Medical Image Reconstruction:
Multiangular Sectional Roentgenography
by Computer

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I. SUMMARY

This report describes progress achieved under the no-cost Memorandum of Agreement between the High Altitude Observatory of the National Center for Atmospheric Research (HAO-NCAR) and the USAF School of Aerospace Medicine, Clinical Sciences Division, Dental Sciences Branch (USAF-SAM-NGD) on the subject of three-dimensional computer analysis of dental and medical radiographs with the goal of improving present medical and dental techniques for internal visualization, diagnosis, and measurement.

We have developed and successfully demonstrated a computer technique which

1) analyzes a single series (or set) of calibrated, but otherwise ordinary, radiographs taken sequentially at uniformly-spaced azimuthal angles around an organism,

2) derives the three-dimensional density distribution within the organism,

3) displays the derived three-dimensional internal structure of the organism by means of computer-drawn serial-section maps of the object. (Each set of serial-section maps displays parallel slices with the same thickness, spacing, and orientation. When displayed in rapid serial sequence through motion picture cinematography, each set of maps provides a step-by-step "journey" through the organism in the specified direction.)

4) prepares an anatomic atlas of serial-section maps for any specified number of orientations, including the transverse, axial, frontal, and sagittal orientations,
5) permits the resolution of low-contrast features by emphasizing a particular density range of soft tissue. (An anatomic atlas can thus be prepared which contains serial-section maps for each orientation and for each density range of tissue.)

We have demonstrated our computer technique for two special cases, an extracted molar tooth and the undissected head of a rat. These organs were chosen because their sizes were comparable with dental X-ray peri-apical film. It remains to clinically evaluate computer-derived sections of larger structures such as the head and trunk of the human body. Also, we have so far derived our numerical data from radiographic films by densitometry. We are now examining television tapes of fluoroscopic scans as a possible alternative to radiographic films.

For the cases discussed in this report, sequences of 60 or 120 radiographs were analyzed. The three-dimensional resolution of the derived density distribution approached 0.5 mm (about 2% of the field of view for each dimension), which is well within diagnostic range. This resolution is a factor four less than the resolution of the original two-dimensional radiographs. If desired, somewhat higher three-dimensional spatial resolution can be achieved from the same data sets, but only by accepting poorer statistics, or equivalently, a higher noise level and noise sensitivity of the solution. (Significant improvement in the resolution of computer-derived internal density maps can of course be realized by including more azimuthal views in the data set.)

Our present goals are to achieve the maximum possible resolution and stability for our three-dimensional mathematical solution, to decrease
to a minimum the turn-around imaging and computing time, and to test our algorithms with more cases to permit clinical evaluation of this technique.

II. HISTORICAL BACKGROUND

1. NEED FOR IMAGE RECONSTRUCTION TECHNIQUES IN SOLAR PHYSICS

During the past decade, new solar coronagraphs on the ground and in earth orbit have made available accurate and frequent (at least daily) measurements of the polarization and brightness of the sun's tenuous outer atmosphere or corona. Because the corona is a semi-transparent (optically thin) atmosphere, we see only projections of the corona on the plane of the sky. Thus, matter which influences coronal measurements may be anywhere along a line of sight through the corona. At the High Altitude Observatory of the National Center for Atmospheric Research (HAO-NCAR) a number of staff scientists suspected that a sequence of projections of the corona (on the plane of the sky) taken during a solar rotation period might contain sufficient information to permit determination of the spatial (three-dimensional) distribution of the coronal electron density. This goal was achieved by Altschuler and Perry (1972) and Perry and Altschuler (1972, 1973). They replaced the unknown spatial distribution of coronal electron density with a linear combination of products of spherical harmonic and radial basis functions and then found the unknown coefficients of that series by using a least-mean-squares technique which compared the calculated and observed projections of the solar corona on the plane of the sky. Improvements in the computer algorithms are still being made in order to achieve three-dimensional maps of maximum resolution and accuracy for a given set of coronal projection data.
The calculated electron densities can be displayed by the computer as

1) numerical tables,
2) line contour drawings for any planar section through the corona,
3) stereoscopic line contour drawings for the global corona,
4) simulated photographs of the electron density distribution of any section through the corona, and
5) simulated photographs of the corona from any orientation, such as from above the ecliptic plane.

In analyzing solar coronal data the complications are

1) irregularly spaced and missing data because of cloudy days, instrumental repairs, non-commensurable terrestrial-solar rotation rates, etc.,
2) variable data quality because of atmospheric transparency, brightness, and sky polarization,
3) the tilt of the solar rotation axis, so that slices perpendicular to the plane of the sky cannot be analyzed independently,
4) an opaque central region (the sun itself) which restricts data to peripheral areas of the projection plane,
5) temporal variations in the structure of the solar corona during the data collection period of about two weeks (corresponding to half a solar rotation) and
6) large density gradients in coronal structures.

As a result of these complications, the problem of verifying the accuracy of the reconstruction algorithm (which determines the three-dimensional...
density distribution from two-dimensional projections) requires the use of artificial models of known density from which projection data can be derived for use with the reconstruction program.

2. MEDICAL APPLICATIONS

In April 1971, after hearing a description of the solar program, B. R. Altschuler (then at Temple University Dental School) now at the U.S.A.F. School of Aerospace Medicine (Brooks Air Force Base, Texas) suggested that the HAO-NCAR algorithms developed to determine coronal density be applied to the analysis of medical X-ray data. He argued that organisms are semi-transparent to X-rays and, like the solar corona, appear in projection on a radiograph. Thus if there exist radiographs showing projections of an organism from several different orientations, it should be possible to apply computer techniques (similar to those developed at HAO-NCAR) to determine the relative X-ray opacity (linear attenuation coefficient)* at each point within the organism, and then to display the results with computer-simulated photographs of sections of the organism at any specified orientation.

Management at both the U.S.A.F. School of Aerospace Medicine and HAO-NCAR assisted the authors in implementing a study to investigate the feasibility of computer analysis of radiographic data for the purpose of improving the accuracy and dependability of radiological diagnosis of pathology. The result of negotiation was a no-cost Memorandum of

*The linear attenuation coefficient (LAC) is the reciprocal of the mean free path of the X-ray photon, and varies with the local density and atomic number of the absorbing medium as well as with the energy of the incident photons. We refer to the spatial distribution of the LAC as the internal density distribution.
Agreement between the U.S.A.F. School of Aerospace Medicine and HAO-NCAR effective 1972 September 14, in which the School of Aerospace Medicine was to supply the required radiographic data and experimental resources, and HAO-NCAR was to provide the mathematical algorithms and necessary computer resources. R. M. Perry, who works only half-time for HAO-NCAR, volunteered his time for developing improved algorithms for medical purposes. M. D. Altschuler is coordinating the project, providing the necessary medical-mathematical interface, and drafting the necessary reports. B. R. Altschuler assembles the required equipment, provides the radiographic subjects and with his medical colleagues evaluates the diagnostic utility of the computer-derived internal density maps.

It was immediately clear that aside from steep density gradients the complications involved in treating solar coronal data would not arise in the analysis of medical data. Thus we would be working with uniformly calibrated projection data taken at equally-spaced azimuthal angles about a rotation axis which was perpendicular to the X-ray beam (or line of sight). The data would not contain central opaque regions which block transmission, and (in many cases) could be free of temporal variations. Here then was not only an opportunity to demonstrate interdisciplinary spin-off but also an opportunity to use essentially perfect data to verify the HAO algorithms.

The result of our first successful effort (Altschuler, et al., 1973, 1974b; Perry, et al., 1974) was a three-dimensional analysis of an extracted tooth which had been radiographed at 60 equally-spaced azimuthal angles (every 30°) around an arc of 180°. The dental X-ray films were digitized with a slow astronomical densitometer for 0.3 mm
resolution. The computer work involved the analysis of individual slices (cross sections) of the organ perpendicular to the rotation axis, and computer reassembly of the stack of derived cross-sectional maps to produce simulated photographs of axial sections (that is, sections containing the axis of rotation).

Recently, we have improved our computer algorithm

1) to use data from 120 radiographs (taken every 1.5° around an arc of 180°) for high resolution reconstruction,

2) to require an average computer time of only 3 seconds per cross-sectional reconstruction,

3) to produce a series of sagittal and frontal sections as well as axial and cross sections,

4) to produce maps with nonlinear density enhancement so that soft tissue structures can be seen.

Motion pictures of each series of sections provide a visual journey through the organism. The improved algorithm has been applied to an undissected rat head, with fast digitization from radiographs to computer tape provided gratis by the Mayo Clinic (Altschuler, et al., 1974a).

During the course of our work, we learned of similar efforts by other researchers on the problem of determining the structure of a semi-transparent object from information contained in projections of that object. An extensive bibliography by Gordon (1972) and reviews of algorithms such as those by Gordon and Herman (1974) and Mersereau and Oppenheim (1974) accelerated communication among researchers who had been working independently on the same problem but in different fields. In July 1974 the first International Workshop on Image Reconstruction took place at the Brookhaven
National Laboratory. Thus the subject matter of this report now belongs to a respectable field of applied mathematics known as Image Reconstruction.

A number of different algorithms for image reconstruction have been described in the literature. These include various types of algebraic reconstruction techniques, Fourier or Fourier-Bessel transform techniques, and least-mean-squares techniques. All algorithms with which we are familiar determine the two-dimensional density distribution of a limited planar region from one-dimensional projection data. The applications of these algorithms in computer programs have met with various degrees of success, and we find no universal agreement (or even consensus) on which algorithm is "best." Our feeling is that the usefulness of an algorithm in terms of speed, resolution, contrast, stability, etc., depends on many factors in addition to that of mathematical rigor, including:

1) the availability and capability of computing and display hardware,
2) whether real-time display is required, and
3) the assistance of a software virtuoso.

These factors make comparisons among algorithms a difficult task.

The technique we describe in this report uses a mathematically rigorous least-mean-squares method similar in philosophy to that described by Marr (1974). We are using one of the fastest commercial computers, the Control Data 7600, with a fast-access large-core memory. Because of the availability of this resource, our goal has been

1) to process real radiographic and fluoroscopic data at maximum possible resolution consistent with numerical
stability (statistical significance),

2) to map the internal density of an object or organism completely in three dimensions, which means the simultaneous analysis of a large set (or stack) of cross-sectional slices of the object, and

3) to reassemble the computer-derived cross-sectional maps of internal density into sections with other orientations (sagittal, frontal, axial, etc.) for convenient diagnostic study. Our goal, to the time of this report, has not been to achieve real-time solutions, although our computation time is being trimmed to allow such a possibility.

Our primary limitations have been

1) lack of in-house hardware for the fast digitization of radiographs and TV tape fluoroscopic images, and

2) a DD80 computer-controlled CRT microfilm output unit which can display only two (black and white) contrast levels, thus requiring the use of dot patterns to simulate gray-scale images.

The second limitation means that in practice we calculate about 15 times more contrast levels than we can effectively display (17 contrast levels for a 128 x 128 grid). Thus either different density ranges in the reconstructed sections must be displayed separately or nonlinear functions must first be convolved with the calculated internal densities to emphasize the density range of diagnostic interest.

At present we are developing even faster reconstruction algorithms with more stability against random noise and with a minimum of artifacts
in the reconstructed sections. We are also writing programs which reassemble and display the derived internal density distributions in different ways in order to enhance contrast (or other features) and improve diagnostic capability. Ultimately we hope to develop software which would enable a centrally located computer to

1) receive radiographic and fluoroscopic TV images,
2) convert the images to digital form,
3) solve for the three-dimensional internal density distribution of the organism, and
4) transmit back to the doctor a set of desired sections (or, if useful, contour displays) of the internal density distribution. The physician or dentist would be able to communicate back and forth with the computer to change orientation, thickness, or density scale and obtain additional maps to visualize areas of particular diagnostic interest without further patient exposure.

We emphasize that the work described here is not the ultimate goal we hope to achieve but only a level at which we feel a progress report is warranted. Consequently, although no attempt has been made to present the computer software in formal style or optimize every DO loop, there is sufficient information in this report that any research group should be able to reproduce our results given a comparable computer and comparable data.
III. THE MEDICAL PROBLEM

The human body is a highly complex dynamic organism consisting of a multitude of structural components composed of varying materials. These components change in dimension through normal flexing, pulsations, and growth, as well as through disease processes. Body organs may also increase or decrease in density due to normal or abnormal metabolism.

The physician, dentist, or other health professional is concerned with diagnosing, treating, and, if possible, preventing disease. The early and correct diagnosis of a possible or current ailment is absolutely essential for intelligent medical treatment. Present diagnostic procedures are no longer exclusively dependent upon subjective symptomatology. Mechanical aids can be used to assist the doctor in establishing norms and detecting abnormalities. The use of X-ray photography, or roentgenography, permits the visualization of the internal structures of the body without surgical dissection.

The standard X-ray image, whether recorded transiently on a fluorescent screen, or permanently on TV tape or photographic film, exhibits light and dark contrasts of internal organs. In essence, the X-ray image is a projection (or shadowgraph) of a three-dimensional volume onto a two-dimensional plane, as in Fig. 3.1. The light intensity of the X-ray image depends on the transparency or opacity (at X-ray wavelengths) of all the matter encountered along the line of projection through the object. The darker portions of the roentgenogram indicate areas with the greatest transparency. Areas of lesser transparency or greater opacity appear proportionately lighter.
Prior to the advent of computer technology and the application of mathematics to medicine, the information derived from the intensity variations recorded on the X-ray film was interpreted by subjective human experience. The medical significance of areas of varying brightness on the X-ray was learned over the years through the recording and analysis of case histories. Comparisons between the X-ray images and the actual anatomic structures were made through dissection and autopsy. Eventually, correlations were found between changes in tissue density seen in the roentgenogram and the disease processes as they actually occurred. This work was facilitated by standardizing the direction in which a particular "view" was to be made of a patient.

Today a body part is oriented in a specific prescribed manner with respect to the X-ray source and the image recorder (film plate, TV camera, etc.). The diagnostician, from years of practice, learns the "normal" anatomy for that particular view and can then detect abnormal configurations (e.g., foreign bodies, calculi, fractured bones, tumors, congenital malformation, and bone degeneration).

Consider the problems involved in interpreting radiographs of the skull. The anatomical configuration of the skull derives from the interconnection of bones of various densities and structural composition, with large cavities and sinus spaces interspersed between the bones. In addition, the head contains tissues that are "soft" to the X-ray beam—such as the brain and blood vessels—where X-ray absorption is less and contrast differences are small. Since the X-ray image superposes in projection all the structures of all the different tissues and bone layers, the interpretation and visualization of subtle variations in soft tissue contrast
are hampered by the large contrast differences of the surrounding bone structure. Such interpretation by even trained radiologists is extremely difficult even when many different X-ray views are examined. A similar problem exists in other areas of the body--such as the chest and the spinal column. Present partial solutions involve injecting radiographic dyes into vessels to enhance visibility of small vessels and soft tissue structures. Diagnosis of hard tissue structures which are superposed, as in the skull, requires taking several views of the organ and making a comparative study. The doctor must compose in his mind the information derived from a group of disconnected views. In some anatomic regions, the diagnostic information is present but cannot be interpreted because the superposition of structures is too complex.

Diagnostic information is also lost because a roentgenogram is a single image of an object. Thus there is no parallax or depth perception from which the 3-D structure of the object can be obtained.

Various approaches have been tried to improve radiography by reconstructing the 3-D structure of an object from radiographs. Stereoscopic radiographic images require a pair of radiographs, but these still present the object only from a single perspective. For every desired change in viewing perspective, a new pair of radiographs must be taken. This limits the versatility of stereoradiography when many viewing positions are needed.

To produce 3-D images with parallax, radiographic-holographic multiplexing may be used. This is a process in which radiographs of an object are taken with a beam perpendicular to the object's rotation axis, as the object rotates through 360° (McCrickerd and George, 1968; Shuttleworth
et al., 1969; Redman, 1969). The developed radiographs are then oriented about the object and rotated about the object's axis to produce a continuous multiangular radiographic presentation, (Hoppenstein, 1971), which may then be stored holographically (Jeong, et al., 1966). Although multiangular views become possible with holographic multiplexing, the X-ray images are projections, thus making difficult the separation of superposed objects in those areas of the body where the tissue of interest is completely surrounded by higher density material.

For enclosed tissue, tomography, a method to section a human organ radiographically, is used. A tomogram is produced by moving the X-ray source and the film at varying speeds over the subject in such a way that one plane through the subject is kept in focus while parallel planes of less interest above and below are blurred. While many experimental refinements have been made, (e.g., Garrison, et al., 1969; Miller, et al., 1971) the tomogram is essentially uni-angular (oriented in one direction), and a new tomogram with further exposure to the patient is required for each desired plane through the subject. Moreover, planes above and below the required section decrease the contrast in the section of interest.

Tomography would be highly useful for visualizing internal organs surrounded by other structures if serial sections could be made quickly and without re-exposing the patient for each section, and if the uni-angular non-parallax limitation could be removed. On the other hand, radiographic multiplexing would be useful in its multiangular capacity if it could penetrate and separate superposed objects. Thus the ideal radiographic display for the diagnostician would require a tomographic
multi-sectioning capability (but without the noise contributed by nearby sections), in combination with the multi-positional viewing capability of radiographic multiplexing. Further, all the data should be obtainable while exposing the patient to only a single series of radiographs. The creation of this multiangular-multiplexing sectional roentgenography becomes possible when large amounts of radiographic data (in digital form) can be analyzed with mathematically sophisticated image reconstruction techniques.

This report describes an image reconstruction technique for obtaining multiangular-multiplexing computer-composed sectional roentgenograms from a single series of ordinary radiographs. After analysis of the radiographs, the computer draws on an oscilloscope successive slices of the object. These are computer-produced internal density maps or sectional roentgenograms of the object. The slice-by-slice sectioning of the object may be recorded on microfilm for playback on a frame-by-frame or motion picture format. Without further exposure to the patient, computer-produced sections of an organ may be derived for any plane, any angle of orientation, and with any thickness per slice. When axial sections (planes containing the rotation axis) are shown rotating at motion picture speed, multiplex radiographs are produced of one distinct layer of the object (rather than of its total volume). By mathematically rotating the object and sectioning slice by slice in planes perpendicular to the rotation axis (or any other angulation desired), multiple serial tomograms are produced. This computer sectioning technique thus combines the best qualities of multiplexing radiography with that of tomography.

Presently available image reconstruction devices, such as the EMI
and ACTA scanners, derive one section from each radiographic series to the patient. Our technique, on the other hand, derives entire sets of serial sections and provides multi-orientational serial sectioning, all from a single radiographic series.

IV. GEOMETRICAL AND MEDICAL TERMINOLOGY

Because the present report includes both medical and mathematical terms, we have inserted this section for reference.

We construct a right-handed orthogonal coordinate system with the z-axis pointing upward out of the top of the human head, the x-axis pointing from the left ear to the right ear, and the y-axis pointing in the direction "straight ahead." We denote the mutually perpendicular xy, xz, and yz planes and all the planes parallel to them as follows:

- Frontal sections--parallel to xz plane
- Sagittal sections--parallel to yz plane
- Transverse or cross-sections--parallel to xy plane.

In medicine, a plane is called a section and is commonly visualized as a thin slice. Serial sections are consecutive parallel thin slices of an object. The rotation axis is used as a reference in taking radiographs at different azimuthal angles and in deriving the computer-drawn sections. An axial section is any plane containing the axis of rotation.

A coordinate system with reference planes such as those just described can be set up for separate organs as well as for the entire organism. Thus the mid-sagittal section through the head (and body) differs from the mid-sagittal section of a particular tooth. To avoid confusion, the axial section corresponding to the mid-sagittal plane
of the entire individual (the yz plane between the eyes) will be called the mid-sagittal axial plane. The axial section corresponding to the mid-sagittal plane of a sub-organ (e.g., the tooth) will be called the sagittal-axial plane of the sub-organ. Thus in any individual, a separate sagittal-axial plane is possible for each organ, but the individual as a whole has only one mid-sagittal axial plane.

For each organ, some convenient transverse plane (perpendicular to the rotation axis) is chosen as a reference (z = 0) plane and called the equatorial plane. The computer-derived transverse sections are numbered consecutively from the equatorial plane, with numbers increasing along the positive z-axis.

Azimuth is the angle between two axial planes fixed to the object, with one of the axial planes being the reference with azimuth 0°. The 0° (180°) azimuthal plane of an object is some convenient axial section which is set perpendicular to the radiographic film plane before beginning the radiographic series. The 90° (270°) azimuthal plane is the axial section perpendicular to the 0° azimuthal plane.

Rotation of the object is clockwise when viewed from above (that is, when viewed from the anterior or cerebral aspect). If the sagittal-axial plane is chosen to be the 0° azimuthal plane in the case of the human head, then a 45° azimuthal plane would pass through the left eye and right ear (approximately). Note that (with a fixed X-ray beam and film holder) the coordinate system effectively rotates counterclockwise as the azimuthal angle increases.

Longitude defines the orientation of the film plane and varies as does the azimuth between 0° and 360°. Our convention is that whenever
the X-ray beam enters the object from the direction of azimuth $\theta^\circ$, the film plane (which is perpendicular to the beam) is at longitude $\phi^\circ$. Longitude is convenient for labelling X-ray projections (Chapter VI) whereas azimuth is convenient for the mathematical analysis (Chapter VIII).

If for the human head the $0^\circ$ ($180^\circ$) azimuthal plane is the mid-sagittal axial plane, then a radiograph taken at $0^\circ$ longitude shows a frontal section in which the nose-to-film distance is greatest (that is, the nose is pointing toward the X-ray beam source). Then $90^\circ$ longitude would correspond to a lateral radiographic view in which the left ear was farthest from the film (the conventional right lateral view).

[Note: Each organ of the body will require a choice of rotation axis, equatorial plane, and $0^\circ$ longitude based on the organ's unique structure; these must be defined in every case.]

Because of differences in anatomic configuration, the conventions in rat anatomy differ somewhat from those described earlier for humans. The x-axis is the same as before (left ear to right ear), but the z-axis (also the rotation axis) now points "straight ahead" in the viewing direction (parallel to the spine or long axis), and the y-axis points from the top of the head downward (perpendicular to the long axis).

With these conventions (Fig. 4.1), the transverse sections of the rat head contain information similar to the frontal sections of the human head, and the frontal sections of the rat head are analogous to the transverse sections of the human head. The choice of rotation axis and reference planes must be defined for each individual species.

Since the computer program provides serial sectioning at any angulation, one further plane must be defined—an oblique plane. An oblique
plane will be defined as a plane intersecting the rotation axis, but not at right angles to it. To define an oblique plane, we must specify:

1) the altitude, that is, the angle between the oblique plane and the equatorial plane,

2) the azimuth corresponding to the direction of greatest positive vertical inclination, and

3) the point of intersection of the plane and the rotation axis.

When viewing a radiograph the following information is basic for understanding what is seen. A completely unexposed film appears translucent or "white." A radio-opaque object which completely blocks a given radiation dose from exposing the film will also appear white. A completely exposed film will appear black, as will a radiolucent substance (for practical purposes--air). Metallic objects will appear white, bone will appear a light gray to white, and soft tissue will appear in darker shades of gray. Unobstructed areas of the film will appear black.

Figure 4.2 is a radiograph of a mandibular molar tooth (i.e., from the lower jaw) shown inverted from its normal orientation in the mouth. Anatomically, the tooth consists of the biting or occlusal surface and the exposed sides which form the crown. The surface layer of material is composed of hydroxyapatite crystals and organic matter and is called enamel—the hardest structure in the human body. The neck or narrow portion of the tooth forms the cemento-enamel junction. The root surfaces are composed of cementum, a bone which is softer and more radiolucent than enamel. The rest of the tooth is composed of dentin, which is even softer. The heart of the tooth, the pulp chamber, contains the pulp (nerve and blood
vessels) which is nearly radiolucent. The root canals are hollow structures containing a blood and nerve connection between the pulp chamber and the rest of the body. There are two roots in mandibular molars and three in maxillary (upper) molars. The hooked root in Fig. 4.2 is the mesial root (closest to the front of the mouth), and the straighter root (on the right of the radiograph) is the distal root (closest to the back of the mouth). The tips of the roots are the apices of the tooth and are at the apical (root) end of the tooth. The lingual side of the tooth is against the tongue while the buccal surface is against the cheek.

In this technical report, the "axis" of the tooth will be defined as a line running occluso-apically and oriented in the center of the occlusal (or x,y) plane (radial midpoint of the biting surface).

V. THE SIGNIFICANCE OF RADIOGRAPHIC DATA

A radiograph records the radiation intensity which emerges from different points of an object when a beam of radiation is passed through the object in a certain direction. The recording of a radiograph involves two considerations: (1) the interaction of the radiation with the object, and (2) the characteristic response of the film, fluoroscopic screen, or other recording device, to the transmitted radiation.

A beam of radiation can be regarded as a stream of high energy photons, all moving rapidly in essentially the same direction. When such a beam passes through an object, a certain percentage of the photons is scattered or absorbed by the object while the remainder passes through unaffected. A radio-opaque object absorbs or scatters almost all the incoming photons; a radiolucent object transmits almost all photons along their original paths.
The fraction of photons transmitted through the object depends on a quantity called the linear attenuation coefficient (LAC)—with a higher LAC corresponding to a smaller fraction of photons transmitted. Thus air, which is nearly radiolucent, has a very low LAC, while metallic objects, which are nearly radio-opaque, have much higher LAC values. A uniform object of LAC unity transmits $e^{-1}$ (or about 37%) of the incident photons through a unit distance. Similarly, for a beam travelling a distance $x$ through a uniform material of LAC $y$, the fraction of photons emerging is $e^{-xy}$. The numerical values of the LAC depend on the choice of unit distance, which is usually standardized to 1 cm. The choice of a different unit distance would require correction of numerical values by a constant factor. In medical applications, however, the particular normalization for the LAC is usually immaterial because relative differences in density permit structures within an object to be distinguished. In practice, we have used a variety of normalizations for different reconstruction algorithms and data sets.

