[2.39(-3)]	[1.87(-3)]	[2.39(-3)]
Cub	3.30(-3)	4.24(-3)	4.24(-3)	3.12(-3)	2.79(-3)	4.49(-3)
	[2.21(-3)]	[2.03(-3)]	[1.70(-3)]	[2.03(-3)]	[1.70(-3)]	[2.43(-3)]
Aki	4.26(-3)	5.60(-3)	5.51(-3)	4.26(-3)	4.26(-3)	4.26(-3)
	[2.44(-3)]	[2.44(-3)]	[2.44(-3)]	[2.44(-3)]	[2.44(-3)]	[2.44(-3)]
But			5.23(-3)			
			[1.88(-3)]			
Geo			5.57(-3)			
			[2.05(-3)]			
Har			6.50(-3)			
			[2.56(-3)]			
Hym	2.23(-3)	3.59(-3)	3.62(-3)	2.16(-3)	1.77(-3)	3.34(-3)
	[1.45(-3)]	[1.38(-3)]	[9.96(-4)]	[1.38(-3)]	[9.96(-4)]	[1.95(-3)]

Table 21. Error measure for the piecewise quadratic Bernstein polynomial interpolant, cosine bell shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	SCM0	SCM1	Hyman	SCM0-EE	SCM1-EE
Ari	8.84(-4)	1.69(-3)	1.95(-3)	1.31(-3)	1.03(-3)	1.10(-3)
	[3.96(-4)]	[3.82(-4)]	[4.55(-4)]	[4.55(-4)]	[3.82(-4)]	[4.55(-4)]
Cub	8.15(-4)	1.50(-3)	2.03(-3)	1.43(-3)	9.86(-4)	1.17(-3)
	[3.59(-4)]	[3.50(-4)]	[5.31(-4)]	[5.31(-4)]	[3.50(-4)]	[5.31(-4)]
Aki	8.48(-4)	1.86(-3)	1.77(-3)	1.36(-3)	1.01(-3)	1.07(-3)
	[3.81(-4)]	[3.66(-4)]	[4.27(-4)]	[4.27(-4)]	[3.66(-4)]	[4.27(-4)]
But			2.01(-3)			
			[3.79(-4)]			
Geo	1.97(-3)	2.01(-3)	2.02(-3)			
	[3.83(-4)]	[3.82(-4)]	[3.96(-4)]			
Har			2.24(-3)			
			[4.29(-4)]			
Hym	3.15(-4)	1.22(-3)	1.67(-3)	9.39(-4)	5.06(-4)	6.47(-4)
	[1.54(-4)]	[1.51(-4)]	[2.93(-4)]	[2.93(-4)]	[1.51(-4)]	[2.93(-4)]

Table 22. Error measures for the Hermite interpolant, Gaussian bell shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	NCM0	NCM1	NCM1-EE
Ari	1.64(-3)	2.17(-3)	1.64(-3)
	[3.89(-4)]	[3.89(-4)]	[3.89(-4)]
Cub	1.50(-3)	2.28(-3)	1.50(-3)
	[3.44(-4)]	[3.44(-4)]	[3.44(-4)]
Aki	1.77(-3)	1.95(-3)	1.77(-3)
	[3.18(-4)]	[3.18(-4)]	[3.18(-4)]
But		2.37(-3)	
		[4.67(-4)]	
Geo		2.31(-3)	
		[4.38(-4)]	
Har		2.52(-3)	
		[5.37(-4)]	
Hym	1.22(-3)	1.92(-3)	1.22(-3)
	[1.41(-4)]	[1.41(-4)]	[1.41(-4)]

Table 23. Error measure for the rational quadratic interpolant, Gaussian bell shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	1.59(-3)	2.30(-3)	2.82(-3)	1.59(-3)	1.59(-3)	1.59(-3)
	[5.17(-4)]	[5.17(-4)]	[5.17(-4)]	[5.17(-4)]	[5.17(-4)]	[5.17(-4)]
Cub	1.01(-3)	1.71(-3)	1.64(-3)	1.01(-3)	1.01(-3)	1.17(-3)
	[4.20(-4)]	[4.15(-4)]	[4.15(-4)]	[4.15(-4)]	[4.15(-4)]	[4.19(-4)]
Aki	1.24(-3)	2.22(-3)	2.52(-3)	1.24(-3)	1.24(-3)	1.24(-3)
	[4.98(-4)]	[4.98(-4)]	[4.98(-4)]	[4.98(-4)]	[4.98(-4)]	[4.98(-4)]
But			2.99(-3)			
			[5.91(-4)]			
Geo			3.08(-3)			
			[6.02(-4)]			
Har			3.49(-3)			
			[7.68(-4)]			
Hym	6.64(-4)	1.53(-3)	1.99(-3)	6.64(-4)	6.64(-4)	7.46(-4)
	[2.15(-4)]	[2.15(-4)]	[2.15(-4)]	[2.15(-4)]	[2.15(-4)]	[2.50(-4)]