An incident beam of intensity $I_o$ travelling a distance $x$ through an object of uniform LAC $y$, will emerge from the object with intensity

$$I = I_o e^{-xy}$$

so that

$$\ln \left( \frac{I_o}{I} \right) = xy.$$

For an irregular object having a variable LAC, the product $xy$ is replaced by the integral of the LAC along the particular line of sight.

The above relation between the LAC and the fraction of radiation transmitted by the object is called the Lambert-Beer law. For the law
to be strictly valid we have to assume that the beam of radiation passing through the object is monochromatic (or that a given volume of the object transmits all wavelengths equally). In practice, an X-ray beam contains a distribution of wavelengths concentrated at some central value, so that to a first-order approximation the Lambert-Beer law appears to hold.

In reconstructing a cross section of an object, we start with data derived from radiographs. This data gives the integral of the internal density (LAC) for a large number of lines of sight passing through the section of interest. The purpose of a reconstruction algorithm is to obtain an estimate of the internal density (LAC) at each point within a section of an object. By assembling a stack of such cross sections, we can map the internal density at every point within the object. Since different LAC values imply different material densities and compositions, a map of the LAC variations within an object reveals the internal structure of the object. Thus the fraction of radiation transmitted through an object in different directions provides enough information to reconstruct the internal density (LAC) distribution within an object, and hence the internal structural configuration of the object.

The transmittance characteristics of an object must be determined from radiographic data. In the cases we have considered so far, this data was obtained by digitizing radiographs. For calibration, it was necessary to relate a given (digitized) film brightness to the amount of radiation incident upon the film during the exposure. To do this, we removed the object and took radiographs at different exposure times for a known beam intensity (thus generating different amounts of radiation) and measured the corresponding film brightness. [Note: We assumed that
the response of the film depended primarily on the amount of radiation (that is, the number of incident photons) so that, for example, the same amount of blackening of the film would result from 0.5 s exposure to a unit intensity beam as from 1.0 s exposure to a beam of half the unit intensity. Because of limitations of our instrumentation we could not test the validity of this assumption, however.

VI. EXPERIMENTAL PROCEDURE

An extracted molar tooth was chosen as a representative organ to test the feasibility of 3-D radiography by computer because it was small, easily obtainable, contained different densities of material, and had a simple anatomic configuration inasmuch as the root shape could be readily visualized in superposed position. Further, the human molar has been anatomically sectioned and documented many times in varied positions, so that its sectional configuration is well known, and its basic structure is familiar to dentists, physicians, and persons lacking specialized medical training.

The head of a rat was used in further studies to test resolution and applications of the technique. The rat head was chosen as an example because its more complicated configuration would allow us to estimate the capabilities of our computer technique in medical applications. Moreover, the rat head was small enough to fit on the same experimental device used for the molar in the second study.

In all the studies reported here, Kodak DF-58 periapical ultra-speed radiographic film was used with the GE dental X-ray unit model 11CE2-2.
1. INITIAL STUDY--MAXILLARY MOLAR

The extracted (3-rooted) upper molar was rotated about a central axis, and radiographed at equiangular intervals.

Since gradations of density were unknown, threaded brass pins were placed at the apices of the roots and at two points 180° apart at the neck of the tooth. These pins would be seen to change relative positions in succeeding films even if the resolution of the computer data was poor. Further, the pins would represent a maximum density value.

An inexpensive record player was used for a turntable, and a sheet of Keuffel and Esser polar coordinate graph paper, cut to the size and shape of a 45 rpm record, was taped to the turntable so that the center of rotation corresponded to the center of the graph paper (Fig. 6.1). A copper band was friction fitted over the turntable center pin. The occlusal surface of the prepared 3-rooted upper molar was luted to the copper band with red cake compound. The tooth-band-turntable assembly now rotated as a unit and the central axis of the tooth coincided with the center of rotation of the turntable.

The tooth was oriented so that the initial radiograph would most closely resemble a frontal plane of the tooth. This allowed the initial computer-derived axial section to be a frontal-axial section. Azimuth intervals of 10° were marked on the graph paper through 360° (in a clockwise direction as viewed from the top of the turntable).

A standard dental X-ray unit containing a long cone tubehead was used for the experimental procedure. The long cone (a 16-inch cylinder) was placed perpendicular to the axis of rotation of the tooth and aligned with the 0° longitude marked on the turntable. A device was
constructed to allow the placement of a new unexposed film at every $10^0$ azimuthal interval without changing the film angulation or the film-to-tube distance. This was initially solved by using a Rinn XCP* bitewing assembly in which the bitewing holder was reversed in position on the bitewing arm (Fig. 6.1). The assembly's ring aligner was taped to the long cone of the X-ray tubehead. This device permitted the changing of films without re-aligning the film-tube position.

Unfortunately, although the film-tube position was fixed, the long cone and the turntable were not connected. When a film was changed in the XCP holder the freestanding long cone shifted slightly in relative position to the turntable. This shift resulted in an apparent change in size and position of the tooth on the exposed X-ray film. The computer, however, was programmed to assume that the rotation axis of the tooth was in the same relative position with respect to the film and that no eccentricity or extraneous motions occurred.

To re-reference the film-to-tooth axis, a lead millimeter-by-millimeter grid was placed in front of the X-ray film. The grid acted as a means to measure and re-position the tooth on the film during data processing. This was only a partial solution, for the grid could only compensate for lateral changes. Any angular changes or object-to-film distance changes could not be corrected.

With the devices in position, a series of radiographs was taken at 70 kV, 15 mA, and 0.5 s exposure. The initial film was exposed at $0^0$ longitude. The turntable was rotated $10^0$, the film changed, and a new

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*Rinn Corporation, 1212 Abbott Drive, Elgin, Illinois, 60120.
film exposed. This process was repeated for every 10° azimuth through 180°. All films were processed by an automatic film developer.

The developed films were mounted in film holders, then sent to HAO. At HAO, the films were painstakingly hand digitized by V. Tisone. Unfortunately, no shade gradation guide (as described in Chapter V) was made for this initial study. Instead a simple linear relation was assumed to hold between the digitized film brightness and the corresponding radiation intensity. Later we found that the relation between film brightness and radiation intensity can be highly nonlinear (see Chapter X).

A sample of the computer results produced in November of 1972, is shown in Fig. 6.2. Figure 6.2 is an enlargement of the microfilm printout of a computer-calculated and oscilloscope-drawn axial map of the tooth, sectioned in a sagittal-axial plane at 90° longitude. The outline of the tooth is visible, and threaded pins are seen to project from either side of the neck of the tooth. The internal structures are not distinguishable. The maps from this first attempt have little value from a diagnostic standpoint because the resolution is poor. Compared with the radiograph of a lower molar in Fig. 4.2, the amount of information derived from the printouts is minimal.

Nevertheless, considering how many unknown variables were introduced by the lack of a shade gradation guide and by the shifting film-tubehead-turntable relationship, the results were still significant because they demonstrated the capability of the computer program to create axial sections of irregular objects. It was now necessary to perfect a more refined experimental device in which all mechanical variables would be controlled.
2. SECOND STUDY--MANDIBULAR MOLAR

A no-cost project was begun between the School of Aerospace Medicine and the High Altitude Observatory. An apparatus was made with considerably more rigidity and accuracy than in the previous study. A salvaged aviation device was used instead of the record turntable. The device was machined to rotate $1.5^\circ$ for every turn of a geared handle and was marked in $4.5^\circ$ increments along a circular edge (Fig. 6.3). A wooden disk was fastened to the device's center of rotation and an ivory cylinder, whose center was pre-drilled, was friction fastened to a nail driven through the center of the wooden disk. The ivory cylinder/disk assembly was rotated to the marked $0^\circ$ setting on the instrument. A thin layer of dental utility wax was placed on the top of the cylinder and a bead of wax was placed in the central fossa of the tooth. The lower molar was positioned on the cylinder with its 5 occlusal cusps resting on the cylinder surface. The sticky utility wax attached the tooth to the cylinder so that all components rotated as one. The entire device was fixed to a plywood baseboard.

From the previous experiment with the turntable, it was learned that the X-ray tube, turntable, and film must be exactly positioned—the tube perpendicular to both the long axis of the tooth and the film plane, and the tooth rotation axis and the film plane parallel to each other. Two 1 x 4 inch boards, each with a semicircular notch cut in one end, were nailed and cross braced to the baseboard in such a manner as to allow the X-ray tube cylinder, when resting in the notches, to align perpendicular to the sample's axis of rotation (Fig. 6.4). Rubber
bands across the top kept the tubehead seated in the semicircular notches. The film holder was constructed of two pieces of thin masonite which were grooved and shaped with a router so that when connected a slot was formed to hold the X-ray. The film holder was connected to the tube cylinder wood supports by masonite boards so that the bottom of the holder just rested against the wooden disk turntable (Fig. 6.3). X,Y adjustment gears on the turntable device allowed fine lateral adjustments for centering the tooth/turntable with the film and cylinder. Tooth/film distance could be adjusted by the turntable gear system. These X,Y adjustments were made and the gears locked to prevent further movement.

The final result was a rigid assembly in which a new film could be repositioned with no movement of the rest of the apparatus. The only movement possible was rotation of the tooth and ivory cylinder in equi-azimuthal increments.

The film-to-tooth and film-to-X-ray-source distances were adjusted as closely as possible to that encountered in a clinical situation with a patient. The tooth was oriented so that 0° longitude was a sagittal-axial plane. The lingual surface was placed closest to the film plane as in the clinical case. In this orientation a radiograph would be produced (Fig. 4.2) that closely resembles the ordinary periapical radiograph familiar to dentists, and the roots of the tooth would not superpose upon one another. The initial computer-derived map would also be a sagittal-axial section at 0° longitude.

The equatorial plane was defined for this tooth as being the top of the ivory cylinder which is in contact with the tooth. All subsequent transverse sections were numbered consecutively so that the
transverse plane containing the root tip had the highest number while the transverse section at the cusp tips had the lowest number.

After the apparatus was tested and the alignment verified radiographically, radiographs were made to determine the best exposure times, contrast, and penetration required. An automatic processor developed all the films in the study to ensure uniformity of shade contrast. It was determined that 75 kV, 15 mA, and 500 ms produced the optimum radiographs.

A series of radiographs was made as the tooth was rotated incrementally. A 75 kV, 15 mA, and 500 ms exposure was used. The initial radiograph was exposed at 0° longitude. It was removed from the film holder and its wrapper numbered with a crayon. A new film was positioned, the tooth rotated clockwise 1.5°, and the second film exposed. This process was repeated through 180° of rotation. Then the tooth was removed and a set of radiographs was made to create a shade-gradation guide.

All films were processed on the same day by a single automatic film processor. It was necessary to maintain the sequence of the films. This required placing the films into the processor on only one track at 2-minute intervals. Films were mounted in marked dental film holders as they were removed from the machine. The films were then sent to HAO where they were hand-digitized.

In June 1973 the Control Data 7600 computer processed the data and calculated transverse serial sections perpendicular to the rotation axis at intervals of 0.3 mm. The computer then processed the data to produce axial sections at 3° azimuthal increments over 180°. Sixty transverse sections and 60 axial sections were calculated.
These individual maps were drawn on an oscilloscope by the computer and the results were recorded on a microfilm strip. Figures 6.5 and 6.6 are the entire microfilm results. Figure 6.5 demonstrates axial sections in a clockwise rotation from $0^\circ$ to $180^\circ$. The tooth map in the upper left corner of Fig. 6.5 is the $0^\circ$ longitude axial section. Serial sections run down the columns. The five cusps in the upper left corner of Fig. 6.6 represent the first transverse section of the tooth just above the top of the ivory cylinder (the equatorial plane). Figures 6.7, 6.8 and 6.9 are enlargements of axial sections at $0^\circ$, $45^\circ$ and $90^\circ$ longitude, respectively. Figures 4.2, 6.10 and 6.11 are enlargements of the periapical radiographs at $0^\circ$, $45^\circ$ and $90^\circ$ longitude orientations respectively. Figures 6.12-6.16 show enlargements of a sequence of transverse sections proceeding from the occlusal surface of the tooth to the apices.

Comparing Figs. 4.2 and 6.7 ($0^\circ$ longitude), Figs. 6.10 and 6.8 ($45^\circ$ longitude), and Figs. 6.11 and 6.9 ($90^\circ$ longitude), we may observe the respective differences between the dental radiograph and the computer derived axial section. The radiograph represents a compression of the entire volume of the tooth onto a 2-D surface. The cusps of the tooth are superposed one upon the other and the enamel tooth surfaces, buccal and lingual, tend to block the view of the internal dentin. The computer drawn sagittal-axial section at $0^\circ$ longitude (Fig. 6.7) is a thin slice through the center of the tooth; superposed cusps are eliminated to reveal only the cusp present in this particular plane, and the mesial root canal appears to close in the axial section, indicating a bend out of the plane. In the radiograph, the root canals appear to be open, and the apex of the mesial root shows a bifurcation of the root canal.
(or two root canals) which is not seen in the computer axial section for $0^\circ$.

Rotating clockwise to $45^\circ$ (Figs. 6.8 and 6.10), the value of the axial sectioning becomes increasingly clearer. The superposed root in the radiograph (Fig. 6.10) is removed from the section, thus allowing clearer visualization. From the axial section (Fig. 6.8), we can assume a curvature in the distal root as it "disappears," having curved out of the 2-D plane of the section—a fact not indicated in the ordinary radiograph. The pulp and dentin are identifiable. The possibility of a carious lesion is noted by the drop in density in the dentin in the center of the film between the enamel and the pulp chamber. The location and height of the mesial-buccal pulp horn can be noted.

Rotating clockwise to $90^\circ$, the mesial root is hooked "toward" us in the axial section (Fig. 6.9). By examining axial sections Figs. 6.7, 6.8 and 6.9 this orientation may be determined. The radiograph (Fig. 6.11), gives no indication of the curvature of the root, while the frontal-axial section (Fig. 6.9) clearly indicates that the root is suspended in the center of the axial plane—meaning that it originates outside the plane. Two distinct root canal orifices become visible in the axial section. Figure 6.15 demonstrates the existence of three root canals in the tooth—two in the mesial and one in the distal root. The dentist knows three canals should be present, but to even the untrained, the number of canals can easily be determined by viewing the computer maps. These details are much harder to discern from the radiographs.

A few of the sections in Figs. 6.5 and 6.6 are enlarged for closer examination. Figures 6.12-6.16 show transverse sections of the tooth. Figure 6.12 demonstrates a section through the tips of the cusps just
above the ivory cylinder. The tooth is oriented at $0^\circ$ longitude, the buccal of the tooth to the bottom of the figure, the mesial to the left. The tooth cusp plane is being viewed from "above," i.e., from the apical or root end of the tooth. The computer serially sections "upward," (i.e., toward the apices) since the tooth is a lower molar that was inverted. The heavy computer "noise" in this section may in part be due to the wax holding the tooth to the cylinder. Figure 6.13 is a section cutting two mesial pulp horns. The tooth's roots bifurcate as seen in Fig. 6.14. As the root narrows the mesial root forms two root canals--a vital consideration in endodontic (root canal) therapy (Fig. 6.15). The apices of the root canals are located as in Fig. 6.16. Knowing the locations of the ends of the canals is of extreme importance when sealing the canal during endodontic treatment.

From these results, a side-by-side comparison of the radiographs and the computer sections allows even the casual observer to obtain far greater information from the computer maps. To the physician, dentist, and other health professionals, it clearly provides a tremendous potential increase in diagnostic information heretofore unavailable. This second study was a vast improvement over the initial experiments since the precision of both the apparatus and the programming was much greater. Computer time was reduced to a total of 8 minutes for all 60 views.

Visualization of both the root structure and the configuration of the pulp horns and canals was not as easy without the computer maps, and for comparable accuracy would require destructive physical sectioning of the tooth with tooth sectioning saws. Because the computer can reorient the sectioning plane at will--providing sagittal, frontal, axial, transverse
or even oblique serial sections, while exposing the subject to irradiation in only one actual series of radiographs, the non-destructive sectioning is highly important for morphologic studies. If the microfilm printout from the computer is recorded on motion picture film or TV tape, a multiplexing effect occurs which is truly dramatic, enhances diagnostic evaluation of the individual sections, and rivals holographic multiangular multiplexing. Since a limitless number of views may be calculated or interpolated by computer, and the computer may reorient the subject and take further views, an almost infinite number of multiangular-multiplexing computer-produced sectional radiographs is possible.

3. MULTIPLEXING THE MULTIANGULAR SECTIONAL ROENTGENOGRAPHS

The computer microfilm printout was in the form of 35 mm film strip. The 35 mm film was mounted in a special motion picture holder. Each 35 mm microfilm frame was photographed with 16 mm motion picture film for one second duration. The microfilm was then refilmed at one-quarter second duration. This procedure was repeated with the microfilm frames filmed in reverse sequence. When the 16 mm motion picture film was run through a standard projector, the viewer observed a multiplexing illusion of the axial planes rotating in sequence through 180°. The sequence slowed, and reversed, producing a dramatic effect. The multiplexing of the transverse sections by motion picture created the illusion that the viewer was traveling through the lower molar from cusps to apex just as X-rays do. Reversing and slowing the film produced equally dramatic results. The total internal information of the tooth being displayed in rapid serial sections allowed the mind to grasp far more vividly the content of the entity than heretofore
possible. Most important, the additional dimension of time was added. The film could be slowed, stopped, or reversed to permit comprehensive diagnostic study of selected regions or individual sections.

4. THIRD STUDY: THE HEAD OF A RAT

The undissected head of a previously euthanized laboratory white rat, fixed in formalin, was used as the radiographic subject. A semi-rigid plastic cylindrical container was used to hold the head. Two steel rings (serving as a density reference) were snugly seated within and on the bottom of the container. The rat head fit tightly against the cylinder walls and the top surface of one ring (Fig. 6.17). The floor of the container was fastened to the calibrated turntable (the same as that used in the mandibular molar study) by means of a pin through its center. The center of rotation of the turntable and cylinder coincided. The long cone of the X-ray unit, the film holder, and the turntable base were rigidly attached so that only the turntable rotated. The film holder was placed parallel to the rotation axis of the cylinder while the long cone was positioned perpendicular to it. A thin steel rod was attached to the film holder as a reference so that it would appear on the bottom of every exposed film parallel to the film edge.

The cylinder assembly was rotated so that the rat head was in a lateral aspect with its left ear closest to the film. This arbitrary initial position was defined as 0° longitude. Any plane passing through, and parallel to the center of rotation of the cylinder, was defined as an axial section. At 0° longitude the axial section parallel to the X-ray film plane very nearly corresponded to the mid-sagittal plane.
of the rat (i.e., a "slice" passing exactly between the eyes from the nose to the back of the head) (Fig. 4.1). In this study, the plane formed by the top of the steel ring within the cylinder was considered as being the equatorial plane, and all subsequent transverse planes were increasingly numbered so that the tip of the nose had the highest number value.

The cylinder assembly containing the rat head was rotated through 180°. At each 1.5° rotational increment, a new film was placed in the film holder and exposed. One hundred and twenty-one radiographs were exposed for the data set. A set of radiographs was also exposed to provide a shade guide. All films were then developed in an automatic film processor. Using equipment designed by the Mayo Foundation (Robb et al., 1973), the developed films were electronically scanned at 200 lines per film and converted to seven track computer tape format. The digitized densitometry data from the radiographs was analyzed by the Control Data 7600 computer at NCAR. The computer calculated internal density maps and displayed them on an oscilloscope. The oscilloscope images were photographed on 35 mm motion picture strip film as the computer drew them.

Over 450 multiangular section maps were calculated by computer. Each section map required about one second of peripheral computer time. Four types of maps were calculated and drawn: axial sections, one at each 1.5° orientation through 180° (Fig. 6.18); serial sagittal sections, from right to left ear (Fig. 6.19); serial frontal sections, ventral to dorsal or from chin to the top of the skull (Fig. 6.20); and transverse sections, caudal to cranial or from back of head to nose (Fig. 6.21).
The total length of the rat skull was 25 mm or about one inch. The individual tooth cusps were less than 0.3 mm wide. Resolution was lost in the scanning of the X-ray film during conversion to digital format, and in the production of the output 35 mm film. Since resolution on the output film was only about 0.5 mm, the resolution of the maps was not sufficient to separate the individual rat teeth. The computer-drawn sections have 17 gray levels, although over 250 gray-scale gradations were calculated by the computer. The prints shown here are several generations removed from the original computer-drawn film, resulting in some further loss of contrast and detail.

Figure 6.18 is an axial section of the rat at 60° longitude. The edge of the premaxilla is seen connecting the incisors and the maxilla. The mandible is present. The tympanic bulla appears like a spiral nebula just above the white-appearing angle of the occipital bone in the bottom of the map.

Figure 6.19 is a sagittal or lateral section about the level of the left ramus of the mandible (the large white circle), and the left middle ear (middle bottom). Note in Fig. 6.19 that the thin snout of the rodent is not present because it is outside the given plane. Subsequent lateral sections (not shown) serially progress to the other side of the head.

Figure 6.20 is a selected map of a frontal section as the computer serially sectioned ventral to dorsal (chin to top of skull). This view is at the level of the mandibular teeth.

The transverse section (Fig. 6.21) is oriented with the dorsal aspect (top of skull) on the left. Figure 6.21 maps a section of each ramus of the mandible (the two long disconnected bones top and bottom), and the zygomatic bones.
VII. MEDICAL APPLICATIONS

Medical applications of the computer technique described in this paper include detection, pinpoint location, and sectional analysis of organs, tumors, aneurysms, ulcers, fractures, calculi, bullets, needles, and blockages in digestive, circulatory, and respiratory tracts. Other possible applications are: millimeter-by-millimeter cross sectioning of alveolar bone and teeth for 3-D diagnostic mapping of caries and periodontal bone defects; transverse sectioning of the vertebrae for spinal injury and pathology; evaluation of arthritic TMJ (temporomandibular joint) and related bony structures; density changes in bone during growth, weightlessness, and stress. Applications may also include real-time positional analysis of probes such as endodontic depth gauges in teeth, or cryogenic probes in the brain and other organs. In the future a sectional analysis could lead to the computer analyzing the location of a brain tumor, directing a probe on a track outside the head to position itself automatically directly above the tumor, and directing the probe to reach the precise location of the tumor center. Radiation therapy devices could be aimed and focused on tumors in the same manner. The computer would determine the optimum beam paths to avoid radiation-sensitive organs.

Further development of computer programming and radiographic electronic preprocessing would allow the doctor to take a series of radiographs, and then immediately obtain computer sectional radiographs of an organ for any required orientations by entering the appropriate coordinates on a console. If the doctor was not satisfied with a particular orientation, he would merely ask the computer to assemble another set of serial sections, at a
new orientation, without any further radiation exposure to the patient. This would allow real-time viewing and precise placement of cryogenic probes and other objects in delicate surgical procedures, and even allow novel three-dimensional studies of the dynamic movements of the heart, larynx, oral structures, etc. Computer-derived topographic mapping of organs is also possible from the same initial series of radiographs. Sequential display of the sectional radiographs by television or motion picture, or by holographic techniques, would enhance visualization.

The resolution of the multiangular sectional radiographs presented in this report is of diagnostic quality. The tomograms were computed in a time frame that is both economical and practical. The large amounts of information that were processed demonstrate that a versatile improved mathematical programming technique has evolved for 3-D internal density mapping and interpretation. Computerized multiangular sectional radiography may solve certain problems of radiographic visualization encountered by the doctor. The distinct advantage of this technique is that it allows the entire area of interest to be scanned at one time, and sectioned continuously in any given direction so that no areas are missed. Any tumor whose size is within the resolution limit of the apparatus may be detected. Computer image contrast enhancement may allow the detection of early pathology presently missed by current radiographic surveys.

VIII. MATHEMATICAL OUTLINE

1. INTRODUCTION

In the experiments previously described, an object is rotated about an axis and radiographed with a beam of parallel X-rays at specified
azimuthal angles in such a way that each X-ray path (or line-of-sight) lies in one and only one plane perpendicular to the rotation axis. As a result, each plane perpendicular to the rotation axis provides a cross section (or cross-sectional slice) of the object which can be analyzed independently of the others.

An X-ray passing through an object diminishes in intensity according to the total amount of matter encountered along its path. Thus the parallel X-rays which traverse a specified cross section of the object (in the direction defined by the azimuthal angle) inscribe a one-dimensional intensity distribution on a radiograph perpendicular to the beam. For each azimuthal angle at which a radiograph was taken, there exists a one-dimensional intensity distribution which corresponds to a projection of the cross section in question. The set of such one-dimensional intensity distributions which appear on radiographs at different azimuthal angles provide data which can be used to determine the two-dimensional internal density (LAC) distribution of the corresponding cross section.

In this chapter we describe our algorithm to derive the internal density structure of a single cross section of an object when given radiographic projections of that section from different azimuthal directions.

Once the density distributions (or maps) are derived for a stack of cross sections, the computer program can reassemble the information and drawn internal density maps for slices of arbitrary orientation through the object (for example, frontal, sagittal, or axial sections).

2. THE MATHEMATICAL PROBLEM

On the x-y plane, let f be defined as a real non-negative function
which can be non-zero only in the circular domain of unit radius

\[ C = \{(x,y) : x^2 + y^2 < 1\} \] (8-2.1)

or (in polar coordinates)

\[ C = \{(r,\theta) : 0 \leq r < 1, \ 0 \leq \theta < 2\pi\} . \] (8-2.2)

Physically, \( f \) represents the internal density (or LAC) distribution of a cross section of the object, and corresponds to the unknown function we are trying to determine. We require that the entire cross section of the object, that is, all regions of non-zero X-ray absorption or scattering, lie within the domain \( C \), so that \( f \equiv 0 \) when \((x,y) \notin C\).

We define the projection function \( g \) so that each of its values \( g(s) \) is the integral of \( f \) along some specified X-ray path (or line of sight) \( s \) passing through \( C \), with the limits of integration determined by the boundaries of \( C \). More precisely, if we specify a ray path \( s \) by the position \((p,\phi)\) (in polar coordinates) of its closest approach to the center of \( C \), as in Fig. 8.1,* then we can write

\[ g(\rho,\mu) = \int_{-R}^{R} f(r,\phi) \, dx \] (8-2.3)

where

\[ R = (1 - \rho^2)^{1/2} \] (8-2.4)

\[ r = (\rho^2 + x^2)^{1/2} = \rho \sec \xi \] (8-2.5)

*This is equivalent to the characterization of \( s \) as the locus of points \((r,\theta)\) in \( C \) satisfying \( r \cos(\theta - \mu) = \rho \), if we use appropriate conventions for the case \( \rho = 0 \).
\begin{align*} 
\theta &= \mu + \xi \\
\xi &= \arctan(x/p) = \arcsin(x/(p^2 + x^2)^{1/2}) 
\end{align*}

(8-2.6) (8-2.7)

Physically, the projection function \( g \) for a given ray path \( s \), [that is, \( g(s) \) or \( g(\rho, \mu) \)] corresponds to the natural logarithm of the ratio of the entering and emerging intensities of a beam of radiation travelling through the object along \( s \) (in accord with the Lambert-Beer law).

A projection of \( f \) is defined as the (one-dimensional) distribution of \( g(\rho, \mu) \) and \( g(\rho, \mu + \pi) \) for some specified constant value of \( \mu \) as \( \rho \) varies over the interval \( 0 \leq \rho < 1 \). Thus for optically thin objects we need consider only projections over the domain \( 0 \leq \mu < \pi \). A finite set of projections (that is, a finite number of \( \mu \)) with intensities measured at a finite number of \( \rho \) in each projection constitutes the data set

\[ D = \left\{ g(\rho_i, \mu_j), g(\rho_i, \mu_j + \pi) : 0 \leq \rho_i < 1, 0 \leq \mu_j < \pi, i, j \in I^+ \right\} \]

(8-2.8) or equivalently

\[ D = \left\{ g(\rho_i, \mu_j) : 0 \leq \rho_i < 1, 0 \leq \mu_j < 2\pi, i, j \in I^+ \right\} \]

(8-2.9)

In other words the data set \( D \) is the set of known values of the projection function \( g \).

The mathematical problem is to determine the two-dimensional internal density (LAC) distribution \( f \), or some approximation to it, from the numerical data values of \( D \).

*An optically-thin object is one in which photons may be scattered or absorbed once but not twice. Thus photons may leave a beam but may not re-enter the beam after having left.*
3. THE STRATEGY

Having defined the mathematical problem, we can outline the strategy to be followed in the next several sections.

Section 4:

The unknown density function \( f(r, \theta) \) is represented by a (real) Fourier series in \( \theta \) with a set of unknown radial functions \( f_{mp}(r) \) as coefficients.

Section 5:

The known projection function \( g(\rho, \mu) \) is analyzed harmonically and expressed as a Fourier series in \( \mu \) with a set of (known) functions \( g_{mp}(\rho) \) as coefficients. Each value of \( g_{mp}(\rho) \), however, turns out to be simply an integral transform of the corresponding radial function \( f_{mp}(r) \).

Section 6:

Each unknown radial function \( f_{mp}(r) \) is approximated by a truncated series, more specifically, by a linear combination of radial basis functions \( f_{mpi}(r) \) which are chosen for convenience. Thus to approximate the unknown function \( f_{mp}(r) \) we need only determine the best set of constant coefficients for a linear combination of the known basis functions \( f_{mpi}(r) \).

Section 7:

Each known function \( g_{mp}(\rho) \) is expressed as a truncated series, corresponding to a linear combination of the functions \( g_{mpi}(\rho) \). Each function \( g_{mpi}(\rho) \) is simply an integral transform of the basis function \( f_{mpi}(r) \). Of importance is the circumstance that the constants for the series of \( g_{mp}(\rho) \) (in terms of \( g_{mpi}(\rho) \)) turn out to be the same as those
for the series of $f_{mp}(r)$ (in terms of $f_{mpi}(r)$).

Section 8:

Treating each harmonic $(m,p)$ separately, we apply a least-mean-squares technique to find the set of constant coefficients for that (truncated) linear combination of $g_{mpi}(\rho)$ which best represents the known function $g_{mp}(\rho)$. The series for $f_{mp}(r)$ is also determined thereby, since the set of constant coefficients is the same. Once the functions $f_{mp}(r)$ are known, the density function $f(r,\theta)$ can be readily synthesized.

Section 9:

We discuss our choice of truncation limits and our choice of basis functions $f_{mpi}(r)$ in the expansion of $f_{mp}(r)$ and show how uniform resolution is achieved in practice.

Section 10:

Finally, we discuss the evaluation of the line-of-sight integrals $g_{mpi}(\rho)$ at each $\rho$ by elementary integration and suitable averaging. These procedures greatly simplify the calculation.

To summarize, we seek the set of constant coefficients which provides the best series representation of $f(r,\theta)$ (in terms of some chosen set of basis functions) consistent with the given data set D.

4. SERIES REPRESENTATION OF THE DENSITY DISTRIBUTION $f(r,\theta)$

The unknown density function $f(r,\theta)$ expressed as a real Fourier expansion in $\theta$ is

$$f(r,\theta) = \sum_{k=1}^{K} f_{mp}(r) E_{p}(m\theta) \quad (8-4.1)$$
where

\[ k \in \mathbb{I}^+ = \text{set of positive integers} \quad (8-4.2) \]

\[ m = \lfloor k/2 \rfloor = \text{the greatest integer not exceeding } k/2 \quad (8-4.3) \]

\[ p = k - 2m \quad \text{so that } p \in \{0,1\} \quad (8-4.4) \]

\[ E_p(m\theta) = \begin{cases} 
\sin m\theta & \text{if } p = 0 \\
\cos m\theta & \text{if } p = 1 
\end{cases} \quad (8-4.5) \]

and the unknown radial functions \( f_{mp}(r) \) are still to be specified.

5. SERIES REPRESENTATION OF THE PROJECTION FUNCTION \( g(\rho, \mu) \)

Inserting Eq. (4.1) into Eq. (2.3), we get

\[
g(\rho, \mu) = \int_{-R}^{R} \left[ \sum_{k=1}^{\infty} f_{mp}(r) E_p(m\theta) \right] dx \quad (8-5.1)
\]

If we now interchange the summation and integral signs, and use the identities

\[
\sin m\theta = \sin(m\xi + m\mu) = \sin m\xi \cos m\mu + \cos m\xi \sin m\mu \quad (8-5.2)
\]

\[
\cos m\theta = \cos(m\xi + m\mu) = \cos m\xi \cos m\mu - \sin m\xi \sin m\mu \quad (8-5.3)
\]

and recognize that symmetry demands

\[
\int_{-R}^{R} f_{mp}(r) \sin m\xi \, dx = 0 \quad (8-5.4)
\]

\[
\int_{-R}^{R} f_{mp}(r) \cos m\xi \, dx = 2 \int_{0}^{R} f_{mp}(r) \cos m\xi \, dx \quad (8-5.5)
\]
we find
\[ g(p,') = k_E p(p) E_p(m) \]  \hspace{1cm} (8-5.6)

where
\[ g_{mp}(p) = 2 \int_0^R f_{mp}(r) \cos m\xi \, dx \]  \hspace{1cm} (8-5.7)

Since \( p \) is constant along the ray path, we can write
\[ g_{mp}(p) = 2 p \int_0^\phi f_{mp}(\rho \sec \xi) \cos m\xi \sec^2 \xi \, d\xi \]  \hspace{1cm} (8-5.8)

where
\[ \phi = \arccos \rho = \arctan \left[ \frac{(1-\rho^2)^{1/2}}{\rho} \right] \]  \hspace{1cm} (8-5.9)

Thus Eq. (5.6) is a Fourier analysis of the projection function \( g(p,\mu) \).

The function \( g_{mp}(\rho) \) can be determined from Eq. (5.6), since \( g(p,\mu) \) is
given by the data. Equation (5.7) or (5.8) shows that each value of
\( g_{mp}(\rho) \) is an integral transform of \( f_{mp}(r) \), more specifically the integral
of \( f_{mp}(r) \cos m\xi \) along the line of sight, \( (\rho,0) \).

6. EXPANDING THE RADIAL FUNCTIONS \( f_{mp}(r) \)

We now approximate each radial function \( f_{mp}(r) \) with a truncated
series corresponding to a linear combination of radial basis functions
\( f_{mpi}(r) \). The basis functions \( f_{mpi}(r) \) are chosen for convenience, and
are linearly independent and orthogonal. Several types of basis
functions have been used in the past. At present we are using pulse
functions, which will be described in Section 9. Then,
f_{mp}(r) = \sum_{i=1}^{I_m} c_{mpi} f_{mpi}(r) \quad (8-6.1)

where $I_m$ is a parameter depending on $m$. The constant coefficients are unknown and will be determined in Section 8 by a least-mean-squares technique.

7. EXPANDING THE HARMONIC COMPONENTS $g_{mp}(\rho)$ OF THE PROJECTION FUNCTION

Substituting Eq. (6.1) into Eq. (5.7) or (5.8) and defining the functions $g_{mpi}$ by

$$g_{mpi}(\rho) = 2 \int_0^R f_{mpi}(r) \cos m\xi \, dr \quad (8-7.1)$$

$$= 2\rho \int_0^\phi f_{mpi}(\rho \sec \xi) \cos m\xi \sec^2 \xi \, d\xi \quad (8-7.2)$$

we can write

$$g_{mp}(\rho) = \sum_{i=1}^{I_m} c_{mpi} g_{mpi}(\rho) \quad (8-7.3)$$

Since the set of coefficients $\{c_{mpi}\}$ is the same in Eq. (7.3) as in Eq. (6.1), the problem of approximating the functions $f_{mp}$ merely requires determining linear combinations of the $g_{mpi}$ which best approximate each function $g_{mp}$.

8. LEAST MEAN SQUARES TECHNIQUE TO DETERMINE THE SET $\{c_{mpi}\}$

An approximation to the function $g_{mp}$ in the form of Eq. (7.3) requires that the constants $c_{mpi}$ be chosen to minimize the norm
\[ E = \| g_{mp}^{*}(\rho) - \sum_{i=1}^{m} C_{mpi} g_{mpi}(\rho) \| \]  \hspace{1cm} (8-8.1)

where the \((L^2)\) norm

\[ \| F \| = \langle F, F \rangle^{\frac{1}{2}} \]  \hspace{1cm} (8-8.2)

is induced by the inner product

\[ \langle F_1, F_2 \rangle = \sum_{n=1}^{N} F_1(\rho_n) F_2(\rho_n) w(\rho_n) \]  \hspace{1cm} (8-8.3)

of any real functions \(F_1\) and \(F_2\) defined on the interval \([0,1]\) with \(0 < \rho_n < 1\) and \(w(\rho_n) > 0\). Our particular choices for the values of \(\rho_n, w(\rho_n)\), and \(N\) in Eq. (8.3) are made so that the inner product will approximate the integral

\[ \langle F_1, F_2 \rangle \approx C \int_{0}^{1} F_1(\rho) F_2(\rho) \, d\rho \]  \hspace{1cm} (8-8.4)

where \(C\) is a normalization constant. Suitable values for \(\rho_n, w(\rho_n), N,\) and \(C\) will be discussed later.

To find the set of coefficients \(\{C_{mp1}, C_{mp2}, \ldots, C_{mpm}\}\) which when inserted into Eq. (7.3) best approximates the function \(g_{mp}(\rho)\) in the sense of minimizing the norm \(E\) of Eq. (8.1), we solve the set of linear equations

\[ \frac{\partial E^2}{\partial C_{mpi}} = 0 \]  \hspace{1cm} (8-8.5)
which can be written in matrix form as
\[ Y = AX \] (8-8.6)

where \( X \) is the column vector containing the \( I_m \) unknown coefficients, with entries
\[ x_i = C_{mpi} \] (8-8.7)

\( Y \) is the column vector with the \( I_m \) entries
\[ y_i = \langle g_{mp}, g_{mpi} \rangle \] (8-8.8)

and \( A \) is the \( I_m \times I_m \) matrix with entries
\[ a_{ij} = \langle g_{mpi}, g_{mpj} \rangle \] (8-8.9)

The inner product defined by Eq. (8.3) or (8.4) may be interpreted as the usual summation (in the least-mean-squares technique) over the given data, in this case the projection data in the range \( 0 \leq p < 1 \).

The solution of Eq. (8.6) is then
\[ X = A^{-1}Y \] (8-8.10)

for nonsingular \( A \). Thus we have determined the coefficients \( C_{mpi} \) which provide the best (least-mean-squares) approximation for \( g_{mp}(\rho) \) in Eq. (7.3) and, at the same time, the best approximation for \( f_{mp}(r) \) in Eq. (6.1). Since the functions \( g_{mp}(\rho) \) are known from m-fold harmonic analysis of the observed projection function \( g(\rho, \mu) \), and since the \( f_{mpi}(r) \) are chosen basis functions from which the \( g_{mpi}(\rho) \) can be found,
the procedure of this section provides a means to estimate the unknown density function \( f(r, \theta) \) using Eqs. (8.7) through (8.10) in Eq. (6.1) and then applying Eq. (4.1).

We now backtrack to specify the basis functions \( f_{\text{mpi}}(r) \) and the truncation limits of the series.

9. **THE BASIS FUNCTIONS \( f_{\text{mpi}}(r) \), TRUNCATION LIMITS, ETC.**

Although several different types of basis functions have been used for the \( f_{\text{mpi}}(r) \), in particular Chebyshev polynomials of the first kind, we find that for computer applications a convenient set of basis functions are the simple step functions defined by

\[
f_{\text{mpi}}(r) = \Delta_i(r) = \begin{cases} 1, & r_i - \frac{1}{2L} \leq r < r_i + \frac{1}{2L} \\ 0, & \text{otherwise} \end{cases}
\]  

(8-9.1)

where

\[ 1 \leq i \leq L = \text{number of radial steps (or annuli)} \]  

(8-9.2)

\[ r_i = 1 - \frac{(i-\frac{1}{2})}{L} = \text{midpoint of the } i^{\text{th}} \text{ radial step} \]  

(8-9.3)

\[ 1/L = \text{width of the step} \]  

(8-9.4)

Note that step \( i = 1 \) is outermost and step \( i = L \) is innermost (so that \( i = L \) corresponds to a circular domain containing the origin). That is,

\[ r_1 > r_2 > \cdots > r_i > r_{i+1} > \cdots > r_L > 0 \]  

(8-9.5)

The quantities \( I_m, L, \rho_n, w(\rho_n), N, \text{ and } C \) which appear in Eqs. (6.1), (9.2), (8.3), and (8.4) are chosen so that

1) The resolution in the resulting density distribution is as
nearly uniform as possible, and is as high as possible consistent with uniformity.

2) The matrix equation (8.6) will not be ill-conditioned.

In this section we discuss the choice of $L$ and $I_m$; in the next section we determine optimal values of $\rho_n$, $w(\rho_n)$, $N$, and $C$.

Let

$$M = \text{the number of projections (or X-ray pictures) originally taken of the objected to be reconstructed.}$$  \hspace{1cm} (8-9.6)

Then we choose

$$L = \left[ \frac{M}{\pi} + 1 \right]$$  \hspace{1cm} (8-9.7)

$$m_i = \min(\lceil n r_i L \rceil, M-1)$$  \hspace{1cm} (8-9.8)

$$I_m = \text{maximum } i \text{ such that } m_i \geq m$$  \hspace{1cm} (8-9.9)

where $i$ and $L$ are related by Eq. (9.2), and

$$[x] = \text{the greatest integer not exceeding } x.$$  \hspace{1cm} (8-9.11)

Thus we have

$$M \geq m_1 > m_2 > \cdots > m_i > m_{i+1} > \cdots > m_L = \min(1, M-1).$$  \hspace{1cm} (8-9.10)

Criteria (9.7) to (9.9) ensure that the number of basis functions $\Delta_i(r) E_p(m \theta)$ available for representing density structures across any unit distance on the disk is approximately constant, and that the resolution of the resulting density distribution $f$ is as high as possible consistent with uniformity.
To see how the principle of uniform resolution leads to our choice of criteria, note first that in determining the density function \( f(r, \theta) \) from \( M \) projections, we cannot do better than to represent \( f(r, \theta) \) by a truncated Fourier series with \( 2M \) basis functions* in azimuthal angle \( \theta \). For uniform resolution, the number of basis functions per unit length must be the same in all directions. Since there are \( 2M/2\pi = M/\pi \) basis functions per unit length in the azimuthal direction at \( r = 1 \), and \( L \) basis functions per unit length in the radial direction, we set

\[
L = \text{integer near } \frac{M}{\pi} \quad (8-9.12)
\]

We choose criterion (9.7) so that \( L \) is never zero even for small \( M \).

Similarly, to maintain uniform resolution and avoid overfitting when expanding \( f_{mp}(r) \) by Eq. (6.1) where \( m \) is given, we must truncate the series at some step \( i = I_m \) of radius \( r_i \) (Eq. 9.3) so that

\[
\frac{2m + 1}{2\pi r_i} \approx L \quad (8-9.13)
\]

Equation (9.13) clearly holds when \( m \) is equal to the \( m_i \) defined by Eq. (9.8). Thus criterion (9.13) is satisfied if \( I_m \) is chosen so that \( m_i \approx m \). This is the reason for choosing \( I_m \) by criterion (9.9). In other words, for a given number \( M \) of projections, a specified \( m \), and uniform resolution, we cannot resolve features with characteristic dimensions smaller than \( 1/L \approx \pi/M \approx \pi r_i / m_i \), where \( r_i \gtrsim m_i / \pi L \). Thus

*Although \( 2M+1 \) basis functions are needed to include all harmonics through order \( M \), only \( 2M \) functions are recoverable because only \( 2M \) values of \( \mu \) can be obtained from \( M \) projections.
if we are given \( m \), we must truncate the series at the step \( i = I_m \) such that \( m_i \) is equal to or slightly greater than \( m \). The quantity \((M-1)\) can be less than or equal to \( \pi r_i L \) only when \( i = 1 \), that is, at the outermost annulus.

Table I gives numerical values of \( L, r_i, \) and \( m_i \) for various \( M \) by means of Eqs. (9.7), (9.3), and (9.8). Table II gives the \( r_i \) and \( m_i \) for \( M=120 \), and Table III shows (for \( M=120 \)) how the \( I_m \) are chosen with Eq. (9.9) and how well criterion (9.13) is satisfied.

10. GAUSSIAN INTEGRATION

If we combine Eqs. (4.1), (6.1), and (9.1), we obtain for the estimation of the density distribution

\[
f(r, \theta) = \sum_{k=1}^{2m_1+1} E_p(m\theta) \sum_{i=1}^{I_m} C_{mpi} \Delta_i(r),
\]

where \( m_1 \) is defined by Eq. (9.8) and (from Eq. (9.10)) is just the largest harmonic order in any summation of the form of Eq. (6.1).

In the series expansion of Eq. (10.1), the unit disk is divided into annular strips of width \( L/L \) centered on circles of radii \( r_i \). Within each annular strip, the density distribution is constant with respect to radial variation. Thus the radii \( r_i \) might seem a natural choice for the values of \( \rho_n \) at which the projection functions are observed. But with such a choice for the \( \rho_n \), the matrices \( A \) (Eq. 8.9) which must be inverted to determine the coefficients \( C_{mpi} \) become hopelessly ill-conditioned when the number of projections exceeds 25 or 30. Ill-conditioning means there exist almost linearly dependent rows of the
matrix $A$, or equivalently, that two or more equations (hyperplanes) of
the matrix equation (8.6) contain nearly the same information (so that
the hyperplanes are almost parallel). The ill-conditioning of $A$ (Eq. 8.9)
arises--when using the basis functions of Eq. (9.1)--because the integrals
of Eq. (7.1) are derived from the summation of many fluctuations along
the line of sight when $m$ is large and $p$ is close to 1. Therefore, for
precision in evaluating the $a_{ij}$ of Eq. (8.9) we must sum over a sufficient
number of $p_n$ so that the inner product defined by Eq. (8.3) is a good
approximation of the integral (8.4). We choose the normalization constant
$C$ of Eq. (8.4) in such a way that if the functions $F_1(p)$ and $F_2(p)$ are
both replaced with the same radial basis function $\Delta_i(r)$ and $p$ is replaced
with $r$, then the integral will equal unity. That is

$$\frac{1}{C} = \frac{1}{L} \int_{0}^{L} [\Delta_i(r)]^2 \, dr = \frac{1}{L} \quad (8-10.2)$$

or

$$C = L \quad (8-10.3)$$

Now, we must increase the number of $p$ values so that the values of
$a_{ij}$ and $y_i$--which must be determined by finite summations (Eqs. 8.3, 8.8,
8.9)--are more nearly what would be obtained by integrating over a con-
tinuum. The actual $p$ values used in the program are defined by

$$p_n = p_{ik} = r_i + \frac{X_{Gk}}{2L} \quad (8-10.4)$$

where

$n, i, k, G \in I^+ = \text{the set of positive integers} \quad (8-10.5)$
n = (i-1) G + k \quad \text{(8-10.6)}

and

x_{Gk} = k^{th} abscissa for Gaussian quadrature of order G \quad \text{(8-10.7)}

so that k ranges from 1 through G. Thus

\begin{align*}
1 & \leq k \leq G \quad \text{(8-10.8)} \\
-1 & < x_{Gk} < 1 \quad \text{(8-10.9)}
\end{align*}

and

\Delta_i(\rho_{jk}) = \delta_{ij} \quad \text{(8-10.10)}

Similarly, the weighting function w is defined by

w(\rho_n) = w(\rho_{ik}) = w_{Gk} \quad \text{(8-10.11a)}

where

w_{Gk} = k^{th} weight for Gaussian quadrature of order G \quad \text{(8-10.11b)}

normalized so that

\begin{align*}
\sum_{k=1}^{G} w_{Gk} &= 1 \quad \text{(8-10.12)}
\end{align*}

The summation limit N of Eq. (8.3) is defined by

N = \max(n) = \max(i) \cdot G = L \cdot G \quad \text{(8-10.13)}

For the present program G = 10 and the resulting matrices A are very well-conditioned.
Having achieved a well-conditioned matrix $A$ by increasing the number of $n_p$, we now note that since in the resulting density distribution any detail smaller than the radial step width $1/L$ is not significant (see Eq. 10.1), we can average the values of $g(\rho_{ik}, \mu)$ over the different values of $k$ for given $i$ and $\mu$. Then

$$g(\rho_{ik}, \mu) = g(\rho_{ij}, \mu) = g(r_i, \mu)$$  \hspace{1cm} (8-10.14)

for all $k$ and $j$ between 1 and $G$. This procedure also effectively averages the harmonic components $g_{mp}$ so that

$$g_{mp}(\rho_{ik}) = g_{mp}(\rho_{ij}) = g_{mp}(r_i)$$  \hspace{1cm} (8-10.15)

for all $k$ and $j$ between 1 and $G$, and there is only one value of the harmonic component $g_{mp}$ for each radius $r_i$.

The averaging procedure of Eq. (10.15) permits us to simplify the calculation by writing the $Y$-vector of Eq. (8.8) as the product of two matrices

$$Y = B_0 V$$  \hspace{1cm} (8-10.16)

so that

$$X = A^{-1} B_0 V$$  \hspace{1cm} (8-10.17)

where $V$ has entries

$$v_i = g_{mp}(r_i)$$  \hspace{1cm} (8-10.18)

and $B_0$ has entries

$$b_{ij}^{(o)} = \sum_{k=1}^{G} g_{mp}(\rho_{jk}) w_{Gk}$$  \hspace{1cm} (8-10.19)
In other words the elements of the $y$-vector, which from Eqs. (8.8) and (8.3) can be written as

$$y_i = \sum_{n=1}^{L} g_{m_p}^{(n)} g_{m_p}^{(n)} w^{(n)} , \quad (8-10.20)$$

can now be separated with the help of Eqs. (10.15) and (10.11) as

$$y_i = \sum_{j=1}^{L} \sum_{k=1}^{G} g_{m_p}^{(n)} g_{m_p}^{(n)} w_{G_k}$$

$$= \sum_{j=1}^{L} g_{m_p}^{(r_j)} \left( \sum_{k=1}^{G} g_{m_p}^{(n)} w_{G_k} \right)$$

$$= \sum_{j=1}^{L} b_{ij}^{(0)} v_j \quad (8-10.21)$$

Thus matrix $A^{-1}$ is an $I_m \times I_m$ square matrix, $B_o$ is an $I_m \times L$ matrix, $V$ is an $L \times 1$ column matrix, and $X$ is an $L \times 1$ solution matrix.