Table 24. Error measure for the rational cubic interpolant version 1, Gaussian bell shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	1.43(-3)	2.15(-3)	2.67(-3)	1.43(-3)	1.43(-3)	1.43(-3)
	[4.78(-4)]	[4.78(-4)]	[4.78(-4)]	[4.78(-4)]	[4.78(-4)]	[4.78(-4)]
Cub	9.04(-4)	1.57(-3)	1.50(-3)	8.98(-4)	8.98(-4)	1.08(-3)
	[3.98(-4)]	[3.91(-4)]	[3.91(-4)]	[3.91(-4)]	[3.91(-4)]	[4.03(-4)]
Aki	1.12(-3)	2.08(-3)	2.38(-3)	1.12(-3)	1.12(-3)	1.12(-3)
	[4.71(-4)]	[4.71(-4)]	[4.71(-4)]	[4.71(-4)]	[4.71(-4)]	[4.71(-4)]
But			2.88(-3)			
			[5.65(-4)]			
Geo			2.94(-3)			
			[5.69(-4)]			
Har			3.35(-3)			
			[7.22(-4)]			
Hym	5.37(-4)	1.39(-3)	1.84(-3)	5.36(-4)	5.36(-4)	6.40(-4)
	[1.91(-4)]	[1.91(-4)]	[1.91(-4)]	[1.91(-4)]	[1.91(-4)]	[2.31(-4)]

Table 25. Error measure for the rational cubic interpolant version 2, Gaussian bell shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	1.60(-3)	2.33(-3)	2.59(-3)	1.60(-3)	1.60(-3)	1.60(-3)
	[5.38(-4)]	[5.38(-4)]	[5.38(-4)]	[5.38(-4)]	[5.38(-4)]	[5.38(-4)]
Cub	9.29(-4)	1.63(-3)	1.59(-3)	9.23(-4)	9.23(-4)	1.55(-3)
	[4.05(-4)]	[3.99(-4)]	[3.99(-4)]	[3.99(-4)]	[3.99(-4)]	[5.21(-4)]
Aki	1.30(-3)	2.26(-3)	2.34(-3)	1.30(-3)	1.30(-3)	1.30(-3)
	[5.37(-4)]	[5.37(-4)]	[5.37(-4)]	[5.37(-4)]	[5.37(-4)]	[5.37(-4)]
But			2.68(-3)			
			[5.54(-4)]			
Geo			2.83(-3)			
			[5.98(-4)]			
Har			3.23(-3)			
			[7.36(-4)]			
Hym	6.13(-4)	1.51(-3)	1.77(-3)	6.13(-4)	6.13(-4)	1.17(-3)
	[2.11(-4)]	[2.10(-4)]	[2.10(-4)]	[2.10(-4)]	[2.10(-4)]	[4.15(-4)]

Table 26. Error measure for the piecewise quadratic interpolant, Gaussian bell shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	SCM0	SCM1	Hyman	SCM0-EE	SCM1-EE
Ari	5.08(-3)	6.30(-3)	6.63(-3)	5.23(-3)	5.12(-3)	4.95(-3)
	[1.73(-3)]	[1.50(-3)]	[1.33(-3)]	[1.33(-3)]	[1.50(-3)]	[1.33(-3)]
Cub	5.04(-3)	5.99(-3)	7.39(-3)	5.57(-3)	5.13(-3)	5.23(-3)
	[1.69(-3)]	[1.48(-3)]	[1.57(-3)]	[1.57(-3)]	[1.48(-3)]	[1.57(-3)]
Aki	3.48(-3)	5.14(-3)	6.56(-3)	4.61(-3)	3.76(-3)	4.08(-3)
	[1.27(-3)]	[1.14(-3)]	[1.46(-3)]	[1.46(-3)]	[1.14(-3)]	[1.46(-3)]
But			7.02(-3)			
			[1.52(-3)]			
Geo	6.90(-3)	6.94(-3)	6.98(-3)			
	[1.48(-3)]	[1.48(-3)]	[1.52(-3)]			
Har			7.49(-3)			
			[1.80(-3)]			
Hym	4.68(-3)	6.14(-3)	7.02(-3)	5.38(-3)	4.80(-3)	4.97(-3)
	[1.57(-3)]	[1.30(-3)]	[1.47(-3)]	[1.47(-3)]	[1.30(-3)]	[1.47(-3)]