In the evaluation of $A$ and $Y$ of harmonic order $m$ by Eqs. (8.9) and (10.21) it is necessary to evaluate the projection functions $g_{m_p}^{(n)}$ at every position $\rho_{1k}$. If Eqs. (9.1) and (7.2) are combined we obtain

$$g_{m_p}^{(n)}(\rho) = 2\rho \int_{0}^{\phi} \Delta_{i}(\rho \sec \xi) \cos m \xi \sec^2 \xi d\xi \quad (8-10.22)$$

$$= \begin{cases} 
2\rho \int_{0}^{\phi/1} \cos m \xi \sec^2 \xi d\xi, & \rho < r_i + \frac{1}{2L} \\
0, & \rho \geq r_i + \frac{1}{2L} 
\end{cases} \quad (8-10.23)$$
where

\[
\phi_{i1} = \begin{cases} 
0, & \rho \geq r_i - \frac{1}{2L} \\
\arccos \left( \frac{\rho}{r_i - \frac{1}{2L}} \right), & \rho < r_i - \frac{1}{2L}
\end{cases}
\]  

(8-10.24)

and

\[
\phi_{i2} = \arccos \left( \frac{\rho}{r_i + \frac{1}{2L}} \right) 
\]  

(8-10.25)

as is clear from Fig. (8.2).

Note that the value of \( g_{mpi} \) of Eq. (10.23) is independent of \( \rho \), (that is, of whether we use sin or cos), which means that the matrix \( A^{-1}B_0 \) is also independent of \( \rho \) and depends only on the harmonic order \( m \).

Equation (10.23) can be evaluated by elementary integration, thereby saving much computing effort in determining \( A \) and \( Y \). Using the identities

\[
\cos (\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta 
\]  

(8-10.26)

with

\[
\alpha = (m - 2)\xi 
\]  

(8-10.27)

\[
\beta = 2\xi 
\]  

(8-10.28)

and

\[
\cos 2\xi = 2 \cos^2 \xi - 1 
\]  

(8-10.29)

we find

\[
\cos m\xi = 4 \cos^2 \xi \cos [(m-2)\xi] - 2 \cos [(m-2)\xi] - \cos[(m-4)\xi] 
\]  

(8-10.30)

If we substitute this identity in Eq. (10.23), we obtain a recursion relation valid for \( m \geq 4 \), namely
\[ g_{m\pi}(\rho) = 8\rho \frac{\sin \left(\frac{(m-2)\phi_{i2}}{m-2}\right) - \sin \left(\frac{(m-2)\phi_{i1}}{m-2}\right)}{m-2} - 2g_{(m-2)\rho_i}(\rho) - g_{(m-4)\rho_i}(\rho) \]  

(8-10.31)

when \( \rho < r_i + \frac{1}{2L} \). Thus if we determine the values of \( g_{m\pi}(\rho) \) for \( m = 0, 1, 2, 3 \), we can then find the values of \( g_{m\pi}(\rho) \) for all other \( m \geq 4 \).

When \( m=0 \), we substitute the indefinite integral

\[
\int \sec^2 \xi \, d\xi = \tan \xi \tag{8-10.32}
\]

in Eq. (10.23) and obtain for \( \rho < r + \frac{1}{2L} \)

\[ g_{0\pi_1}(\rho) = 2 \left( \sqrt{\frac{r^2(2) - \rho^2}{r^2(1) - \rho^2}} \right) \]  

(8-10.33)

where

\[ r(1) = \max \left\{ r_i - \frac{1}{2L}, \rho \right\} = \rho \sec \phi_{i1} \]  

(8-10.34)

\[ r(2) = r_i + \frac{1}{2L} = \rho \sec \phi_{i2} \]  

(8-10.35)

Similarly, when \( m=1 \) we substitute the indefinite integral

\[
\int \cos \xi \sec^2 \xi \, d\xi = \int \sec \xi \, d\xi = \ln \left( \frac{1 + \sin \xi}{\cos \xi} \right) \tag{8-10.36}
\]

in Eq. (10.23) and obtain for \( \rho < r_i + \frac{1}{2L} \)

\[ g_{1\pi_1}(\rho) = 2\rho \ln \left( \frac{\cos \theta_1 + \sin \theta_2}{\cos \theta_1 - \sin \theta_2} \right) \]  

(8-10.37)
where

\[ \theta_1 = \frac{1}{2} (\phi_{i1} + \phi_{i2}) \]  
\[ \theta_2 = \frac{1}{2} (\phi_{i2} - \phi_{i1}) \]  

(8-10.38) (8-10.39)

When \( m=2 \), we use the identity (10.29) in Eq. (10.23) to obtain

\[ g_{2\pi_1}(\rho) = 4\rho (\phi_{i2} - \phi_{i1}) - g_{0\pi_1}(\rho) \]  

(8-10.40)

When \( m=3 \), we use the identity

\[ \cos 3\xi = 4 \cos^3 \xi - 3 \cos \xi \]  

(8-10.41)

in Eq. (10.23) to obtain

\[ g_{3\pi_1}(\rho) = 8\rho (\sin \phi_{i2} - \sin \phi_{i1}) - 3 g_{1\pi_1}(\rho) \]  

(8-10.42)

We have thus completed the evaluation of the projection functions by elementary integration using Eqs. (10.33), (10.37), (10.40), (10.42) and the recursion equation (10.31). Because the integrals of Eq. (10.23) do not have to be evaluated by numerical integration, the matrices \( A \) and \( B_0 \) can be very rapidly assembled.

Moreover, with our choice of radial basis functions (Eq. 9.1), the values of \( g_{mp_i}(\rho_n) \) of Eq. (10.23) are independent of \( \rho \), and thus depend only on the indices \( m, i, \) and \( n \). To compute the unknown density distribution of an object, we first use the stored values of \( g_{mp_i}(\rho) \) to calculate and store the matrix \( A^{-1}B_0 \) for each different harmonic order \( m \). Then to find the unknown density function \( f(r,\theta) \) for each slice of the object we (1) read in the vector \( V \) for each harmonic of the data, (2) assemble \( X \) by Eq. (10.17), (3) use the components of the vectors \( X \) to
find the functions $f_{mp}(r)$ with Eq. (6.1), and (4) assemble $f(r,\theta)$ with Eq. (4.1).

**IX. COMPUTER PROGRAM**

1. **THE TECHNIQUE: PROGRAM IMPLEMENTATION**

Program SECTION reconstructs a three-dimensional object from a set of projections (X-ray pictures of the object, digitized and processed according to Lambert-Beer's law). The projections are taken at equally spaced angles as the object is rotated through $180^\circ$. The plane of each projection is parallel to the rotation axis. Parallel (non-diverging) geometry is assumed for the lines of sight in each projection.

In the description which follows, bracketed numerals refer to card numbers of locations in the program where the corresponding operations described in the text are performed. A sequence of cards is indicated by a hyphenated pair of numerals--thus the extent of the entire program is [1-485]. The Appendix provides a listing of the program; card numbers are in the left hand margin.

By Eq. (8-10.17) a set of solution coefficients, the vector $X$, is evaluated by a matrix-vector multiplication. The vector $V$ in this multiplication is determined from the observed projection function $g$ by $m$-fold harmonic analysis, and thus will vary from section to section as the object is reconstructed. On the other hand the matrix $B$ defined by $B=A^{-1}B_0$, and which thus appears in Eq. (8-10.17), is independent of the observed projection function $g$. Thus its value can be computed, stored and used repeatedly as different sections of the object are reconstructed.
In the program the B matrices for the different harmonic orders \( m \) are computed in subroutine FIRST and stored in large-core memory (LCM). Then in subroutine RECON these matrices are recovered from LCM and used repeatedly to reconstruct the different sections of the object.

The first major operation in subroutine FIRST is to determine Gaussian coefficients corresponding to the \( x_{Gk} \) and \( w_{Gk} \) (Eqs. (8-10.7), (8-10.11b)) for \( G=10 \). (In the program \( G \) is denoted by MXNT.) The quantities \( x_{Gk} \) and \( w_{Gk} \) are evaluated from declared data and stored in the arrays RVL and WVL respectively [52-54]. The normalization used in the program is the following

\[
RVL(k) = \frac{1 + x_{Gk}}{2} \quad (9-1.1)
\]

\[
WVL(k) = w_{Gk} \quad (9-1.2)
\]

Next the program reads certain parameters from the data set describing the object to be reconstructed [55]. These parameters are as follows:

- \( NVIEWS = \) Twice the number of projections originally taken of the object (=\( 2M \)) \quad (9-1.3)
- \( IRMAX = \) The parameter \( L \) of Eq. (8-9.7) \quad (9-1.4)
- \( NSECT = \) The number of sections of the object represented by the data \quad (9-1.5)
- \( IPAK = \) The number of 60-bit computer words occupied by \( (NVIEWS) \) data words packed in 8-bit format \quad (9-1.6)

For best results the value of \( IRMAX \) should be approximately what would be determined from \( NVIEWS (=2M) \) by Eq. (8-9.7), so that

\[
IRMAX \approx \frac{NVIEWS}{2\pi} + \frac{1}{2} \quad . \quad (9-1.7)
\]
The program will still execute if IRMAX departs greatly from the relation suggested by Eq. (1.7), but if IRMAX is too large the resolution in the reconstructed sections will be highly non-uniform.

The acceptable limits for the four parameters are:

- \( \text{NVIEWS} - 1\text{-}260 \)
- \( \text{IRMAX} - 1\text{-}41 \)
- \( \text{NSECT} - \geq 1 \) (no upper limit)
- \( \text{IPAK} - 1\text{-}35 \)

This means that the maximum number of projections the present program can handle is 130 and the maximum number of points per digitized video line of the object is 82. If the above limits are exceeded, a diagnostic is printed and execution terminates.

The next major step is to determine the quantities \( I_m \) (Eq. 8-9.9). These are found by first defining an array MAXM, implementing Eq. (8-9.8) [72-73]:

\[
\text{MAXM}(i) = m_i + 1
\]

The values of \( I_m \) are then computed by implementing Eq. (8-9.9), and stored in the array NRAD [74-77]:

\[
\text{NRAD}(M) = I_m
\]

where

\[
M = m + 1
\]

Meanwhile the total number of all entries in all the B matrices is computed and stored in NSUM [76]:
Thus NSUM is the number of values which must be ultimately written in LCM by subroutine FIRST. To insure that the B matrices will be written in LCM and not on a slow-access disk file, the desired LCM space is reserved with (dummy) BRANWT's [78-80].

The program next computes the values of \( g_{mpj}(\rho_{ik}) \) for \( m = 0,1,2, \) and 3, implementing Eqs. (8-10.33), (8-10.37), (8-10.40), and (8-10.42), respectively, for all the different values of \( \rho_{ik} \) (Eq. 8-10.4). Since the value of \( g_{mpj}(\rho_{ik}) \) is independent of \( p \), only one value is stored for each pair of \( p \) values. Moreover by Eqs. (8-10.4) and (8-10.23), \( g_{mpj}(\rho_{ik}) \) is zero whenever \( \rho_{ik} > r_j + \frac{1}{2L} \), or \( j > i \), so only those values of \( g_{mpj}(\rho_{ik}) \) are stored for which \( j \leq i \).

In the DO loop, [81-97], values of \( g_{mpj}(\rho_{ik}) \) are computed. In this loop the order of variation of the indices, from fastest to slowest, is \( j, i, m, \) and \( k \). (Since \( g_{mpj} \) is independent of \( p \), index \( p \) never appears.) The computed values of \( g_{mpj}(\rho_{ik}) \) are stored in the array FNT. Meanwhile, values of \( \sin \phi_{j2} \) and \( \cos \phi_{j2} \), where \( \phi_{j2} \) is defined by Eq. (8-10.25), are computed and stored in both the arrays SCA and SCM. Later the arrays FNT, SCA, and SCM will be used in a recursive scheme to compute values of \( g_{mpj}(\rho_{ik}) \) for \( m \geq 4 \) (as in Eq. 8-10.31).

When all values of \( g_{mpj}(\rho_{ik}), \sin \phi_{j2}, \) and \( \cos \phi_{j2} \) for a given value of \( k \) have been computed and stored in the appropriate arrays, these arrays are copied for later retrieval [96-97], \( k \) is incremented, and the process

\[
NSUM = L \sum_{m=0}^{m_1} I_m = \text{IRMAX} \times \sum_{M=1}^{\text{MMAX}} \text{NRAD}(M) \quad (9-1.11)
\]
is repeated until k reaches the Gaussian index G.

The variables used in this DO loop are related to those of the present text as follows:

<table>
<thead>
<tr>
<th>text:</th>
<th>DO loop [81-97]:</th>
</tr>
</thead>
<tbody>
<tr>
<td>G [gaussian index]</td>
<td>MXNT</td>
</tr>
<tr>
<td>k</td>
<td>INT</td>
</tr>
<tr>
<td>i</td>
<td>I</td>
</tr>
<tr>
<td>i - j + 1</td>
<td>J</td>
</tr>
<tr>
<td>$\rho_{ik}$</td>
<td>XR</td>
</tr>
<tr>
<td>$r_j + 1/2L$</td>
<td>YR</td>
</tr>
<tr>
<td>$(i(i-1))/2 + j$</td>
<td>IJ</td>
</tr>
<tr>
<td>$g_{mpj}(\rho_{ik})$</td>
<td>FNT(IJ,M)</td>
</tr>
<tr>
<td>m+1</td>
<td>M=1,2,3, or 4</td>
</tr>
<tr>
<td>$\phi_{j1}$ (from Eq. 8-10.24)</td>
<td>T1</td>
</tr>
<tr>
<td>$\phi_{j2}$ (from Eq. 8-10.25)</td>
<td>T2</td>
</tr>
<tr>
<td>$\sin \phi_{j2}$</td>
<td>SCM(IJ,1)=SCA(IJ,1)</td>
</tr>
<tr>
<td>$\cos \phi_{j2}$</td>
<td>SCM(IJ,2)=SCA(IJ,2)</td>
</tr>
<tr>
<td>$\sqrt{r_{(1)}^2-\rho_{ik}^2}$ (from Eq. 8-10.33)</td>
<td>RT1</td>
</tr>
<tr>
<td>$\sqrt{r_{(2)}^2-\rho_{ik}^2}$ (from Eq. 8-10.33)</td>
<td>RT2</td>
</tr>
<tr>
<td>$\cos \theta_1+\sin \theta_2$</td>
<td>RATIO</td>
</tr>
<tr>
<td>$\cos \frac{\theta_1}{1}-\sin \frac{\theta_2}{1}$ (from Eq. 8-10.37)</td>
<td>RATIO</td>
</tr>
</tbody>
</table>
From Eqs. (8-10.24) and (8-10.25) we have

\[
\phi_{j1} = \begin{cases} 
\phi(j+1)2', & j < i \\
0, & j = i 
\end{cases}
\]  

(9-1.28)

This fact is exploited in the DO loop to reduce computations for quantities that depend on \( \phi_{j1} \) or \( \phi_{j2} \). For example, when index \( J \) (Eq. 1.15) is greater than 1, \( T_1 \) (Eq. 1.21) is simply the value of \( T_2 \) obtained for the previous (smaller) value of \( J \).

When the values of \( g_{m\rho j}(\rho_{ik}) \) have been computed for all values of \( j, i, \) and \( k \), for \( m = 0,1,2, \) and 3, the subroutine is ready to proceed with its primary task—the computation and storage of the \( B \) matrices for the solution coefficients of the reconstructed density (Eq. 8-10.17). This is accomplished in the DO loop extending over [99-144].

In the first part of this loop, extending to [127] the values of matrices \( A \) and \( B_0 \) are determined for the given value of \( m = M-1 \) in the program) implementing Eqs. (8-8.3, 8-10.4, 8-10.6, 8-10.13 and 8-10.19). First the arrays \( A \) (which stores matrix \( A \)) and \( B \) (which stores matrix \( B_0 \)) are initialized to zero [101-105]. The actual computation of the \( A \) and \( B_0 \) matrices begins at [106].

Values of the functions \( g_{m1\rho j}(\rho_{ik}), \sin(m_2\phi_{j2}), \cos(m_2\phi_{j2}), \sin \phi_{j2}, \) and \( \cos \phi_{j2} \) are retrieved with BRANRD and stored in the arrays FNT, SCM, and SCA (cf. pages 64, 66), implementing Eqs. (1.19), (1.23) and (1.24). Because of the recursive nature of this DO loop, at this point [107], the array FNT will contain the following values:
\[ FNT(IJ,K) = \begin{cases} 
\frac{g(K-1)p_j(\rho_{ik})}{M \leq 4} \\
\frac{g(M+K-5)p_j(\rho_{ik})}{M > 4, M \text{ odd}} \\
\frac{g(M+K-4)p_j}{M > 4, M \text{ even, } K \text{ odd}} \\
\frac{g(M+K-6)p_j}{M > 4, M \text{ even, } K \text{ even}} 
\end{cases} \tag{9-1.29} \]

where \( IJ, i, \) and \( j \) are related as in Eq. (1.18) and \( K = 1, 2, 3, \) or 4.
Similarly the array SCM will contain the values:
\[ SCM(IJ,1) = \begin{cases} 
\sin \phi_j, M \leq 4 \ (m \leq 3) \\
\sin((M-4)\phi_j) = \sin((m-3)\phi_j), M > 4 
\end{cases} \tag{9-1.30} \]
\[ SCM(IJ,2) = \begin{cases} 
\cos \phi_j, M \leq 4 \ (m \leq 3) \\
\cos((M-4)\phi_j) = \cos((m-3)\phi_j), M > 4 \ (m > 3) 
\end{cases} \tag{9-1.31} \]

while the array SCA will contain values defined by Eqs. (1.23) and (1.24) as before (where \( M = m+1 \) as in Eq. 1.20).

If \( M \leq 4 \) the array FNT already contains the needed values of \( g_{mpj}(\rho_{ik}) \) so Eqs. (8-10.19) and (8-8.9) are implemented in [108-112] and matrices \( A \) and \( B_0 \) are computed immediately.

However if \( M > 4 \) it is necessary to implement Eq. (8-10.31) to determine the correct values of \( g_{mpj}(\rho_{ik}) \) before determining matrices \( A \) and \( B_0 \). Before Eq. (8-10.31) can be implemented it is necessary to determine values of \( \sin((m-2)\phi_j) \) and \( \sin((m-2)\phi_j) \) (cf. Eq. 8-10.31). From Eqs. (1.23), (1.24) and (1.28) it is evident that only the latter expression, \( \sin((m-2)\phi_j) \), must be computed for a given value of \( j \). This together with \( \cos((m-2)\phi_j) \) is computed from the arrays SCM and SCA in [113-116]
using the recursion relations

\[
\sin(x+\phi) = \sin x \cos \phi + \cos x \sin \phi \quad (9-1.32)
\]

\[
\cos(x+\phi) = \cos x \cos \phi - \sin x \sin \phi \quad (9-1.33)
\]

with

\[x = (m-2)\phi_j \quad (9-1.34)\]

and

\[\phi = \phi_j \quad (9-1.35)\]

Equations (8-10.31), (8-10.19), and (8-8.9) are then implemented in [117-125]. Recursively computed values of \(g_{\text{mpj}}(\rho_{ik})\), \(\sin((m-2)\phi_j)\), and \(\cos((m-2)\phi_j)\) are stored in the arrays FNT and SCM, respectively, and are copied for later retrieval [126]. Values of \(g_{\text{mpj}}\), \(\sin((m-2)\phi_j)\) and \(\cos((m-2)\phi_j)\) for \(j > I_m + 1\) are not used and therefore are not computed. At this point [126] the values stored in the FNT and SCM arrays agree with Eqs. (1.29)-(1.31), except that \(M\), in effect, has been incremented by 1.

Finally at [128], matrices \(A\) and \(B_0\) have been computed for the given value of \(M\), and it remains to determine the solution matrix \(B = A^{-1}B_0\). First it is necessary to invert matrix \(A\). This is accomplished by first calling subroutine ORTHO to determine orthonormalization coefficients \(C_{ij}\) for the functions \(g_{\text{mpj}}\). These coefficients have the property that the functions \(\phi_i\) defined by

\[
\phi_i = \sum_{j=1}^{i} C_{ij} g_{\text{mpj}} \quad (9-1.36)
\]

where \(1 \leq i \leq I_m\), form an orthonormal set under the inner product defined
by Eq. (8-8.3). Moreover, it can be shown that the entries $a^*_{ij}$ of the matrix $A^{-1}$ are determined from the $C_{ij}$ by

$$a^*_{ij} = a^*_{ji} = \sum_{k=i}^{I_m} C_{ik} C_{kj} \text{ (assuming } i \geq j)$$  \hspace{1cm} (9-1.37)

The above equation is implemented in [129-133]; the entries $C_{ik}$ and $a^*_{ij}$ are stored in arrays $C$ and $A$, respectively.

The solution matrix $B$ is determined in the DO loop [134-143]. A given entry $b_{ij}$ of matrix $B$ is stored in ARRAY (MIJ), [143]. When ARRAY is filled with $B$ entries it is copied into LCM and refilled until all the $B$ matrices, for all orders $m$, have been computed and stored in LCM.

The indexing scheme used at [143] for entry $b_{ij}$, order $m$, is defined by

$$MIJ = (L \cdot \sum_{m_0=0}^{m-1} I_{m_0}) + (i-1)L + j - P \cdot H$$  \hspace{1cm} (9-1.38)

where $L$ and $i_m$ are defined by Eqs. (8-9.7) and (8-9.9), respectively, $P$ is the number of times that ARRAY has already been filled and copied into LCM, and $H$ is the number of entries required to fill ARRAY—in this case, 11,000. Thus every entry in every $B$ matrix has a unique location in LCM.

Other variables used in this DO loop are related to those of the present text as follows:
When the B matrices for all orders \( m \) have been computed, the remaining B entries are copied into LCM [144]. The program then prints the absolute value of the first entry of the first B matrix, the maximum absolute value of any entry of any B matrix (showing whether any B matrices are ill-conditioned), and the execution time required for subroutine FIRST. The program also prints the number of projections originally taken of the object (=NVIEWS/2) and the number of points in a video scan line of data (=2*IRMAX).

In subroutine RECON the data for a given section of the object to be reconstructed is read from logical unit 3 [234-235], unpacked [238-239], and stored in the array XRAY. Thus XRAY now contains projectional information for all the different angular orientations and the radial positions of the object.
given section. The values stored in XRAY are defined by

\[ XRAY(i,j) = g(r_i, \mu_j) \]  \hspace{1cm} (9-1.46)

where \( g \) is the observed projection function,

\[ i' = L - i + 1 \]  \hspace{1cm} (9-1.47)

and

\[ \mu_j = 2\pi(j-1)/NVIEWS \]  \hspace{1cm} (9-1.48)

\( L \) and \( NVIEWS \) are defined as in Eqs. (8-9.7) and (1.3), respectively, with \( 1 \leq i \leq L \), and \( 1 \leq j \leq NVIEWS \).

Next the data for a given radial position \( r_i \), (all angular orientations \( \mu_j \)) is transferred to the complex array FTO and its (forward) Fourier transform is taken using the fast-transform routine FOURT [247].* This Fourier transform (which is now stored in FTO) is converted to a real (sine-cosine) transform and stored in the array FT [246-251]. When this process has been completed for all radial positions \( r_i \), the FT array contains the following values

\[ FT(i,j) = g_{mp}(r_i) \]  \hspace{1cm} (9-1.49)

where \( g_{mp} \) is the harmonic component of the observed projection function \( g \) (Eq. 8-5.7), \( i' \) is defined as before, and

\[ j = 2m + p \]  \hspace{1cm} (9-1.50)

*Subroutine FOURT is an NCAR library routine which is numbered separately in the program listing of the Appendix.
Now the coefficients stored in the FT array are ready to be converted into Fourier coefficients of the reconstructed density. First, values of $g_{mp}(r_i)$ extending over all radii $r_i$, for a given value of $p$ and $m$, are collected and stored in the array $FR$, with the order reversed so that

$$FR(i) = g_{mp}(r_i) = V_i$$  \hspace{1cm} (9-1.51)

($V_i$ is the vector entry defined by Eq. 8-10.18.) Meanwhile, entries of the $B$ matrix for the given order $m$ have been read from LCM and stored in the array $D$. Elements stored in $D$ are related to the entries $b_{ij}$ of matrix $B$ by

$$D(IJ) = b_{ij}$$  \hspace{1cm} (9-1.52)

where

$$IJ = (i-1)I + j$$  \hspace{1cm} (Eq. 1.43)

Finally, Eq. (8-10.17) is implemented and coefficients $C_{mpi}$ of the reconstructed section are computed and again stored in FT. When this process is completed the FT array contains the following:

$$FT(i,j) = \begin{cases} 
C_{mpi'}, i' \leq I_m \\
0, \text{ otherwise}
\end{cases}$$  \hspace{1cm} (9-1.53)

where $i'$ and $j$ are defined as in Eqs. (1.47) and (1.50) respectively.

To convert the coefficients $C_{mpi}$ to densities of the reconstructed section, values of $C_{mpi}$ corresponding to a particular radial position $r_i$, (for all values of $p$ and $m$) are retrieved, converted to complex form,
and stored in the array FTO [270-274]. A reverse Fourier transform is taken using FOURT, which converts the quantities stored in FTO into (real) density values [275]. At this point

\[ FTO(j) = f(r_i, \mu_j) \]  

so that the observed projection function \( g \) (Eq. 1.46) has now been converted to the density function \( f \), representing the reconstructed object. Values of \( FTO(j) \) which represent the density of a section of the object along a particular radius \( r_i \), are then converted to 16 bit format and written on logical unit 5 [281].

When all sections of the object have been reconstructed, the number of sections reconstructed (NSECT), the average amount of time required to reconstruct a section, and the maximum density value computed, are printed.

2. THE DATA

Data representing projections of the object to be reconstructed are stored on logical unit 3. Each logical record of data contains projectional values for one section of the object. The values stored in this logical record correspond to \( g(r_i, \mu_j) \) (cf. Eq. 1.46), with \( j \) the most rapidly varying index. Thus each record is divided into blocks of data consisting of all the values of \( g(r_i, \mu_j) \) for a particular radius \( r_i \). The radii \( r_i \) extend from near the axis of rotation \( (r_L = r_{L1}) \) to the right-hand edge of the field of view \( (r_1 = r_{L1}) \). The data within each block extend over all angular orientations \( \mu_j \), which cover 360° (Eq. 1.48). Data values for \( \mu_j \geq 180^\circ \) correspond to the left-hand side of the field of view in the original projections.
The data are written on 7-track tape, 800 BPI, odd parity (binary), packed 7\(\frac{1}{2}\) words of data per 60-bit computer word, ordered low to high. Each data word is 8 bits in length. The parameters defining the dimensions of the data array are NVIEWS (the number of angles \(\psi_j\)), IRMAX (the number of radii \(r_i\)), and IPAK (the number of computer words occupied by (NVIEWS) words of data). The total number of data values for the entire object is NVIEWS·IRMAX·NSECT and the number of computer words occupied by this data is IPAK·IRMAX·NSECT.

Each logical record of data consists of a 4-word header followed by NVIEWS·IRMAX packed data values so that the number of 60-bit computer words occupied by the record is 4+IPAK·IRMAX. The data values begin with bit 1 of word 5. The 4-word header contains, in order, the values of NVIEWS, NSECT, IPAK, and IRMAX, for the given record. All headers of all records should be identical. When the first header of the first record is read in subroutine FIRST the program has all the information needed to read in all the data.

3. THE RESULTS

In subroutine RECON the data records were read and unpacked, the data were analyzed, and numerical values representing the digitized sections of the reconstructed object were stored (in 16-bit format) on logical unit 5. In subroutine PLOT this digitized information is normalized to 8-bit format (linearly adjusted so that the maximum density value is \(2^8-1 = 255\)) and stored on logical unit 4 [329].

The format of the digitized values stored on unit 4 is identical to that of the data stored on unit 3. The values stored in a given
logical record correspond to \( C \ f(r_{i,, j}) \), where \( j \) is again the most rapidly varying index and \( C \) is a normalization constant. Each record is preceded by a 4-word header identical to that of the data records on unit 3. The density values are packed 7½ per computer word, ordered low to high, beginning in word 5 as before. The number of density values per record is NVIEWS·IRMAX (IPAK·IRMAX computer words) and the number of records (sections) is NSECT.

The densities stored on unit 4 are tabulations of the different sections in polar format. Subroutine PLOT also converts each reconstructed section into rectangular format, suitable for plotting as a gray-scale image on microfilm or printout paper. First the densities for a given section (calculated in RECON and written on unit 5) are collected and stored in the array DNS [320-328]. The values stored in DNS are defined by

\[
DNS(i,j) = f(r_{i,, j}) \tag{9-3.1}
\]

analogous to Eq. (1.46). Then the densities are converted into rectangular format by interpolating over the DNS array. The interpolated densities are then normalized to the range 0-1, and stored in the array ZMAP, which is dimensioned 128 x 128. The values stored in ZMAP are determined by

\[
ZMAP(i,j) \approx \begin{cases} 
C_{01} f(r_{ij}, \theta_{ij}), & r_{ij} < 1 \\
0, & \text{otherwise}
\end{cases} \tag{9-3.2}
\]

where \( 1 \leq i \leq 128, 1 \leq j \leq 128, C_{01} \) is a normalization constant which limits the maximum value of ZMAP\((i,j)\) to 1,
\[ r_{ij} \cos \theta_{ij} = (i-64.5)/128 \quad (9-3.3) \]

and

\[ r_{ij} \sin \theta_{ij} = (j-64.5)/128 \quad (9-3.4) \]

The "approximately equals" sign in Eq. (3.2) is used because ZMAP is computed by interpolation. The ZMAP array represents the normalized density function \( C_{01} f \) at the centers of squares which form a 128 x 128 grid that encloses the unit disk.

If we define a quantity \( P_k \) for \( 1 \leq k \leq 128 \) by

\[ P_k = (k-64.5)/128 \quad (9-3.5) \]

we can determine \( r_{ij} \) and \( \theta_{ij} \) respectively, from \( i \) and \( j \), by

\[ r_{ij} = \sqrt{P_i^2 + P_j^2} \quad (9-3.6) \]

\[ \theta_{ij} = \begin{cases} \arctan (P_j/P_i), & P_i > 0, P_j > 0 \text{ (quadrant I)} \\ \pi - \arctan (-P_j/P_i), & P_i < 0, P_j > 0 \text{ (quadrant II)} \\ \pi + \arctan (P_j/P_i), & P_i < 0, P_j < 0 \text{ (quadrant III)} \\ 2\pi - \arctan (-P_j/P_i), & P_i > 0, P_j < 0 \text{ (quadrant IV)} \end{cases} \quad (9-3.7) \]

where the function \( \arctan \) has the range \((0, \pi/2)\) for the arguments indicated.

By Eqs. (1.4), (1.47), (1.48) and (3.1) it is evident that the arguments of the density function \( f \) are related to those of the array DNS by
when the right-hand expression is defined, that is, when \( r \cdot \text{IRMAX} + 1 \) is an integer between 1 and IRMAX and \( \frac{\theta \cdot \text{NVIEWS}}{2\pi} + 1 \) is an integer between 1 and NVIEWS. The interpolation scheme used to determine ZMAP from DNS is defined by

\[
ZMAP(i,j) = F(i-Aij)DNS(k_0, h_0) + h_{ij} DNS(k_0, h_1), \quad r_{ij} < 1
\]

\[
0, \text{ otherwise}
\]  

(9-3.9)

where

\[
k_0 = \text{the greatest integer not exceeding } r_{ij} \cdot \text{IRMAX} + 1
\]

(9-3.10)

\[
h_0 = \text{the greatest integer not exceeding } \frac{\theta_{ij} \cdot \text{NVIEWS}}{2\pi} + 1
\]

(9-3.11)

\[
h_1 = \begin{cases} 1, & j_0 = \text{NVIEWS} \\ h_0 + 1, & \text{otherwise} \end{cases}
\]

(9-3.12)

and

\[
A_{ij} = \frac{\theta_{ij} \cdot \text{NVIEWS}}{2\pi} + 1 - j_0
\]

(9-3.13)

This interpolation scheme is suggested by the fact that the density function \( f(r,\theta) \) varies not as a continuous function of \( \theta \) but as a discontinuous step function of \( r \).

To implement the above interpolation scheme for the ZMAP array it is necessary to evaluate the quantities \( r_{ij} \) and \( \theta_{ij} \) for each of
the 16,384 points \((i,j)\) at which the ZMAP array is to be determined. However the amount of storage (or computation) required for this evaluation can be greatly reduced by exploiting symmetry relations among the values of \(r_{ij}\) and \(\theta_{ij}\) for different \(i\) and \(j\).

By Eq. (3.5) we have
\[
P(129-k) = -P_k.
\] (9-3.14)

If for the pair \((i,j)\) we define values \(i_0\) and \(j_0\) by
\[
i_0 = \max(64.5 + |i-64.5|, 64.5 + |j-64.5|)
\] (9-3.15)

and
\[
j_0 = \min(64.5 + |i-64.5|, 64.5 + |j-64.5|)
\] (9-3.16)

so that \(P_{i_0} > 0, P_{j_0} > 0\), from Eqs. (3.5) and (3.14)-(3.16) we can deduce that
\[
P_{i_0} = \max(|P_i|, |P_j|)
\] (9-3.17)

and
\[
P_{j_0} = \min(|P_i|, |P_j|)
\] (9-3.18)

Thus by Eqs. (3.6), (3.17) and (3.18) we obtain
\[
r_{ij} = r_{i_0j_0}
\] (9-3.19)

Moreover by Eq. (3.7) we have, since \(P_{i_0} > 0, P_{j_0} > 0\),
\[
\theta_{i_0j_0} = \pi/2 - r_{j_0i_0}
\] (9-3.20)
By applying Eqs. (3.5), (3.7), (3.17), (3.18) and (3.20) we obtain

\[ \theta_{ij} = q + \theta_o \]  \hspace{1cm} (9-3.21)

where

\[ q = \begin{cases} 
0, & \text{if } (P_i, P_j) \text{ is in quadrant I} \\
\frac{\pi}{2}, & \text{if } (P_i, P_j) \text{ is in quadrant II} \\
\pi, & \text{if } (P_i, P_j) \text{ is in quadrant III} \\
\frac{3\pi}{2}, & \text{if } (P_i, P_j) \text{ is in quadrant IV} 
\end{cases} \]  \hspace{1cm} (9-3.22)

and

\[ \theta_o = \begin{cases} 
\theta_{ij0}, & \text{if } P_i \cdot \text{sign}(P_j) \geq P_j \cdot \text{sign}(P_i) \\
\frac{\pi}{2} - \theta_{ij0} = \theta_{ij0}, & \text{otherwise} 
\end{cases} \]  \hspace{1cm} (9-3.23)

The values of \( r_{ij} \) and \( \theta_{ij} \) needed for the interpolation scheme of Eq. (3.9), can be determined from the values of \( r_{ij0} \) and \( \theta_{ij0} \) by Eqs. (3.19) and (3.21)-(3.23). This means that to implement Eq. (3.9) it is only necessary to store the values of \( r_{ij0} \) and \( \theta_{ij0} \). This reduces by nearly a factor of 8 the amount of storage that would be required if all the 32,768 values of \( r_{ij} \) and \( \theta_{ij} \) had to be stored in the program.

To implement Eq. (3.9) and interpolate a reconstructed section over rectangular coordinates, the program first computes normalized values of \( r_{ij0} \) and \( \theta_{ij0} \) \([311-314]\). These normalized values are stored in the arrays RCOR and TCOR, respectively. The normalization used is the following, suggested by Eq. (3.8):
\[
\begin{align*}
\text{RCOR}(IJ) &= \begin{cases} 
\frac{r_{i_0j_0} \cdot \text{IRMAX} + 1}{r_{i_0j_0}}, & r_{i_0j_0} < 1 \\
0, & \text{otherwise}
\end{cases} \\
\text{TCOR}(IJ) &= \frac{\theta_{i_0j_0} \cdot \text{NVIEWS}}{2\pi}, & r_{i_0j_0} < 1
\end{align*}
\]

and

\[
IJ = (i_0(i_0-1))/2 + j_0
\]

When \( r_{i_0j_0} \geq 1 \) we set RCOR(IJ) to 0 to indicate that ZMAP(i,j) should be set to zero as required by Eq. (3.9). The reason that values of TCOR are not incremented by 1 as might be expected from Eq. (3.9) is that these values are sometimes subtracted to compute the correct normalized values of \( \theta_{ij} \).

Finally Eq. (3.9) is implemented in the DO loop [333-342]. The rectangular array ZMAP representing the reconstructed section is interpolated from the polar array DNS. Variables in the DO loop are related to those of the text as follows:

<table>
<thead>
<tr>
<th>Text:</th>
<th>DO loop [333-342]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i )</td>
<td>( I )</td>
</tr>
<tr>
<td>( 0.5+</td>
<td>i-64.5</td>
</tr>
<tr>
<td>( P_i \cdot \text{IRMAX} )</td>
<td>( XI )</td>
</tr>
<tr>
<td>( 1, P_i \geq 0 \ (i &gt; 64) )</td>
<td>( IX )</td>
</tr>
<tr>
<td>( 0, P_i &lt; 0 \ (i \leq 64) )</td>
<td>( IX )</td>
</tr>
<tr>
<td>( j )</td>
<td>( J )</td>
</tr>
<tr>
<td>( 0.5+</td>
<td>j-64.5</td>
</tr>
</tbody>
</table>
\( P_j \cdot \text{IRMAX} \)
\[
\begin{align*}
1, P_j \geq 0 \ (j > 64) & \quad = \ YJ \\
0, P_j < 0 \ (j \leq 64) & \quad = \ JY
\end{align*}
\]
(9-3.33)
\( P_i \cdot \text{sign}(P_j) \cdot \text{IRMAX} \)
\[
= \ XSY \
(9-3.35)
\]
\( P_j \cdot \text{sign}(P_i) \cdot \text{IRMAX} \)
\[
= \ YSX \
(9-3.36)
\]
\( (i_o(i_o-1))/2 + j_o \)
\[
= \ IJ \quad \text{(Eq. 3.26)}
\]
\( k_o \) (from Eq. 3.10)
\[
= \ IRAD \
(9-3.37)
\]
\[
\frac{\theta i_o j_o \cdot \text{NVIEWS}}{2\pi} 
= \ THO \
(9-3.38)
\]
\[
\frac{\alpha \cdot \text{NVIEWS}}{2\pi} + 1
= \ Q(IX,JY) \
(9-3.39)
\]
\[
\frac{\theta i_j \cdot \text{NVIEWS}}{2\pi} + 1
= \ THETA \
(9-3.40)
\]
\( h_o \) (from Eq. 3.11)
\[
= \ ITH \
(9-3.41)
\]
\( h_1 \) (from Eq. 3.12)
\[
= \ ITH_1 \
(9-3.42)
\]
\( \Delta_{ij} \) (from Eq. 3.13)
\[
= \ DTH \
(9-3.43)
\]
\( C_{01} \) (normalization constant)
\[
= \ RDMX \
(9-3.44)
\]

When the ZMAP array has been computed it may be plotted as a grayscale image on film or print paper (see below). In addition, values of the ZMAP array are normalized to integers in the range 0-16 [346-350] and written on logical unit 6 [351]. The normalization used is the following:

\[
IZ = v + \text{(greatest integer not exceeding 16 Z)} \quad \text{(9-3.45)}
\]

where \( Z \) is a given element of the ZMAP array and \( v \) is a random quantity whose value is 0 or 1, chosen so that the average value of \( IZ \) over a large number of identical \( Z \) values is nearly equal to 16 \( Z \).
When the normalization of the ZMAP array is completed the normalized values are stored on logical unit 6, one logical record per section of the object. Each logical record (representing one section) consists of a 4-word header followed by $12 \times 128 = 1536$ words containing intensities packed 12 values per word, that is, 5 bits per intensity, ordered low to high. Each record is partitioned into 12-word blocks, each containing one video line of the reconstructed section. Within each of these blocks the last $4/3$ words are empty so that the number of values per block (that is, per video line) is $12 \times (32/3) = 128$. The intensities are arranged so that the $j^{th}$ video line (12-word block) contains the normalized values of $ZMAP(i,j)$ for $1 \leq i \leq 128$, in order. Thus the first 5 bits of the first 12-word block represent the lower-left corner of the field of view, and the last 5 bits of the nonempty portion of the last 12-word block (bits 36-40 of word 11) represent the upper-right corner.

The 4-word header at the beginning of each logical record contains in order, the number of sections (NSECT), the number of computer words per video line (12), the number of 5-bit intensities per line or the number of lines per section (128), and the number of radial positions $r_i$ in the density values stored in polar format on unit 4, IRMAX. (IRMAX can be used with NSECT to determine relative distances within and between the different sections of the object.)

4. PLOTTING THE RECONSTRUCTED SECTIONS

The program provides the option of plotting gray-scale or halftone pictures of the reconstructed sections on film or print paper. The controlling parameter is IOPT which is defined in subroutine FIRST [48].
The following options are available.

<table>
<thead>
<tr>
<th>IOPT</th>
<th>Option</th>
</tr>
</thead>
<tbody>
<tr>
<td>≤ 0</td>
<td>plot sections on microfilm</td>
</tr>
<tr>
<td>= 1</td>
<td>plot sections on printer</td>
</tr>
<tr>
<td>≥ 2</td>
<td>ignore all plotting</td>
</tr>
</tbody>
</table>

The printing option is useful on debugging runs. The printer-produced gray-scale pictures provide a crude but quick indication of whether the data have been properly reduced and scaled, and whether the program itself is functioning correctly. (These pictures will be negatives of those that are plotted on film.) Unless the number of sections is small the printing option is normally used with the parameter NSKIP, defined in subroutine FIRST, \[49\], set to a value greater than 1. The program will then reconstruct sections at intervals of NSKIP, beginning with the first section and skipping the ones in between. (This will also cause the program to redefine NSECT to the value \((\text{NSECT}-1)/\text{NSKIP} + 1\), \[231\], so that NSECT equals the number of sections actually reconstructed.) Thus, if the total number of sections is 83 and NSKIP=20 the program will reconstruct only sections 1, 21, 41, 61, and 81.

NSKIP may be used with any value of IOPT. The default values of IOPT and NSKIP are 0 and 1, respectively, so that all available sections are reconstructed and plotted on microfilm unless specified otherwise. The gray-scale pictures (whether on film or print paper) are oriented so that the origin is in the center and the x-axis is horizontal.

X. EXPERIMENTAL TESTING OF PROGRAM SECTION

To gain some idea of the practical value of program SECTION in
actual clinical use, we reconstructed cross sections of an object using digitized radiographic data. The object, a rat's head, was placed in a polyethylene cup and mounted on a turntable. A series of 120 radiographs was taken at intervals of $1.5^\circ$ while the object was rotated to obtain full $180^\circ$ coverage.

The 120 radiographs of the rat head that were used in this analysis were converted to digital format at the Mayo Clinic (Robb, et al., 1973). The pictures were scanned with a 285 x 200 raster (57,000 points per picture) and intensities, normalized to the range 0-255, were recorded in 8-bit format on tape. As each picture was being scanned it was simultaneously displayed on a TV screen in the format shown in Fig. 10.1.

In Fig. 10.1 the rat head can be seen as a ghostly profile near the center of the picture (top of skull to the right) while the enclosing white rectangle indicates the total area of the picture scanned (285 points in the horizontal direction x 200 lines in the vertical). The metal ring at the base of the rat head and the metal rod (below the ring) attached to the radiographic film holder are both seen as white streaks at the lower edge of the white rectangle. The thin white line in the upper part of the picture shows the current scanning position of the instrument. In this case, line 105 (the 71st line from the top) has just been scanned. Scanning proceeds from the top edge of the picture down, left to right. The variation of intensities across the line is shown by the dotted graph below (partially obscured by the bright area at the base of the picture). The scale of this graph is established by the white line at the left edge of the picture, indicating the ordinate of the maximum possible intensity (255) from the zero level (located at the lower edge
of the white rectangle). In this case the digitized intensities along
the line have maximum, minimum, and average values of 211, 25, and 50
units, respectively.

After the 120 radiographs were digitized, a number of steps were
necessary to convert the data into a suitable format for the reconstruc-
tion algorithm. These steps included (1) converting the digitized bright-
ness values into logarithmic units of radiation exposure, (2) correcting
the orientation of each digitized picture to make the scan lines perpen-
dicular to the rotation axis of the subject, so that similarly numbered
lines in different pictures would represent views, at different angular
orientations, of the same section through the object, (3) reducing each
digitized picture from the original 285 x 200 array into a 78 x 83 array
suitable for use in the reconstruction program, and (4) separating and
rearranging the data so that lines from different pictures representing
different views of the same section of the object would be stored together
as an array.

The conversion of the digitized film intensities into logarithmic
units of radiation exposure was accomplished as follows:

(1) A series of blank radiographs was exposed for time intervals
covering the range 0-1 s and brightness values were obtained by
digitization (see Table IV).

(2) From this discrete set of values, a smooth curve giving the
radiation exposure I as a function of film brightness was deter-
mined using splines under tension. In this manner I was expressed
as the time required to produce given blackening of the film for a
unit intensity of radiation exposure. This curve was then con-
verted into logarithmic units by taking \( \ln(I_0/I) \), where \( I_0 \) was the brightness value corresponding to the 1 s exposure time used in the radiographs of the rat head. The final calibration curve obtained is shown in Fig. 10.2.

(3) A background intensity ("black level") representing \( I_0 \) was determined for a given scan line by averaging the first and last 5 intensities on the line. Then all intensities of the given line were linearly adjusted using the "white level" of 454 as a fixed point, so that the background value coincided with the value of 30 shown in table IV for 1 s exposure. We assumed that the white level or intensity that would have been measured for a radio-opaque object, was the same for all scan lines and was equal to the value of 454 given for 0 s exposure in table IV.

In actual tests, white levels measured for the metal objects at the base of the rat head in several pictures were constant to within about 1% of the value shown in table IV. Black levels for the 120 radiographs varied over the approximate range of 20-40, about 10% of the typical range of intensities measured for the non-opaque region of a picture (that is, excluding the metal objects).

(4) The adjusted intensities for a given scan line were converted to logarithmic exposure values (values of \( \ln(I_0/I) \)) using the calibration curve shown in Fig. 10.2, and were then normalized to the range 0-255. Normalized intensities were then stored in 8-bit format on tape.

When the pictures had been reduced to logarithmic units of radiation exposure each picture was bilinearly interpolated (in the manner of
Eqs. (10.2)-(10.4) below) to obtain a picture with all scan lines perpendicular to the rotation axis of the subject. It was assumed that this rotation axis coincided with the central axis of the metal ring (cylinder) at the base of the rat head. Since parallel geometry is assumed in each picture, the position of the true axis of rotation is immaterial provided that it is parallel to the assumed axis.

The metal ring appears as a white region of nearly rectangular shape in each radiograph (see Fig. 10.1), with the lower portion of the ring obscured by the metal rod that was attached to the film holder. The upper left and right corners of this region, which were easily distinguished from the surrounding dark background in the digitized pictures, were used as reference points, and coordinate axes coinciding with (a) the line segment through the points, and (b) the perpendicular bisector of this line segment, were used as x and y axes, respectively. Thus the y axis coincided with the projection of the assumed rotation axis in each picture.

While each picture was being oriented by interpolation it was also cropped to remove the metallic objects at the base of the rat head and to reduce the black background at the sides of each picture (thus improving spatial resolution by reducing the field of view). This procedure reduced the array dimensions from the original 285 x 200 to 251 x 166. By measuring a photograph like that shown in Fig. 10.1, it was determined that a picture cropped as above would have its horizontal and vertical dimensions in the ratio of approximately 1:1.06. To maintain uniformity of resolution in the reconstructed sections and to minimize the amount of computation we decided that the number of points across the field of view (that is, in the horizontal dimension) in the final reduced pictures should be approximately
times the number of views (see chapter VIII). In this case we chose to have 78 points across the field of view to match the 120 views that were taken. To maintain uniformity of resolution between the horizontal and vertical dimensions the number of lines in each picture was reduced to 83 so that the ratio between vertical and horizontal dimensions reflected the shape of each picture (that is, 83/78 ≈ 1.06).

This final reduction in the size of the picture array (from 251 x 166 to 78 x 83) was accomplished by averaging neighboring points in the larger array to obtain intensities for the smaller array. The averaging process was mathematically equivalent to the following:

1. Let $P_{IJ}$ denote the picture intensity for point $(I,J)$ in the larger (251 x 166) array. We assume that values of $P_{IJ}$ represent a function defined over a rectangularly spaced set of discrete points $(x_I, y_J)$ in the plane, where

$$x_I = (I-.5)/251$$

and

$$y_J = (J-.5)/166$$

Thus the point $(x_I, y_J)$ is in the unit square bounded by the lines $x=0, x=1, y=0, y=1$.

2. The function defined for discrete points $(x_I, y_J)$ by the above is extended to a function $f(x, y)$ defined for arbitrary points $(x, y)$ in the unit square by bilinear interpolation of the intensities $P_{IJ}$. That is, $f(x, y)$ is defined by

$$f(x, y) = P_{IJ} \cdot (1-\Delta x) \cdot (1-\Delta y) + P_{I', J} \cdot \Delta x \cdot (1-\Delta y)$$

$$+ P_{IJ'} \cdot (1-\Delta x) \cdot \Delta y + P_{I', J'} \cdot \Delta x \cdot \Delta y$$

(10.2)
\[ I' = I + 1 \quad \text{(or } I' = I \text{ if } I = 251) \]
\[ J' = J + 1 \quad \text{(or } J' = J \text{ if } J = 166) \]
\[ (10.3) \]

I and J are chosen so that \( x_{I} \leq x \leq x_{I+1} \), (or \( x_{I} \leq x \) if \( I = I' \)), \( y_{J} \leq y \leq y_{J+1} \), (or \( y_{J} \leq y \) if \( J = J' \)) and

\[ \Delta x = x - x_{I} \]
\[ \Delta y = y - y_{J} \]
\[ (10.4) \]

(3) Let \( Q_{ij} \) denote the picture intensity for point \((i,j)\) in the smaller \((78 \times 83)\) array. Then \( Q_{ij} \) is defined by

\[
Q_{ij} = \left( \int_{x_0}^{x_1} \int_{y_0}^{y_1} f(x,y)dydx \right) / \left[ (x_1-x_0)(y_1-y_0) \right]
\]

where
\[
x_0 = (i-1)/78, \quad x_1 = i/78 \]
\[
y_0 = (j-1)/83, \quad y_1 = j/83 \]
\[ (10.5) \]

so that \( Q_{ij} \) is the average value of the interpolated picture \( f(x,y) \) over a unit cell corresponding to point \((i,j)\) in the smaller array.