Table 27. Error measures for the Hermite interpolant, triangle shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	NCM0	NCM1	NCM1-EE
Ari	6.75(-3)	7.33(-3)	6.36(-3)
	[1.58(-3)]	[1.18(-3)]	[1.18(-3)]
Cub	6.17(-3)	7.99(-3)	5.87(-3)
	[1.51(-3)]	[1.21(-3)]	[1.21(-3)]
Aki	5.26(-3)	7.16(-3)	5.26(-3)
	[1.17(-3)]	[1.17(-3)]	[1.17(-3)]
But		8.12(-3)	
		[1.80(-3)]	
Geo		7.95(-3)	
		[1.68(-3)]	
Har		8.47(-3)	
		[2.08(-3)]	
Hym	6.69(-3)	7.83(-3)	6.60(-3)
	[1.40(-3)]	[1.31(-3)]	[1.31(-3)]

Table 28. Error measure for the rational quadratic interpolant, triangle shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	6.02(-3)	7.20(-3)	7.76(-3)	6.02(-3)	5.60(-3)	6.02(-3)
	[1.80(-3)]	[1.80(-3)]	[1.39(-3)]	[1.80(-3)]	[1.39(-3)]	[1.80(-3)]
Cub	4.84(-3)	5.86(-3)	5.74(-3)	4.69(-3)	4.38(-3)	4.73(-3)
	[1.63(-3)]	[1.49(-3)]	[1.18(-3)]	[1.49(-3)]	[1.18(-3)]	[1.50(-3)]
Aki	3.80(-3)	5.73(-3)	7.62(-3)	3.80(-3)	3.80(-3)	3.80(-3)
	[1.57(-3)]	[1.57(-3)]	[1.57(-3)]	[1.57(-3)]	[1.57(-3)]	[1.57(-3)]
But			8.50(-3)			
			[1.97(-3)]			
Geo			8.58(-3)			
			[1.99(-3)]			
Har			9.43(-3)			
			[2.54(-3)]			
Hym	4.03(-3)	5.58(-3)	5.68(-3)	3.96(-3)	3.65(-3)	4.64(-3)
	[1.37(-3)]	[1.30(-3)]	[9.88(-4)]	[1.30(-3)]	[9.88(-4)]	[1.31(-3)]

Table 29. Error measure for the rational cubic interpolant version 1, triangle shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	5.73(-3)	6.97(-3)	7.51(-3)	5.73(-3)	5.30(-3)	5.73(-3)
	[1.71(-3)]	[1.71(-3)]	[1.27(-3)]	[1.71(-3)]	[1.27(-3)]	[1.71(-3)]
Cub	4.83(-3)	5.80(-3)	5.67(-3)	4.65(-3)	4.28(-3)	4.58(-3)
	[1.63(-3)]	[1.45(-3)]	[1.08(-3)]	[1.45(-3)]	[1.08(-3)]	[1.44(-3)]
Aki	3.46(-3)	5.47(-3)	7.37(-3)	3.46(-3)	3.46(-3)	3.46(-3)
	[1.46(-3)]	[1.46(-3)]	[1.46(-3)]	[1.46(-3)]	[1.46(-3)]	[1.46(-3)]
But			8.28(-3)			
			[1.85(-3)]			
Geo			8.34(-3)			
			[1.88(-3)]			
Har			9.19(-3)			
			[2.41(-3)]			
Hym	3.92(-3)	5.49(-3)	5.59(-3)	3.84(-3)	3.53(-3)	4.39(-3)
	[1.35(-3)]	[1.26(-3)]	[9.50(-4)]	[1.26(-3)]	[9.50(-4)]	[1.23(-3)]

Table 30. Error measure for the rational cubic interpolant version 2, triangle shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.

Slope	No limiter	NCM0	NCM1	NCM0-EE	NCM1-EE	NCC1
Ari	5.32(-3)	6.55(-3)	6.71(-3)	5.32(-3)	4.93(-3)	5.32(-3)
	[1.70(-3)]	[1.70(-3)]	[1.31(-3)]	[1.70(-3)]	[1.31(-3)]	[1.70(-3)]
Cub	4.77(-3)	5.81(-3)	5.63(-3)	4.60(-3)	4.28(-3)	4.09(-3)
	[1.61(-3)]	[1.45(-3)]	[1.12(-3)]	[1.45(-3)]	[1.12(-3)]	[1.37(-3)]
Aki	3.72(-3)	5.71(-3)	6.69(-3)	3.72(-3)	3.72(-3)	3.72(-3)
	[1.58(-3)]	[1.58(-3)]	[1.58(-3)]	[1.58(-3)]	[1.58(-3)]	[1.58(-3)]
But			7.11(-3)			
			[1.58(-3)]			
Geo			7.33(-3)			
			[1.72(-3)]			
Har			8.07(-3)			
			[2.14(-3)]			
Hym	3.82(-3)	5.48(-3)	5.41(-3)	3.76(-3)	3.43(-3)	3.31(-3)
	[1.33(-3)]	[1.26(-3)]	[9.33(-4)]	[1.26(-3)]	[9.33(-4)]	[1.09(-3)]

Table 31. Error measure for the piecewise quadratic Bernstein polynomial interpolant, triangle shape, using 10-point resolution. See Table 2 caption for an explanation of the numbers.