Finally, when the digitized pictures were correctly oriented and reduced dimensionally, the digitized data were rearranged so that corresponding scan lines in different pictures, representing different views of the same section of the object, would be stored together in an array. Since the final pictures were dimensioned \(78 \times 83\), there were 83 such arrays representing 83 sections of the object, and this arrangement of the data permitted section-by-section reconstruction of the object.
A number of difficulties encountered in the process of obtaining the above set of digitized intensities should be mentioned. These included:

(1) a wobble or variation of the inclination of the rotation axis of the object of the order of 1-2 degrees while the radiographs were being taken,

(2) a somewhat longer than optimal radiographic exposure time, which reduced the contrast between regions of low absorption, such as soft tissue, and the surrounding background on the radiographs, and

(3) a dependency of the measured intensity at a given point on the average brightness of the picture being digitized, this being caused by the electronics of the digitizing system that was used.

Because of the difficulty of obtaining and digitizing the radiographs it was not possible to eliminate these problems at the time of their discovery, although every effort was made, where possible, to correct their effects on the reduced data set.

The wobble or tilt in the rotation axis was caused by an insecure mounting, or fastening to the turntable, of the polyethylene cup containing the rat head. As the object was rotated, there apparently was a gradual increase in the extent of the tilt (which was detected by observing that the upper edge of the metal ring at the base of the rat head appeared slightly curved in a number of radiographs, the curvature increasing with rotation angle up to $180^\circ$). Figure 10.1 shows the $180^\circ$ orientation (not one of the 120 radiographs used in the study since its mirror image
is the $0^\circ$ orientation) with approximately the maximum amount of curvature that was observed. In the absence of a tilt, the upper edge of the ring would have appeared as a straight line in projection, as was observed in the early radiographs of small rotation angle.

Although the exact path of motion of the rotation axis could not be determined from the single series of radiographs taken, the extent of the tilt could be roughly estimated by microscopic measurements of the extent of the curvature of the edge of the ring as seen in different radiographs. When such measurements were made, we estimated that the maximum extent of the tilt was roughly 1-2 degrees out of perpendicular. A $2^\circ$ tilt would have caused a maximum displacement of a point in the object seen in projection of approximately .018 of the horizontal field of view. On the other hand, the separation of resolution elements in the final digitized images was $1/78$th or about .013 of the horizontal field in both dimensions. Thus while the presence of the tilt may have caused some slight distortion in the reconstructed sections, this distortion could not have been serious given the scale of resolution that was used.

A more serious problem was caused by the choice of exposure time (1 s), which made the radiographs quite dark and probably obscured much of the soft tissue structure that might otherwise have been seen. The extent of this problem is shown by the extreme steepness of the calibration curve in Fig. 10.2 at its lower left extremity where a small change in the digitized film intensity produces a very large difference in the logarithmically reduced value. It now appears that the optimal exposure time for the radiographs would have been in the range of 0.5 to 0.8 s, but this could not have been anticipated at the time the radiographs were taken.
A third problem was caused by the response of the digitizer itself to the background brightness of a picture. It was possible to increase the measured intensity at a given point on a picture merely by interposing a black object over an area of the picture not currently being scanned. Thus the intensity measured by the digitizer at a given point was somewhat dependent upon the average brightness of the picture being digitized.

The radiographs of the rat head itself all appeared to have very nearly the same average brightness. In this case the dependency factor was essentially constant, as suggested by the small variance in the measured intensities of the black, and especially, the white levels on the different radiographs. On the other hand there was great variation in the average brightness of the calibration exposures, so special precautions were taken to make the background brightness as similar as possible to that of the rat head exposures.

To adjust this background brightness, a rat-head radiograph was superimposed over each calibration exposure before digitization, and intensities were measured through the white area at the base of the rat head. This white area was found to absorb about 15% of the incident light, causing a corresponding reduction in the measured intensities (later corrected).

Originally, the digitizer had been adjusted to provide the maximum possible range of intensities for the relatively dark rat head. Thus in many cases the calibration intensities would have been larger than the maximum recordable value of 255, and it was necessary to reset the gain of the instrument (by a linear factor) to bring these intensities within the recordable range.
Finally, when the calibration intensities were measured, it was necessary to convert these values into the digital units of the original radiographs. This conversion involved (1) a correction for the 15% absorption of the white area of the radiograph used to control the background brightness, and (2) a correction for the resetting of the gain of the digitizer. When these corrections were applied it was found that the calibration intensities ranged in value from about 60 (black, or 1 s exposure) to 454 (white or 0 s exposure). The black value of 60 was slightly outside the range of values of approximately 20-40 found on the radiographs. To provide a more realistic comparison with the radiograph intensities, the scale of the calibration intensities was linearly adjusted downward so that the black value became a more typical 30, while the white level of 454 remained fixed. The final adjusted calibration intensities are shown in table IV. These values were used in plotting the calibration curve of Fig. 10.2. This final adjustment did not affect the reduction of film intensities to logarithmic units since, in fitting the black level of a given radiographic scan to that of the calibration curve, a compensating adjustment was automatically made.

XI. SOFT TISSUE IMAGE ENHANCEMENT

The output oscilloscope, which displayed the serial sections of the undissected rat head, condensed the 256 gray levels into only 17 shade levels. This resulted in a loss of soft tissue definition in the internal density maps because the number of shade levels in the soft tissue range was only two or three out of the 17 used to make the map. To display soft tissue, the number of the oscilloscope shade levels reserved for the
soft tissue density range was greatly increased, so that small variations in density would be made visible. Since the total brightness variation from black to white available on the oscilloscope remained the same, it was necessary to decrease the number of shade levels for densities outside the density range of soft tissue. The relation between a calculated gray level and an oscilloscope shade level for the soft tissue enhancement is given by the solid curve of Fig. 11.1. The corresponding relation without soft tissue enhancement is given by the dashed straight line in the figure.

A complete series of axial, sagittal, transverse, and frontal maps were calculated using the image enhancement indicated by Fig. 11.1.

Figure 11.3 shows the soft-tissue image-enhanced serial section at the same orientation as the non-enhanced section of Fig. 11.2. Captions were placed on Fig. 11.3 (by an artist) to illustrate the structures visible. The bone structures, whose density contrast is strongly compressed, appear almost uniformly white. The orbit, however, which is not apparent in Fig. 11.2 suddenly becomes visible in the enhanced image Fig. 11.3. The brain tissue is also enhanced. Denser brain tissue appears white at the lower portion of the photograph.

The soft-tissue image enhancement of Fig. 11.3 shows much gray-level information which is not visible in the original unenhanced image, Fig. 11.2. Computer image enhancement which allows the separation of minutely differing gray levels may thus considerably improve the diagnostic value of reconstructed sections. A similar enhancement applied directly to the digitized radiographic or fluoroscopic images may reduce the radiation dosage needed for diagnostic quality.
Fig. 3.1 A representative medical radiograph.

Fig. 4.1 Anatomic planes defined for the rat.
Fig. 4.2 Periapical radiograph of inverted mandibular molar (in vitro) oriented at $0^\circ$ longitude for molar study.

Fig. 6.1 Rotation apparatus for initial study.
Fig. 6.2  Sagittal-axial section through maxillary molar (first attempt).

Fig. 6.3  Improved rotation device used in molar and head studies.
Fig. 6.4 Arrangement of radiographic apparatus for molar and head studies.
Fig. 6.5  Computer-derived (serial) axial section maps of tooth through $180^\circ$ of rotation with increments of $3^\circ$. 
Fig. 6.6 Computer-derived (serial) transverse section maps of tooth.
Fig. 6.7  Computer-derived axial section mapped at $0^\circ$ longitude.

Fig. 6.8  Computer-derived axial section mapped at $45^\circ$ longitude.
Fig. 6.9  Computer-derived axial section mapped at $90^\circ$ longitude.

Fig. 6.10  Radiograph of tooth oriented at $45^\circ$ longitude.
Fig. 6.11  Radiograph of tooth oriented at $90^\circ$ longitude.

Fig. 6.12  Computer-derived transverse section through cusp tips.
Fig. 6.13  Computer-derived transverse section through mesial pulp horns.

Fig. 6.14  Computer-derived transverse section through roots.
Fig. 6.15  Computer-derived transverse section through roots near apices.

Fig. 6.16  Computer-derived transverse section at root apices.
Fig. 6.17  Detail of rat head/cylinder/film plane relationship at 0° longitude.

Fig. 6.18  Computer-derived axial section map of rat head at 60° longitude.
Fig. 6.19  Computer-derived sagittal section map at level of left ramus.

Fig. 6.20  Computer-derived frontal section map at level of mandible.
Fig. 6.21  Computer-derived transverse section map at level of ramus.

Fig. 8.1  Geometry of the line of sight ($\rho, \mu$).
path 1:

\[ \phi_{11} = 0 \text{ because } \rho \geq r_1 - \frac{1}{2L} \]

path 2:

\[ \phi_{11} = \arccos\left(\frac{\rho}{r_1 - \frac{1}{2L}}\right) \text{ because } \rho < r_1 - \frac{1}{2L} \]

Fig. 8.2 Visualization of double value of Eqs. (10.24) and (10.25).
Fig. 10.1 Radiograph in the process of digitization.

Fig. 10.2 Calibration curve used to convert digitized film intensities into logarithmic units of radiation exposure.
Fig. 11.1  Relation between calculated gray level and oscilloscope shade level for soft tissue enhancement (solid curve) and normal contrast (dashed line).

Fig. 11.2  Computer-derived normal contrast axial section at $40.5^\circ$ longitude.
Fig. 11.3 Computer-derived contrast-enhanced axial section corresponding to Fig. 11.2, indicating soft tissue of eyeball. (Because of printing and copying processes this figure does not show as many contrast levels as the original microfilm output of the computer oscilloscope.)
<table>
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<th>$M/\pi$</th>
<th>$M/\pi + 1$</th>
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<th>M-1</th>
<th>$r_1$</th>
<th>...</th>
<th>$r_2$</th>
<th>...</th>
<th>$r_L$</th>
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<td>...</td>
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TABLE II
For $M = 120$, $r_i$ and $m_i$ determined from Eqs. (9.3) and (9.8) are listed

|M = 120, $L = 39$|

<table>
<thead>
<tr>
<th>$i$</th>
<th>$r_i$</th>
<th>$m_i = \lfloor nr_iL \rfloor$</th>
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<td>0.654</td>
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<tr>
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<tr>
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For $M = 120$, $L = 39$: Then given $m$, the cutoff $I_m$ listed here is found from Table II and criterion (8-9.9).

<table>
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<th>$m$</th>
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<th>$r_{I_m}$</th>
<th>$r_{I_m}^{2m}$</th>
<th>$(2m+1)^{2m}$</th>
<th>$(2m+1)_{I_m}$</th>
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<td>37.69</td>
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<td>37.69</td>
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Note: $m=0$ corresponds to the constant (in the azimuthal sense) of the series of Eq. (4.1).
TABLE IV

Exposure Times and Digitized Intensities for Control Group of Blank Radiographs

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<td>1.000</td>
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THIS PROGRAM RECONSTRUCTS A 3-DIMENSIONAL OBJECT FROM A SET OF PROJECTIONS (X-RAY PICTURES OF THE OBJECT PROCESSED ACCORDING TO LAMBERT-BEER'S LAW). THE PROJECTIONS ARE TAKEN AT EQUALLY SPACED ANGLES OVER 180 DEGREES. THE PLANE OF EACH PROJECTION IS PARALLEL TO THE ROTATION AXIS. PARALLEL GEOMETRY IS ASSUMED FOR THE LINES OF SIGHT IN EACH PROJECTION.

AUTHOR - R. MICHAEL PERRY, NATIONAL CENTER FOR ATMOSPHERIC RESEARCH, BOULDER, COLORADO 80302

HISTORY - THIS PROGRAM IS AN IMPLEMENTATION OF A RECONSTRUCTION ALGORITHM DEVELOPED BY THE AUTHOR IN MARCH, 1974.

COMPUTER - CDC 7600 PROGRAMMING LANGUAGE - FORTRAN

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COMMON DUMMY(143)

CALL OVERLAY(1,0,FIRST)
CALL OVERLAY(2,0,RECON)
CALL OVERLAY(3,0,PLOT)
IF(I.EQ.1)CALL BRANCK(1)

END

LENGTH OF ROUTINE SECTION 35

VARIABLE ASSIGNMENTS

DUMMY - OC 0 FIRST - 30 RECON - 27 PLOT - 25 I - 25

SUBROUTINES CALLED

OVERLAY BRANCK EXIT

COMMON BLOCKS AND LENGTHS
- 217

COMPILE TIME = 10 MILLISECS
SUBROUTINE FIRST
OVERLAY(10, FIRST)

THIS SUBROUTINE COMPUTES AND STORES NUMERICAL COEFFICIENTS
WHICH ARE RETRIEVED AND USED EACH TIME A SECTION OF THE OBJECT
IS RECONSTRUCTED.

COMMON PI, OPT, NVIEWS, NSECT, IPAK, IRMAX, LIMJH, LIM2, LIMIN, DMAX
COMMON LIMAX, MMAX, NSKIP, NRAD(130)
DIMENSION MAX(41), ARRAY(11000), A(861), B(861), C(861)
DIMENSION FNT(861,4), SCM(861,2), SCA(861,2)
DIMENSION BK(41), RVL(10), WVL(10)
DATA (RVL(I), I=1,9)/.97390 65285 17172,.86506 33666 88985,.67940 95
1682 99024,.43339 53941 29247,.14887 43389 81631/
DATA (WVL(I), I=1,5)/.06667 13443 08688,.14945 13491 50581,.21908 63
1625 15929 67926 67193 09996,.29552 42241 14753/
DATA MXNTVBIPB/10,260,41935/
DATA NAMFLIMF, LIMF2/200,6888,5166/
DATA TWOPI/6.28318 53071 7959/
TIMEO=TIMEF(1.)
IOPT=0
NSKIP=1
PI=3.14159 26535 8979 $ LIMA=11000
MXP1=MNX+1
MXH=MXT/2 $ DO 33 INT=1,MXH $ RVL(IN)=5*(1.-RVL(IN))
133 RVL(MXP1-INT)=1.-RVL(INT) $ WVL(IN)=WVL(IN)*.5
33 WVL(MXP1-INT)=WVL(IN)
CALL RDTAPE(3,1,2,NVIEWS,4) $ CALL IOWAIT(3,NST,4)
IF(NST-1)37,38938
38 PRINT 39 $ STOP
39 FORMAT(//,10X,*BAD FIRST READ ON UNIT 3*)
37 IF(NSECT.GE.1)GO TO 46 $ PRINT 47,NSECT $ STOP
47 FORMAT(//,10X,*NSECT=*,I5,* NSECT MUST BE GT. ZERO*)
46 IF(1.LE.NVIEWS.AND.NVIEWS.LE.NVB)GO TO 41
42 FORMAT(//,10X,*NVIEWS=*,I5,* PERMISSIBLE RANGE IS 1 -*,I4)
41 IF(1.LE.IRMAX.AND.IRMAX.LE.IRB)GO TO 43
44 FORMAT(//,10X,*IRMAX=*,I5,* PERMISSIBLE RANGE IS 1 -*,I4)
43 IF(1.LE.IPAK.AND.IPAK.LE.IPB)GO TO 40
45 PRINT 44,IRMAX,IRB $ STOP
40 REWIND 3 $ LIMJH=IPAK*IRMAX+4 $ NVH=NAMES/2 $ IR2=2*IRMAX
13 MAXM(IR)=MINO(MMMAX)
13 IR=IPMAX
20 NSUM=O $ DO 18 M=1,MMAX
14 IF(MAXM(IR).GE.M)GO TO 20 $ IR=IR-1 $ GO TO 14
18 NSUM=NSUM+IR
17 XJ=IRMAX-.5 $ M=PI*XJ+1. $ MMAX=MINO(NVH*M)
16 DO 13 IR=1,IRMAX $ JR=IRMAX-IR+1 $ XJ=JR-.5 $ M=PI*XJ+1.
13 MAXM(IR)=MINO(M+MMAX)
14 IR=IRMAX $ NSUM=0 $ DO 18 M=1,MAX
15 IF(MAXM(IR).GE.M)GO TO 20 $ IR=IR-1 $ GO TO 14
18 NSUM=NSUM+IR
17 XJ=IRMAX-.5 $ M=PI*XJ+1. $ MMAX=MINO(NVH*M)
16 DO 13 IR=1,IRMAX $ JR=IRMAX-IR+1 $ XJ=JR-.5 $ M=PI*XJ+1.
13 MAXM(IR)=MINO(M+MMAX)
14 IR=IRMAX $ NSUM=0 $ DO 18 M=1,MAX
15 IF(MAXM(IR).GE.M)GO TO 20 $ IR=IR-1 $ GO TO 14
18 NSUM=NSUM+IR
17 XJ=IRMAX-.5 $ M=PI*XJ+1. $ MMAX=MINO(NVH*M)
16 DO 13 IR=1,IRMAX $ JR=IRMAX-IR+1 $ XJ=JR-.5 $ M=PI*XJ+1.
13 MAXM(IR)=MINO(M+MMAX)
14 IR=IRMAX $ NSUM=0 $ DO 18 M=1,MAX
15 IF(MAXM(IR).GE.M)GO TO 20 $ IR=IR-1 $ GO TO 14
18 NSUM=NSUM+IR
17 XJ=IRMAX-.5 $ M=PI*XJ+1. $ MMAX=MINO(NVH*M)
16 DO 13 IR=1,IRMAX $ JR=IRMAX-IR+1 $ XJ=JR-.5 $ M=PI*XJ+1.
13 MAXM(IR)=MINO(M+MMAX)
14 IR=IRMAX $ NSUM=0 $ DO 18 M=1,MAX
15 IF(MAXM(IR).GE.M)GO TO 20 $ IR=IR-1 $ GO TO 14
18 NSUM=NSUM+IR
17 XJ=IRMAX-.5 $ M=PI*XJ+1. $ MMAX=MINO(NVH*M)
16 DO 13 IR=1,IRMAX $ JR=IRMAX-IR+1 $ XJ=JR-.5 $ M=PI*XJ+1.
DO 1 J=1,I
$ YR=RRCP*(IRJ+J)
$ IJ=I2J-J
IF(J.EQ.1)T1=RT1=0.
$ RT2=2.*SQRT(Y*R-Y)
FNT(IJ,1)=RT2-RT1
$ RT1=RT2
IF(MMAX.LE.1)GO TO 2
T2=ACOS(X/R)
$ C1=COS(.5*(T1*T2))
$ S1=SIN(.5*(T2-T1))
RATIO=(C1+S1)/(C1-S1)
FNT(IJ,2)=2.*XR*ALOG(RATIO)
$ IF(MMAX.LE.2)GO TO 2
FNT(IJ,3)=4.*XR*(T2-T1)-FNT(IJ,1)
$ IF(MMAX.LE.3)GO TO 2
IF(J.EQ.1)S12=0.
$ S1=S12
$ S2=SIN(T2)
FNT(IJ,4)=8.*XR*(S2-S1)-3.*FNT(IJ,2)
$ S12=S2
SCA(IJ,1)=SCM(IJ,1)
$ SCA(IJ,2)=SCM(IJ,2)=COS(T2)
2 T1=T2
1 CONTINUE
NAMNT=NAMF+INT
$ CALL BRANWT(NAMNT,FNTLIMF)
36 CALL BRANCK(NAMNT)
NAME=1
$ MM2=0
AMAX=0.
DO 4 M=1,MMAX
$ IMAX=NRAD(M)
$ MA=IRMAX*M
LIMR=(IMAX*IMAX+IMAX)/2
$ M1=2-(M-2*(M/2))
$ M2=M1+2
DO 23 IJ=1,LIMR
23 A(IJ)=0.
DO IR=1,IRMAX
$ I2=(IR*IR-IR)/2
$ I=MINO(IR,IMAX)
DO 24 J=1,I
$ IJ=I2+J
24 B(IJ)=0.
DO INT=1,MXNT
$ NAMNT=NAMF+INT
$ CALL BRANRD(NAMNT,FNTLIMF)
$ CALL BRANCK(NAMNT)
WV=WVL(INT)
$ IF(M.GT.4)GO TO 5
DO IR=1,IRMAX
$ I2=(IR*IR-IR)/2
$ I=MINO(IR,IMAX+1)
DO J=1,I
$ IJ=I2+J
$ JK=J2+K
5 A(JK)=A(JK)+WV*B(IJ)
6 A(JK)=A(JK)+WV*B(IJ)
$ RV=RVL(INT)
DO IR=1,IRMAX
$ I2=(IR*IR-IR)/2
$ I=MINO(IR,IMAX+1)
DO J=1,I
$ IJ=I2+J
$ SUM=0.
DO K=1,IMAX
$ K2=(K*K-K)/2
$ JK=J2+K
8 A(JK)=A(JK)+WV*B(IJ)
12 CALL ORTHO(A,C,BK,IMAX,LIMF)
3 CONTINUE
CALL BRANWT(NAMNT,FNTLIMF)
$ CALL BRANCK(NAMNT)
APPROXIMATE
PROGRAM LOCATION

140  S1240    MIJ=MM2+IJ $ IF(MIJ.LE.LIMA) GO TO 11
141  S1241    CALL BRANWT(NAME+ARRAY+LIMA)
142  S1252    CALL BRANCK(NAME) $ NAME=NAME+1 $ MM2=MM2-LIMA $ MIJ=1
143  S1276    11 ARRAY(MIJ)=SUM
144  S1301    CALL 8RANWT(NAMEARRAYLIMA)
145  S1310    CALL BRANCK(NAME) $ NAME=NAME+1 $ MM2=MM2-LIMA $ MIJ=1
146  S1310    30 FORMAT(/,10X,*FIRST INV. COEF. = *,E12.4,* MAX. ABS. VAL. OF CO
147  S1310    1EF. IS*,E12.4)
148  S1320    11 ARRAY(MIJ)=SUM
149  S1320    CALL 8RANWT(NAMEARRAYLIMA)
150  S1320    CALL BRANCK(NAME)
151  S1325    PRINT 30,NVH $ PRINT 319IR2
152  S1332    50 FORMAT(/,1OX,*NUMBER OF PROJECTIONS COVERING 180 DEGREES IS*,I4)
153  S1332    51 FORMAT(/,1OX,*NUMBER OF POINTS PER VIDEO LINE, IN EFFECT, IS*,I4)
154  S1332    RETURN
155  S1332    END

LENGTH OF ROUTINE FIRST 51553

VARIABLE ASSIGNMENTS

MAXM - 0  ARRAY - 51  A - 25441  B - 27176  C - 30733  FNT - 32470
SCH - 41254  SCA - 44546  RK - 50046  RVL - 50111  WVL - 50123  PI - 0C 0
IOPT - 1C 0  VIEW - 2C 0  ISELECT - 3C 0  IPAK - 4C 0  IREMAX - 5C 0  LIMJH - 6C 0
LIM2 - 7C 0  LIMIN - 10C 0  DMAX - 11C 0  LIMA - 12C 0  MMAX - 13C 0  NSKIP - 14C 0
NRA - 15C 0  MXNT - 50135  NVB - 50136  IRB - 50137  IPB - 50140  NAMF - 50141
LIMF - 50142  LIMF2 - 50143  TMOP - 50144  TIMF0 - 51432  MXP1 - 51431  MXXH - 51430
INT - 51427  NST - 51426  NVH - 51425  IR2 - 51424  XR - 51423  M - 51422
IR - 51421  JR - 51420  NSUM - 51417  RPRG - 51416  NAMAX - 51415  NAMF - 51414
RV - 51413  X - 51412  IRJ - 51411  XR - 51410  X2 - 51407  IZJ - 51400
J - 51405  YR - 51404  IJ - 51403  T1 - 51402  RT1 - 51401  RT2 - 51400
T2 - 51377  C1 - 51376  S1 - 51375  TRATIO - 51374  S12 - 51373  S2 - 51372
NAMNT - 51371  MM2 - 51370  AMAX - 51367  IMAX - 51366  MA - 51365  LIMR - 51364
M1 - 51363  M2 - 51362  IZ - 51361  WV - 51360  J2 - 51357  BIJ - 51356
K - 51355  IK - 51354  JK - 51353  C2 - 51352  R - 51351  F1 - 51350
F2 - 51347  SUM - 51346  K2 - 51345  KI - 51344  KJ - 51343  IZ2 - 51342
JJ - 51341  ALLI - 51340  MIJ - 51337  T - 51336

SUBROUTINES CALLED

TIMEF RDTAPE IOWAIT OUTPTC STOP REWINM BRANWT BRANCK SQRTF ACOSF COSF SINF LOGF
BRANOR ORTHO

COMMON BLOCKS AND LENGTHS
- 217

COMPILE TIME = 395 MILLISECS
SUBROUTINE ORTHO(A,B,WORK,N,N2)
OVERLAY(1.0,FIRST)

THIS SUBROUTINE COMPUTES GRAM-SCHMIDT ORTHONORMALIZATION CO-
EFFICIENTS FROM VALUES CONTAINED IN A AND STORES THESE COEFFI-
CIENTS IN B. A, B, AND WORK (A SCRATCH-SPACE ARRAY) ARE DIMEN-
SIONED N2, N2, AND N, RESPECTIVELY. N2 AND N MUST BE RELATED BY

N2 = (N*(N+1))/2

INNER PRODUCTS OF FUNCTIONS F1, ... , FN ARE STORED IN A. THE
VALUE STORED IN A(I,J) IS THE INNER PRODUCT OF FI AND FJ, WHERE

IJ = (I*(I-1))/2 + J

THE ORTHONORMALIZATION COEFFICIENTS STORED IN B HAVE THE PROPERTY
THAT THE FUNCTIONS GI DEFINED BY

GI = SUM, J=1 TO I, OF B(I,J)*FJ

FORM AN ORTHONORMAL SET UNDER THE INNER PRODUCT INDICATED IN A.

IF THE VALUE OF A(I,I) REPRESENTING THE SQUARED NORM OF THE
FUNCTION FI, IS LESS THAN ZERO A DIAGNOSTIC IS PRINTED AND EXECU-
TION TERMINATES. THIS ALSO HAPPENS IF A(1) IS ZERO. AS A PRECAUTION AGAINST ILL-CONDITIONING THE RATIO

(NORM(FI-FILS)/NORM(FI))**2

IS EVALUATED FOR EACH FUNCTION FI. HERE FILS IS THE LEAST-MEAN
SQUARE APPROXIMATION TO FI IN TERMS OF FJ, FOR J LESS THAN I.
IF THIS RATIO IS LESS THAN EPS THE COEFFICIENTS B(I,J) FOR THE
GIVEN VALUE OF I ARE ALL SET TO ZERO. OTHERWISE B(I,J) WILL CON-
TAIN THE USUAL ORTHONORMALIZATION COEFFICIENTS.

DIMENSION A(N2),B(N2),WORK(N)
DATA EPS=1.1171,E-10,1,1/
IF(A(1).GT.0.)8,9
9 PRINT 10,IIA(II) $ STOP
10 FORMAT(/'ORTHO ERR MESSAGE - NORM OUT OF RANGE FOR ORDER I=*'
11 I=1,8
12 IF(A(I).LT.1.E-10)GO TO 9
13 B(I)=1. $ IF(I.EQ.1)GO TO 7 $ IM=I-1
14 DO 2 J=1,IM $ SUM=0. $ J2=(J*J-J)/2
15 DO 3 JP=1,J $ IJP=(I*IM-J)/2+JP
16 SUM=SUM-A(IJP)*B(JJP)
17 3 WORK(J)=SUM
18 DO 4 J=1,IM $ SUM=SUM-A(J)*B(J)
19 4 IF(SNORM.LT.1.E-10)GO TO 9
20 R=SNORM/SNO $ IF(R.EQ.0.)SNORM=1./SORT(SNORM)
21 IF(R.EQ.0.)SNORM=1./SORT(SNORM)
22 DO 5 IP=1,IM $ SUM=SUM-A(IP)*B(IP)
23 5 B(IP)=SUM
24 DO 6 IP=1,IM $ SUM=SUM-A(IP)*B(IP)
25 6 IF(IP.EQ.1)GO TO 7 $ J2=I2-1
26 DO 7 J=1,J $ J2=I2-1-1+J $ JJP=J2+JP
27 SUM=SUM-A(J)*B(JJP)
28 7 IF(IP.EQ.1)GO TO 9 $ SNORM=SNORM/SN0
29 IF(R.EQ.0.)SNORM=1./SORT(SNORM)
30 IF(R.EQ.0.)SNORM=1./SORT(SNORM)
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<tr>
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<td>155 RETURN</td>
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<td>156 END</td>
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LENGTH OF ROUTINE ORTHO 234

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SUBROUTINES CALLED

OUTPTC  SQRTF  QRQRS

COMPILE TIME = 66 MILLISECS
SUBROUTINE RECON
OVERLAY(2,0,RECON)
THIS SUBROUTINE DOES THE ACTUAL RECONSTRUCTION OF THE OBJECT
USING THE COEFFICIENTS COMPUTED IN SUBROUTINE FIRST.
APPROXIMATE PROGRAM LOCATION

LENGTH OF ROUTINE RECON 71215

VARIABLE ASSIGNMENTS

ND - 0 WORK - 1 0 - 1 XRAY - 30015 FT - 3222 ARRAY - 30015
FR - 55405 IDNS - 55456 IDN - 63144 ID2 - 63550 JHED - 63656 JXRAY - 63662
JX - 66515 JSL - 66560 FTO - 67164 CCP - 71061 CCN - 71063 RNW - 71065
P1 - 0C 0 10PT - 1C 0 NVIEWS - 2C 0 NSECT - 3C 0 IPAK - 4C 0 IRMAX - 5C 0
LIMJH - 6C 0 LIM2 - 7C 0 LIMIN - 1C 0 DMAX - 1C 0 LIMA - 1C 0 MMAX - 1C 0
NSKIP - 14C 0 TRAD - 15C 0 ROOF - 70174 LTST - 71126 TIM0 - 71125 ISECT - 71124
NST - 71123 I - 71122 12 - 71121 J - 71120 IJ - 71117 JV - 71116
ISK - 71115 M - 71114 MM - 71113 ML1 - 71112 ML0 - 71111 NAME - 71110
MM2 - 71107 MAX - 71106 MA - 71105 MJJ - 71104 L - 71103 ML - 71102
JJ - 71101 II - 71100 SUM - 71077 11 - 71076 IK0 - 71075 N - 71074
MN - 71073 DN - 71072 K - 71071 IK - 71070 T - 71067

SUBROUTINES CALLED
ROTAPE IOWAIT OUTPTC STOP GBYTES FOURSE BRAN0 BRANCK SBYTES WRTAPE REWINM

TIMEF RDTAPE IOWAIT

COMMON BLOCKS AND LENGTHS

- 217

COMPILE TIME = 242 MILLISECONDS
FUNCTION OVERLAY(3,0,PLOT)

THIS SUBROUTINE OUTPUTS THE RESULTS OF THE RECONSTRUCTION OF THE OBJECT DONE IN SUBROUTINE RECON.

COMMON PI, IOPT, NVIEWS, NSECT, IPAK, IRMAX, LIMJH, LIM2, LIMIN, DMX

DIMENSION ZMAP(128,128), DNS(41,260)

DIMENSION RCOR(2080), TCOR(2080), Q(292)

DIMENSION IDNS(2870), IDN(260), ID2(70), JDN(35), JHED(4), JDNS(1435)

DIMENSION KHED(4), KDNS(12,128), KDN(128), KD2(12)

DATA TWOPILIMK/6.28318 53071 7959, 1540/

VIEWS = NVIEWS $ FN = .?5 * VIEWS $ VEW = VIEWS / IOPT $ RMAX = IRMAX

RDMX = 1./DMAX $ SCAL = 255.*RDMX $ RRE = .1.

JHED(1) = NVIEWS $ JHED(2) = NSECT $ JHED(3) = IPAK $ JHED(4) = IRMAX

JHED(1) = NSECT $ JHED(2) = 128 $ JHED(3) = 12 $ JHED(4) = IRMAX

DO 1 I = 1, 64 $ I2 = (I * I - I) / 2 $ XI = I -.5 $ X2 = XI * XI

DO 1 J = 1, I $ IJ = I2 + J $ YJ = J -.5 $ Y2 = YJ * YJ

RR = RMAX * SQRT(X2 + Y2) $ IF (RR.GE.RMAX) KR = -1. $ RCOR(IJ) = RR + 1.

TCOR(IJ) = ATAN(YJ/XI) * VEW


DO 3 ISECT = 1, NSECT $ CALL RDTAPE(511, 2, IDNS, LIMIN)

CALL IOWAIT(511, NSECT, LIMIN) $ IF (NST = 1) 10, 11, 11

1 PRINT I2, ISECT $ STOP

DO 4 IR = 1, IRMAX $ IJO = (IR - 1) * LIM2 $ DO 13 J = 1, LIM2 $ IJ = IJO + J

13 ID2(J) = IDNS(IJ) $ CALL GBYTES(ID2, IDN, 0, 9016, NVIEWS)

DO 14 J = 1, NVIEWS

14 DNS(IR, J) = IDN(J)

DO 15 J = 1, NVIEWS

15 IDN(J) = SCAL * IDN(J) + .5

CALL SBYTES(JD, LIMJH, 0, 8, 0, NVIEWS)

12 = (12 - 1) $ IPAK = 0 J = 1 + IPAK $ IJ = IJ + J

14 DO 19 J = 1, NVIEWS

19 DO 20 I = 1, 128 $ Z = 16.*ZMAP(IJ) $ DZ = Z - AINT(Z)

20 IZ = Z $ IF (DZ.GE.0.) IZ = IZ + 1

CALL SBYTES(KDN, IZ, 0, 5, 0, 128)

DO 20 K = 1, 12

20 KDNS(K, J) = KD2(K)

CALL PHOMAP(ZMAP, 128, 128, 0.91, 0.0, 1.0, 0.) $ CALL LABEL(ISECT)

CALL FRAME $ GO TO 19

2 CALL PRNTMAP(ZMAP, 128, 128, 0.0, 1.0, 0.0, 0.0, 0.) $ CALL LABEL(ISECT)
APPROXIMATE PROGRAM LOCATION
112014
112021
112023
112026
112027
CALL WRTAPE(NU,1,2,KHEDLIMK) $ CALL IOWAIT(NU,NST,LIMK)
IF(NST-1) 317,17
3 CONTINUE
RETURN
END

LENGTH OF ROUTINE PLOT 112141

VARIABLE ASSIGNMENTS

<table>
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<tr>
<th>Variable</th>
<th>Assignments</th>
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SUBROUTINES CALLED

REWINM SQRTF ATANF ROTAPE IOWAIT OUTPTC STOP GBYES SBYTES WRTAPE PHOMAP LABEL FRAME

COMMON BLOCKS AND LENGTHS

- 12

COMPILE TIME = 192 MILLISECS
SUBROUTINE PRNTMAP(ZMAP, M, N, X0, X1, Y0, Y1, Z0, Z1)
OVERLAY(3,0,PLOT)

C
C THIS SUBROUTINE PRINTS A RECTANGULAR GRAY-SCALE PATTERN CORRESPONDING TO THE ZMAP ARRAY, OVER THE RANGE Z0 TO Z1. INTENSITIES PLOTTED ARE LINEARLY PROPORTIONAL TO THE VALUES OF ZMAP-Z0 EXCEPT THAT SATURATION OCCURS AT THE CORRESPONDING LIMIT WHEN ZMAP FALLS OUTSIDE THE RANGE Z0 TO Z1. DARKER INTENSITIES CORRESPOND TO HIGHER VALUES OF ZMAP EXCEPT WHEN Z1 IS LESS THAN Z0. FOR THIS CASE THE INTENSITIES ARE PROPORTIONAL TO Z0-ZMAP SO THAT HIGHER VALUES GIVE LIGHTER INTENSITIES. IF Z0=Z1 THE PROGRAM USES THE MINIMUM AND MAXIMUM VALUES OF THE ZMAP ARRAY IN PLACE OF Z0 AND Z1, RESPECTIVELY.

THE PARAMETERS X0, X1, Y0, AND Y1 SPECIFY HOW MUCH OF THE AVAILABLE SPACE IS USED FOR PLOTTING THE PICTURE. THE SCALING IS LINEAR-LINEAR WITH X AND Y VALUES LIMITED TO THE RANGE 0. TO 1. THAT IS IF X0=0., AND X1=1., ALL THE AVAILABLE SPACE IS USED AND A SQUARE PICTURE IS PLOTTED. HOWEVER IF X0=0., X1=1., AND Y1=0.5 ONLY THE LOWER HALF OF THE AVAILABLE FIELD IS USED AND THE PLOTTED PICTURE IS TWICE AS WIDE AS IT IS TALL.

DIMENSION LVLS(10), CONLVL(10), IMAP(108) * ZMAP(MN)
DATA NLVL(1), 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5 /
DATA (CONLVL(I), I=1, 10) / 0.1, 2, 4, 6, 8, 10 /
DATA (LVLS(I), I=1, 10) / 55B, 46B, 50B, 32B, 02B /
PRINT 9
IF(IFRST.NE.0) GO TO 7 $ IFRST=1

DO 11 I=1, NLVL $ J=LVLS(I) $ CALL GBYTE(J, J, 54, 60)
11 LVLS(I)=J.OR.I55
SCAL=CONLVL(NLVL)

IMIN=108.*X0+1. $ IMAX=108.*X1+1. $ JMIN=64.*Y0+1. $ JMAX=64.*Y1+1.
IMIN=MAXO(IMIN, 1) $ IMAX=MINO(IMAX, 108) $ JMIN=MAXO(JMIN, 1) $ JMAX=MINO(JMAX, 64)
IF(IMIN.GT.IMAX.OR.JMIN.GT.JMAX) RETURN

ZMIN=ZO $ ZMAX=Z1 $ IF(ZMIN.EQ.ZMAX)GO TO 3 $ ZMIN=ZMAX=ZMAP(1, 1)

DO 211 I=1, N $ DO 11 J=1, N $ Z=ZMAP(I, J)
11 ZMIN=AMIN1(ZMIN, Z)
211 ZMAX=AMAX1(ZMAX, Z)

IL=IMAX-IMIN+1 $ JL=JMAX-JMIN+1

A=YJمدن+.5 $ NO=A $ AN=NO $ DN=A-AN $ M1=MINO(M1, NO+1) $ M0=MAXO(M0, NO-1)

DO 311 DZ=(Z-CONLVL(IO))/(CONLVL(I1)-CONLVL(IO))$ I=I0+1
311 IMAP(I)=LVLS(I0)

PRINT 89 (IMAP(I), I=IMIN, IMAX)
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<tr>
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<th>APPROXIMATE PROGRAM LOCATION</th>
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**LENGTH OF ROUTINE PRNTMAP 660**

**VARIABLE ASSIGNMENTS**

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<th>CONLVL</th>
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<th>NLVL</th>
<th>T</th>
<th>J</th>
<th>SCAL</th>
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<th>JMAX</th>
<th>SCAL</th>
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<th>DM</th>
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<tr>
<td>555</td>
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</tbody>
</table>

**SUBROUTINES CALLED**

OUTP TCGBYTE PRANF QORSO

**COMPILE TIME = 204 MILLISECS**
SUBROUTINE PHOMAP(ZMAP,M,N,X0,X1,Y0,Y1,Z0,Z1)
OVERLAY(3,Y0,PLOT)

THIS SUBROUTINE PLOTS ON MICROFILM A RECTANGULAR GRAY-SCALE PATTERN CORRESPONDING TO THE ZMAP ARRAY OVER THE RANGE Z0 TO Z1. INTENSITIES PLOTTED ARE LINEARLY PROPORTIONAL TO THE VALUES OF ZMAP-Z0 EXCEPT THAT SATURATION OCCURS AT THE CORRESPONDING LIMIT WHEN ZMAP FALLS OUTSIDE THE RANGE Z0 TO Z1. LIGHTER INTENSITIES CORRESPOND TO HIGHER VALUES OF ZMAP EXCEPT WHEN Z1 IS LESS THAN Z0. FOR THIS CASE THE INTENSITIES ARE PROPORTIONAL TO Z0-ZMAP SO THAT HIGHER VALUES GIVE DARKER INTENSITIES. IF Z0=Z1 THE PROGRAM USES THE MINIMUM AND MAXIMUM VALUES OF THE ZMAP ARRAY IN PLACE OF Z0 AND Z1, RESPECTIVELY.

THE PARAMETERS X0, X1, Y0, AND Y1 SPECIFY HOW MUCH OF THE AVAILABLE SPACE IS USED FOR PLOTTING THE PICTURE. THE SCALING IS LINEAR-LINEAR WITH X AND Y VALUES LIMITED TO THE RANGE 0. TO 1. Thus if X0=0, X1=1, and Y0=0, Y1=1, ALL THE AVAILABLE SPACE IS USED AND A SQUARE PICTURE IS PLOTTED. HOWEVER IF X0=Y0=0., XI=1., AND Y1=.5 ONLY THE LOWER HALF OF THE AVAILABLE FIELD IS USED AND THE PLOTTED PICTURE IS TWICE AS WIDE AS IT IS TALL.

DIMENSION ZMAP(M,N)
DATA SCAL/16./
IMIN=128.*XO+1. $ IMAX=128.*X1+1.
JMIN=128.*Y0+1. $ JMAX=128.*Y1+1.
IMIN=MAXO(IMIN,IMIN) $ IMAX=MINO(IMAX,IMAX)
JMIN=MAXO(JMIN,JMIN) $ JMAX=MINO(JMAX,JMAX)
IF(IMIN.GT.IMAX.OR.JMIN.GT.JMAX) RETURN
ZMIN=Z0 $ ZMAX=Z1 $ IF(ZMIN.EQ.ZMAX)RETURN $ RDZ=1./(ZMAX-ZMIN)
IL=IMAX-IMIN+1 $ XL=IL $ YL=JMAX-JMIN+1 $ YN=YN
A=YN+1 $ N=0 $ AN=AN $ DN=DN $ NI=MINO(N,N+1)
M0=MAXO(NO(NO)
M0=MAXO(M0(M0)
P1=ZMAP(M0,N0) $ P2=ZMAP(M1,N0) $ P3=ZMAP(M0,N1) $ P4=ZMAP(M1,N1)
P1=P1*0.1+DM $ P2=P2*0.1+DM $ P3=P3*0.1+DM $ P4=P4*0.1+DM
Z=Z1-ZMIN $ Z=SCAL*AMAX1(1,Z)
IF(ZMIN.GT.0.5) RETURN $ CALL GRAY(I,J,Z)
CONTINUE
RETURN
END.
<table>
<thead>
<tr>
<th>Monitor</th>
<th>271</th>
<th>RDZ</th>
<th>270</th>
<th>IL</th>
<th>267</th>
<th>XL</th>
<th>266</th>
<th>YL</th>
<th>265</th>
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</tr>
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<tbody>
<tr>
<td>YN</td>
<td>263</td>
<td>YJ</td>
<td>262</td>
<td>A</td>
<td>261</td>
<td>N0</td>
<td>260</td>
<td>AN</td>
<td>257</td>
<td>DN</td>
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<tr>
<td>N1</td>
<td>255</td>
<td>X1</td>
<td>254</td>
<td>II</td>
<td>253</td>
<td>40</td>
<td>252</td>
<td>AM</td>
<td>251</td>
<td>DM</td>
</tr>
<tr>
<td>M1</td>
<td>247</td>
<td>P1</td>
<td>246</td>
<td>P2</td>
<td>245</td>
<td>P3</td>
<td>244</td>
<td>P4</td>
<td>243</td>
<td>IZ</td>
</tr>
<tr>
<td>DZ</td>
<td>241</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
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</tr>
</tbody>
</table>

SUBROUTINES CALLED

RANF  GRAY  QBURSD

COMPILE TIME = 148 MILLISECS
SUBROUTINE GRAY(I,J,IZ)

OVERLAY(3,0,PLOT)

DIMENSION IFOT(16),JFOT(16)

DATA(IFOT(I), I=1,16)/1959195939793,79397939791,5,91,959919593979397931/

DATA(JFOT(I), I=1,16)/1959195939793,79397939791,5,91,959919593979397931/

IF(IZ.EQ.0)RETURN $ I1=8*(I-1) $ J1=8*(J-1)

DO 2 II=1,IZ $ MX=II+IFOT(II) $ MY=J1+JFOT(II)

CALL POINT(MX,MY) $ RETURN

END

LENGTH OF ROUTINE GRAY 107

VARIABLE ASSIGNMENTS

I - 0 J - 0 IZ - 0 IFOT - 0 JFOT - 20 I1 - 100

J1 - 77 XI - 76 MX - 75 MY - 74

SUBROUTINES CALLED

POINT Q8QRSD

COMPILE TIME = 15 MILLISECS
SUBROUTINE LABEL(II)
OVERLAY(39,0PLOT)
1 FORMAT(1X,I39,1X)
CALL OPTION(0,1*0,0)
ENCORE(5,1*1)
MX=9 $ MY=41 $ CALL PWRT(MX,MY,5HCROSS,5,2,0)
MY=9 $ CALL PWRT(MX,MY,7HSECTON,7,2,0) $ MX=951 $ MY=41
CALL PWRT(MX,MY,SHFRAHE,5,2,0) $ MY=9 $ CALL PWRT(MX,MY,1,5,2,0)
CALL OPTION(0*0,0,0)
RETURN
END

LENGTH OF ROUTINE LABEL 121

VARIABLE ASSIGNMENTS
II - 0 I - 103 MX - 102 MY - 101

SUBROUTINES CALLED
OPTION OUTPTS PWRT QRORSO

COMPIL TIME = 17 MILLISECS
SUBROUTINE FOURT(DATA, NN, NDM, ISIGN, IFORM, WORK)
OVERLAY(2, 0, RECON)

THE COOLEY-TUKEY FAST FOURIER TRANSFORM IN USASI BASIC FORTRAN
TRANSFORM (J1, J2, ..., JN) = SUM (DATA(I1, I2, ..., IN) * W1**((I2-1) * (J2-1) + ...)
WHERE I1 AND J1 RUN FROM 1 TO NN(1) AND W1 = EXP(ISIGN*2*PI/SQRT(-1)/NN(1)), ETC. THERE IS NO LIMIT ON THE DIMENSIONALITY (NUMBER OF SUBSCRIPTS) OF THE DATA ARRAY. IF AN INVERSE TRANSFORM (ISIGN = 1) IS PERFORMED UPON AN ARRAY OF TRANSFORMED DATA, THE ORIGINAL DATA WILL REAPPEAR MULTIPLIED BY NN(1)*NN(2)*..., THE ARRAY OF INPUT DATA MUST BE IN COMPLEX FORMAT. HOWEVER, IF ALL IMAGINARY PARTS ARE ZERO (I.E. THE DATA ARE DISGUISED REAL), RUNNING TIME IS CUT UP TO FORTY PERCENT. (FOR FASTEST TRANSFORM OF REAL DATA, NN(1) SHOULD BE EVEN.) THE TRANSFORM VALUES ARE ALWAYS COMPLEX AND ARE RETURNED IN THE ORIGINAL ARRAY OF DATA, REPLACING THE INPUT DATA. THE LENGTH OF EACH DIMENSION OF THE DATA ARRAY MAY BE ANY INTEGER. THE PROGRAM RUNS FASTER ON COMPOSITE INTEGERS THAN ON PRIMES, AND IS PARTICULARLY FAST ON NUMBERS RICH IN FACTORS OF TWO.

TIMING IS IN FACT GIVEN BY THE FOLLOWING FORMULA. LET NTOT BE THE TOTAL NUMBER OF POINTS (REAL OR COMPLEX) IN THE DATA ARRAY, THAT IS, NTOT = NN(1)*NN(2)*... DECOMPOSE NTOT INTO ITS PRIME FACTORS, SUCH AS 2**K2 * 3**K3 * 5**K5 * ... LET SUM2 BE THE SUM OF ALL THE FACTORS OF TWO IN NTOT, THAT IS, SUM2 = 2*K2. LET SUMF BE THE SUM OF ALL OTHER FACTORS OF NTOT, THAT IS, SUMF = 3*K3*5*K5*... THE TIME TAKEN BY A MULTIDIMENSIONAL TRANSFORM ON THESE NTOT DATA POINTS IS T = NTOT * (T1 + T2*SUM2 + T3*SUMF). ON THE CDC 3300 (FLOATING POINT ADD TIME = SIX MICROSECONDS), T = 3000 + NTOT*(600+40*SUM2+175*SUMF) MICROSECONDS ON COMPLEX DATA.

IMPLEMENTATION OF THE DEFINITION BY SUMMATION WILL RUN IN A TIME PROPORTIONAL TO NTOT*(NN(1)+NN(2)+ ...). FOR HIGHLY COMPOSITE NTOT, THE SAVINGS OFFERED BY THIS PROGRAM CAN BE DRAMATIC. A ONE-DIMENSIONAL ARRAY 4000 IN LENGTH WILL BE TRANSFORMED IN 4000*(600+40*SUM2+175*SUMF) MICROSECONDS ON COMPLEX DATA.

THE FAST FOURIER TRANSFORM PLACES THREE RESTRICTIONS UPON THE DATA,
1. THE NUMBER OF INPUT DATA AND THE NUMBER OF TRANSFORM VALUES MUST BE THE SAME.
2. BOTH THE INPUT DATA AND THE TRANSFORM VALUES MUST REPRESENT EQUISPACED POINTS IN THEIR RESPECTIVE DOMAINS OF TIME AND FREQUENCY. CALLING THESE SPACINGS DELTAT AND DELTAF, IT MUST BE TRUE THAT DELTAF = 2*PI/NN(1)*DELTAT. OF COURSE, DELTAT NEED NOT BE THE SAME FOR EVERY DIMENSION.
3. CONCEPTUALLY AT LEAST, THE INPUT DATA AND THE TRANSFORM OUTPUT REPRESENT SINGLE CYCLES OF PERIODIC FUNCTIONS.

THE CALLING SEQUENCE IS--
CALL FOURT(DATA,NN,NDIM,ISIGN,IFORM,WORK)

DATA IS THE ARRAY USED TO HOLD THE REAL AND IMAGINARY PARTS OF THE DATA ON INPUT AND THE TRANSFORM VALUES ON OUTPUT. IT IS A MULTIDIMENSIONAL FLOATING POINT ARRAY, WITH THE REAL AND IMAGINARY PARTS OF A DATUM STORED IMMEDIATELY ADJACENT IN STORAGE (SUCH AS FORTRAN IV PLACES THEM). NORMAL FORTRAN ORDERING IS EXPECTED, THE FIRST SUBSCRIPT CHANGING FASTEST. THE DIMENSIONS ARE GIVEN IN THE INTEGER ARRAY NN, OF LENGTH NDIM. ISIGN IS -1 TO INDICATE A FORWARD TRANSFORM (EXPONENTIAL SIGN IS -) AND +1 FOR AN INVERSE TRANSFORM (SIGN IS *). IFORM IS +1 IF THE DATA ARE COMPLEX, 0 IF THE DATA ARE REAL. IF IT IS 0, THE IMAGINARY PARTS OF THE DATA MUST BE SET TO ZERO. AS EXPLAINED ABOVE, THE TRANSFORM VALUES ARE ALWAYS COMPLEX AND ARE STORED IN ARRAY DATA.

WORK IS AN ARRAY USED FOR WORKING STORAGE. IT IS FLOATING POINT REAL, ONE DIMENSIONAL OF LENGTH EQUAL TO TWICE THE LARGEST ARRAY DIMENSION NN(I) THAT IS NOT A POWER OF TWO. IF ALL NN(I) ARE POWERS OF TWO, IT IS NOT NEEDED AND MAY BE REPLACED BY ZERO IN THE CALLING SEQUENCE. THUS, FOR A ONE-DIMENSIONAL ARRAY, NN(1) ODD, WORK OCCUPIES AS MANY STORAGE LOCATIONS AS DATA. IF SUPPLIED, WORK MUST NOT BE THE SAME ARRAY AS DATA. ALL SUBSCRIPTS OF ALL ARRAYS BEGIN AT ONE.

EXAMPLE 1. THREE-DIMENSIONAL FORWARD FOURIER TRANSFORM OF A COMPLEX ARRAY DIMENSIONED 32 BY 25 BY 13 IN FORTRAN IV.

DIMENSION DATA(32,25,13),WORK(50),NN(3)

COMPLEX DATA

DATA NN/32,25,13/

DO 1 I=1,32

DO 1 J=1,25

DO 1 K=1,13

1 DATA(I,J,K)=COMPLEX VALUE

CALL FOURT(DATA,NN,NN,ISIGN,IFORM,WORK)

EXAMPLE 2. ONE-DIMENSIONAL FORWARD TRANSFORM OF A REAL ARRAY OF LENGTH 64 IN FORTRAN IV.

DIMENSION DATA(2,64)

DO 2 I=1,64

DATA(1,I)=REAL PART

DATA(2,I)=0.

CALL FOURT(DATA,64,1,0,0)

THERE ARE NO ERROR MESSAGES OR ERROR HALTS IN THIS PROGRAM. THE PROGRAM RETURNS IMMEDIATELY IF NDIM OR ANY NN(I) IS LESS THAN ONE.

PROGRAM BY NORMAN BRENNER FROM THE BASIC PROGRAM BY CHARLES RADER, JUNE 1967. THE IDEA FOR THE DIGIT REVERSAL WAS SUGGESTED BY RALPH ALTER.

THIS IS THE FASTEST AND MOST VERSATILE VERSION OF THE FFT KNOWN TO THE AUTHOR. A PROGRAM CALLED FOUR2 IS AVAILABLE THAT ALSO PERFORMS THE FAST FOURIER TRANSFORM AND IS WRITTEN IN USASI BASIC FORTRAN. IT IS ABOUT ONE THIRD AS LONG AND RESTRICTS THE DIMENSIONS OF THE INPUT ARRAY (WHICH MUST BE COMPLEX) TO BE POWERS OF TWO. ANOTHER PROGRAM, CALLED FOUR1, IS ONE TENTH AS LONG AND RUNS TWO THIRDS AS FAST ON A ONE-DIMENSIONAL COMPLEX ARRAY WHOSE...
LENGTH IS A POWER OF TWO.

REFERENCES--
IEEE AUDIO TRANSACTIONS (JUNE 1967) SPECIAL ISSUE ON THE FFT.

DIMENSION DATA(1)*NN(1)*IFACT(32)*WORK(1)

DATA TWOPI/6.2831853071796/ RTHLF/0.70710678118655/

DATA NPREV/0/NPO/0/

IF(NDIM-1)920191

NTOT=2

DO 2 IDIM=1,NDIM

IF(NN(IDIM))920,920,2

NTOT=NTOT*NN(IDIM)

MAIN LOOP FOR EACH DIMENSION

NP1=2

DO 910 IOIM=1,NDIM

N=NN(IDIM)

NP2=NP1*N

IF(N-1)920,90095

IS N A POWER OF TWO AND IF NOT, WHAT ARE ITS FACTORS

M=N

NTWO=NP1

IF=1

IDIV=2

IOUOT=M/IDIV

IREM=M-IDIV*IQUOT

IF(IOUOT-IDIV)50,11,11

IF(IREM)20,12920

NTWO=NTWO+NTWO

IFACT(IF)=IDIV

IF=IF+1

M=IOUOT

GO TO 30

IDIV=IDIV+2

GO TO 30

IFACT(IF)=M

SEPARATE FOUR CASES--

1. COMPLEX TRANSFORM OR REAL TRANSFORM FOR THE 4TH, 9TH, ETC.
DIMENSIONS.

2. REAL TRANSFORM FOR THE 2ND OR 3RD DIMENSION. METHOD--
TRANSFORM HALF THE DATA, SUPPLYING THE OTHER HALF BY CONJUGATE SYMMETRY.

3. REAL TRANSFORM FOR THE 1ST DIMENSION, N ODD. METHOD--
SET THE IMAGINARY PARTS TO ZERO.

4. REAL TRANSFORM FOR THE 1ST DIMENSION, N EVEN. METHOD--
TRANSFORM A COMPLEX ARRAY OF LENGTH N/2 WHOSE REAL PARTS ARE THE EVEN NUMBERED REAL VALUES AND WHOSE IMAGINARY PARTS ARE THE ODD NUMBERED REAL VALUES. SEPARATE AND SUPPLY THE SECOND HALF BY CONJUGATE SYMMETRY.

ICASE=1
IFMIN=1
I1RNG=NP1
IF(101M-4)71,100,100
IF(IFORM)72,72,100
ICASE=2
ICASE=3
ICASE=4
IFMIN=2
NTWO=NTWO/2
N=N/2
NP2=NP2/2
NTOT=NTOT/2
I=1
DO 80 J=1,NTOT
DATA(J)=DATA(I)
I=I+2
SHUFFLE DATA BY BIT REVERSAL, SINCE N=2**K. AS THE SHUFFLING CAN BE DONE BY SIMPLE INTERCHANGE, NO WORKING ARRAY IS NEEDED
NP2=NP2/2
IF(NP2)200,110,110
J=1
DO 120 I2=1,NP2
IF(J-I2)120,130,130
I1MAX=I2+NP1-2
DO 120 I3=I2,NP2,2
DO 125 J3=I3-I2
DATA(J3+1)=DATA(J3)
DATA(J3)=DATA(J3+1)
M=NP2/2
IF(M-NP1)=150,140,140
J=J+M
GO TO 300

SHUFFLE DATA BY DIGIT REVERSAL FOR GENERAL N

NWORK=2*N

DO 270 I1=1,NP1,2

DO 270 I3=I1,NTOT,NP2

J=I3

DO 260 I=1,NWORK,2

IF(ICASE=3)210,220,210

WORK(I)=DATA(J)

WORK(I*1)=DATA(J+1)

GO TO 230

WORK(I)=DATA(J)

WORK(I+1)=0.

IFP2=NP2

IF=IFMIN

IFP1=IFP2/IFACT(IF)

J=J+IFP1

IF(J-I3-IFP2)=260,250,250

J=J-IFP2

IFP2=IFP1

IF=IF+1

IF(IFP2-NP1)=260,260,240

CONTINUE

I2MAX=I3+NP2-NP1

I=1

DO 270 I2=I3,I2MAXNP1

DATA(I2)=WORK(I)

DATA(I2+1)=WORK(I+1)

I=I+2

MAIN LOOP FOR FACTORS OF TWO, PERFORM FOURIER TRANSFORMS OF
LENGTH FOUR, WITH ONE OF LENGTH TWO IF NEEDED. THE TWIDDLE FACTOR
W=EXP(ISIGN*2*PI*SQRRT(-1)*M/(4*MMAX)). CHECK FOR W=ISIGN*SQRRT(-1)
AND REPEAT FOR W=W*(+1ISIGN*SQRRT(-1))/SQRRT(2).

IF(NTWO-NP1)=600,600,305

IPAR=NTWO/NP1

IF(IPAR-2)=3509330,320

IPAR=IPAR/4

GO TO 310

DO 340 I1=11I1RNG,2

DO 340 K1=I1,NTOT,NP1TW

K2=K1+NP1

TEMPR=DATA(K2)

TEMP=DATA(K2+1)

DATA(K2)=DATA(K1)+TEMPR

DATA(K2+1)=DATA(K1+1)-TEMP

DATA(K1)=DATA(K1)+TEMPR

DATA(K1+1)=DATA(K1+1)+TEMP

MMAX=NP1

IF(MMAX-NTWO/2)=370,600,600
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<th>CARD NUMBER</th>
<th>APPROX MATF PROGRAM LOCATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>277</td>
<td>370 LMAX=MAX0(NP1TW+MMAX/2)</td>
</tr>
<tr>
<td>278</td>
<td>DO 570 L=NP1*LMAX,NP1TW</td>
</tr>
<tr>
<td>279</td>
<td>M=L</td>
</tr>
<tr>
<td>280</td>
<td>IF(MMAX-NP1) GE 20, 420, 350</td>
</tr>
<tr>
<td>281</td>
<td>380 THETA=-TWOPI<em>FLOAT(L)/FLOAT(4</em>MMAX)</td>
</tr>
<tr>
<td>282</td>
<td>IF(ISIGN) GE 400, 390, 390</td>
</tr>
<tr>
<td>283</td>
<td>390 THETA=-THETA</td>
</tr>
<tr>
<td>284</td>
<td>400 WR=COS(THETA)</td>
</tr>
<tr>
<td>285</td>
<td>WI=SIN(THETA)</td>
</tr>
<tr>
<td>286</td>
<td>410 W2R=WR<em>WR-WI</em>WI</td>
</tr>
<tr>
<td>287</td>
<td>W2I=2.<em>WR</em>WI</td>
</tr>
<tr>
<td>288</td>
<td>W3R=W2R<em>WR-W2I</em>WI</td>
</tr>
<tr>
<td>289</td>
<td>W3I=W2R<em>WI+W2I</em>WR</td>
</tr>
<tr>
<td>290</td>
<td>420 DO 530 I1=1,11RNG,2</td>
</tr>
<tr>
<td>291</td>
<td>KMIN=I1*IPARM</td>
</tr>
<tr>
<td>292</td>
<td>IF(MMAX-NPI) GE 330, 430, 440</td>
</tr>
<tr>
<td>293</td>
<td>430 KMIN=I1</td>
</tr>
<tr>
<td>294</td>
<td>440 KDF=IPARM*MMAX</td>
</tr>
<tr>
<td>295</td>
<td>450 KSTEP=KDF<em>4</em>KDF</td>
</tr>
<tr>
<td>296</td>
<td>IF(KSTEP-NTWO) GE 460, 460, 530</td>
</tr>
<tr>
<td>297</td>
<td>460 DO 520 K1=KMIN+NTOT*KSTEP</td>
</tr>
<tr>
<td>298</td>
<td>K2=K1*KDF</td>
</tr>
<tr>
<td>299</td>
<td>K2=K2*KDF</td>
</tr>
<tr>
<td>300</td>
<td>K4=K3*KDF</td>
</tr>
<tr>
<td>301</td>
<td>IF(MMAX-NPI) GE 470, 470, 490</td>
</tr>
<tr>
<td>302</td>
<td>470 U1R=DATA(K1)+DATA(K2)</td>
</tr>
<tr>
<td>303</td>
<td>U1I=DATA(K1)+DATA(K2+1)</td>
</tr>
<tr>
<td>304</td>
<td>U2R=DATA(K3)+DATA(K4)</td>
</tr>
<tr>
<td>305</td>
<td>U2I=DATA(K3)+DATA(K4+1)</td>
</tr>
<tr>
<td>306</td>
<td>U3R=DATA(K1)+DATA(K2)</td>
</tr>
<tr>
<td>307</td>
<td>U3I=DATA(K1)+DATA(K2+1)</td>
</tr>
<tr>
<td>308</td>
<td>IF(ISIGN) GE 471, 472, 472</td>
</tr>
<tr>
<td>309</td>
<td>471 U4R=DATA(K3)+DATA(K4)</td>
</tr>
<tr>
<td>310</td>
<td>U4I=DATA(K4)+DATA(K3)</td>
</tr>
<tr>
<td>311</td>
<td>GO TO 510</td>
</tr>
<tr>
<td>312</td>
<td>472 U4R=DATA(K4+1)+DATA(K3+1)</td>
</tr>
<tr>
<td>313</td>
<td>U4I=DATA(K3)+DATA(K4)</td>
</tr>
<tr>
<td>314</td>
<td>GO TO 510</td>
</tr>
<tr>
<td>315</td>
<td>480 T2R=W2R<em>DATA(K2)-W2I</em>DATA(K2+1)</td>
</tr>
<tr>
<td>316</td>
<td>T2I=W2R<em>DATA(K2+1)+W2I</em>DATA(K2)</td>
</tr>
<tr>
<td>317</td>
<td>T3R=W3R<em>DATA(K3)-W3I</em>DATA(K3+1)</td>
</tr>
<tr>
<td>318</td>
<td>T3I=W3R<em>DATA(K3+1)+W3I</em>DATA(K3)</td>
</tr>
<tr>
<td>319</td>
<td>T4R=DATA(K4)+DATA(K3)+DATA(K4)</td>
</tr>
<tr>
<td>320</td>
<td>T4I=W3R<em>DATA(K4+1)+W3I</em>DATA(K4)</td>
</tr>
<tr>
<td>321</td>
<td>U1R=DATA(K1)+T2R</td>
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<tr>
<td>322</td>
<td>U1I=DATA(K1)+T2I</td>
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<td>U2R=T3R+T4R</td>
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<td>U2I=T3I+T4I</td>
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<td>U3R=DATA(K1)+T2R</td>
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<td>326</td>
<td>U3I=DATA(K1)+T2I</td>
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<td>327</td>
<td>IF(ISIGN) GE 490, 500, 500</td>
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<td>328</td>
<td>490 U4R=T3I+T4I</td>
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<td>329</td>
<td>U4I=T4R-T3R</td>
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<td>330</td>
<td>GO TO 510</td>
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<td>331</td>
<td>500 U4R=T4I-T3I</td>
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<tr>
<td>332</td>
<td>U4I=T3R-T4R</td>
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514 510 DATA(K1)=U1R+U2R
514 510 DATA(K1+1)=U1I+U2I
514 510 DATA(K2)=U3R+U4R
514 510 DATA(K2+1)=U3I+U4I
514 510 DATA(K3)=U1R-U2R
514 510 DATA(K3+1)=U1I-U2I
514 510 DATA(K4)=U3R-U4R
514 510 DATA(K4+1)=U3I-U4I
520 KDIFF=KSTEP
520 KMIN=4*(KMIN-I1)+I1
520 GO TO 450
530 CONTINUE
540 M=M+LMAX
550 IF(M-MMAX)540,540,570
550 IF(ISIGN)550,560,570
560 TEMPL=WR
560 WR=(WR+WI)*RTHLF
560 WI=(WI-TEMPL)*RTHLF
560 GO TO 410
570 CONTINUE
580 IPAR=3-IPAR
590 MMAX=MMAX+MMAX
600 GO TO 360
610 C MAIN LOOP FOR FACTORS NOT EQUAL TO TWO. APPLY THE TWIDDLE FACTOR
620 W=EXP(ISIGN*2*PI*SQRT(-1)*(J1-I1)*(J2-I2)/(IFP1+IFP2)), THEN
630 C PERFORM A FOURIER TRANSFORM OF LENGTH IFACT(IF), MAKING USE OF
640 C CONJUGATE SYMMETRIES.
650 C
660 IF(NTWOD-NP2)605,700,700
670 IFP1=NTWO
680 IFP1=NTWO
690 IFP1=NTWO
700 IFP1=NTWO
710 IF=INON2
720 NR1N1=NP1/2
730 IF2=IFACT(IF)*IFP1
740 IFJMIN=NP1+1
750 IF(JMIN-IFP1)615,615,640
760 DO 635 J=JMIN,IFP1+NP1,1
770 THEA=-2*OP1*FLOAT(J1-I1)/FLOAT(IFP1)
780 IF(ISIGN)625,620,620
790 THEA=THEA
800 WSPI=COS(THEA)
810 WSPI=SN(THEA)
820 WR=WSPI
830 WI=WSPI
840 J2MIN=J1+IFP1
850 J2MAX=J1+IFP2-IFP1
860 DO 635 J2=J2MIN,J2MAX,IFP1
870 I1MAX=J2+I1RNG-2
880 DO 630 J1=I1+I1MAX,2
890 DO 630 J3=I1+J1TOT,IFP2
900 TEMP=DATA(J3)
910 DATA(J3)=DATA(J3)+WP*DATA(J3+1)+WI
DATA(J3+1)=TEMPR*WI+DATA(J3+1)*WR
TEMPR=WR
WR=WR*WSTPR-WI*WSTPI
WI=TEMPR*WSTPI+WI*WSTPR
THETA=-TWOPI/FLOAT(FACT(TF))
IF(ISIGN)650,645,645
THETA=-THFTA
WSTPR=COS(THETA)
WSTPI=SIN(THETA)
J?RNG=IFPI*(1+IFACT(IF)/2)
00 695 I1=1,I1RNG,2
DO 695 I3=I1,NTOTNP2
J2MAX=I3+J2RNG-IFP1
DO 690 J2=I39J2MAXIFP1
J1MAX=J2+IFP1-NP1
DO 680 J1=J2,J1MAXNP1
J3MAX=J1+NP2-IFP2
DO 680 J3=Jl1J3MAXIFP2
JMIN=J3-J2+I3
JMAX=JMIN+IFP2-IFP1
I=1+(J3-I3)/NP1HF
IF(J2-I3)655,6559665
SJRMR=0.
SUMI=0.
DO 660 J=JMIN,JMAX9IFP1
SUMR=SUMR+DATA(J)
SUMI=SUMI*DATA(J+1)
WORK(I)=SUMR
WORK(I+1)=SUMI
GO TO 680
ICONJ=1+(IFP2-2*J2+I3+J3)/NP1HF
J=JMAX
SUMR=DATA(J)
SUMI=DATA(J+1)
OLDSR=0.
OLDSI=0.
J=J-IFP1
TEMPR=SUMR
TEMPI=SUMI
SUMR=2TOWR*SUMR-OLDSR+DATA(J)
SUMI=2TOWR*SUMI-OLDSI+DATA(J+1)
OLDSR=TEMPR
OLDSI=TEMPI
J=J-IFP1
IF(J-JMIN)675,675,670
TEMPR=WR*SUMR-OLDSR+DATA(J)
TEMPl=Wl*SUMI
WORK(I)=TEMPR*TEMPl
WORK(ICONJ)=TEMPR*TEMPl
TEMPR=WR*SUMI-OLDSI+DATA(J+1)
TEMPl=Wl*SUMR
WORK(I+1)=TEMPR*TEMPl
WORK(ICONJ+1)=TEMPR-TEMPl
CONTINUE
IF(J2-I3)685,685,686
WR=WSTPR
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<td>WI=WSTPI</td>
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<td>GO TO 690</td>
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<td>1169</td>
<td>TEMPR=WR</td>
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<td>1170</td>
<td>WR=WR<em>WSTPR-WI</em>WSTPI</td>
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<td>1171</td>
<td>WI=TEMPR<em>WSTPI+WI</em>WSTPR</td>
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<td>1172</td>
<td>TOWR=WR+WR</td>
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<td>1204</td>
<td>I=1</td>
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<td>1205</td>
<td>I2MAX=I3+NP2-NP1</td>
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<td>1206</td>
<td>00 695 I2=I3*I2MAX+NP1</td>
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<td>1207</td>
<td>DATA(I2)=WORK(I)</td>
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<td>DATA(I2+1)=WORK(I+1)</td>
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<td>1209</td>
<td>DO</td>
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<td>IF=IF*1</td>
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<td>I2=I3,I2MAX-NP1</td>
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<td>DATA(I2)=WORK(I)</td>
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<td>DATA(I2+1)=WORK(I+1)</td>
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<td>I=1+2</td>
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<td>IFP1=IFP2</td>
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<tr>
<td>1216</td>
<td>IF(IFP1-NP2)&lt;09700,700</td>
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<td>1217</td>
<td>COMPLETE A REAL TRANSFORM IN THE 1ST DIMENSION, N EVEN, BY CONJUGATE SYMMETRIES.</td>
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<td>1218</td>
<td>GO TO (900,800,9900,701),ICASE</td>
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<td>1219</td>
<td>NHALF=N</td>
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<td>N=N+N</td>
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<td>1221</td>
<td>THETA=-TWOPI/FLOAT(N)</td>
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<td>1222</td>
<td>IF (ISIGN)=703+702+702</td>
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<td>THEETA=THETA</td>
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<td>WSTPR=COS(THETA)</td>
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<td>WSTPI=SIN(THETA)</td>
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<td>WR=WSTPR</td>
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<td>WI=WSTPI</td>
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<td>IMIN=3</td>
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<td>JMIN=2*NHALF-1</td>
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<td>1230</td>
<td>GO TO 725</td>
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<td>J=JMIN</td>
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<td>00 720 I=IMIN*NTOT+NP2</td>
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<td>SUMR=(DATA(I)+DATA(J))/2</td>
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<tr>
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<td>SUMI=(DATA(I+1)+DATA(J+1))/2</td>
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<td>DIFR=(DATA(I)-DATA(J))/2</td>
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<td>DIFI=(DATA(I+1)-DATA(J+1))/2</td>
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<td>TEMPR=WR<em>SUMI+WI</em>DIFR</td>
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<td>TFMPI=WI<em>SUMI-WR</em>DIFI</td>
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<td>DATA(I)=SUMR+TEMPI</td>
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<td>DATA(I+1)=DIFI+TEMPI</td>
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<td>DATA(J)=SUMR-TEMPI</td>
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<td>1242</td>
<td>DATA(J+1)=-DIFI+TEMPI</td>
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<td>1243</td>
<td>J=J+NP2</td>
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<tr>
<td>1244</td>
<td>IMIN=IMIN+2</td>
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<tr>
<td>1245</td>
<td>JMIN=JMIN-2</td>
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<tr>
<td>1246</td>
<td>TEMPR=WR</td>
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<tr>
<td>1247</td>
<td>WR=WR<em>WSTPR-WI</em>WSTPI</td>
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<td>1248</td>
<td>WI=TEMPR<em>WSTPI+WI</em>WSTPR</td>
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<tr>
<td>1249</td>
<td>IF(IMIN-JMIN)=710+730+740</td>
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<tr>
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<td>IF(ISIGN)=731+740+740</td>
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<td>NTOT=NTOT+NP2</td>
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<tr>
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<td>NTOT=NTOT+NTOT</td>
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</table>
J = NTOT + 1
IMAX = NTOT / 2 + 1
MIN = IMAX - 2 * NHALF
I = IMIN
GO TO 755
DATA(J) = DATA(I)
DATA(J+1) = -DATA(I+1)
I = 1 + 2
J = J - 2
IF (I - IMAX) GOTO 760, 760
DATA(J) = DATA(IMIN) - DATA(IMIN+1)
DATA(J+1) = 0.
IF (I - J) GOTO 770, 780, 780
DATA(J) = DATA(I)
DATA(J+1) = DATA(I+1)
I = I - 2
J = J - 2
IF (I - IMIN) GOTO 775, 775, 765
DATA(J) = DATA(IMIN) + DATA(IMIN+1)
DATA(J+1) = 0.
IMAX = IMIN
GO TO 745
DATA(1) = DATA(1) DATA(2)
DATA(2) = 0.
GO TO 900
COMPLETE A REAL TRANSFORM FOR THE 2ND OR 3RD DIMENSION BY
IF (I1NRG - NP1) GOTO 805, 900, 900
DO 800 I3 = 1, NTOT - NP2
I2MAX = I3 + NP2 - NP1
DO 800 I2 = I3, I2MAX + NP1
IMIN = I2 + I1NRG
IMAX = I2 + NP1 - 2
JMAX = 2 * I3 + NP1 - IMIN
IF (I2 - I3) GOTO 820, 820, 810
JMAX = JMAX + NP2
IF (I1IM - 2) GOTO 850, 850, 830
J = JMAX + NP0
DO 840 I = I3, IMIN + IMAX * 2
DATA(I) = DATA(J)
DATA(I+1) = -DATA(J+1)
I = J - 2
J = JMAX
DO 860 I = NPO
DATA(I) = DATA(J)
DATA(I+1) = -DATA(J+1)
J = J - NP0
END OF LOOP ON EACH DIMENSION
NP0 = NP1
NP1 = NP2
NPREV = N
RETURN
APPROXIMATE PROGRAM LOCATION 1523

LENGTH OF ROUTINE FOURT 1777

VARIABLE ASSIGNMENTS

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SUBROUTINES CALLED

| COSF | 08QR5D |
| SINF | 08QR5D |

COMPILE TIME = 601 MILLISECS

TOTAL CPU TIME IN MILLISECONDS = 2010
PPU TIME IN MILLISECONDS = 1654
PAGES PRINTED = 28
CARDS PUNCHED = 1
TOTAL RESOURCES USED = 1.58

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SYSTEM TAPE D88 GENERATED 06/04/75 AT 08/12 FROM SOURCE TAPE E60

LIST OF ROUTINES MODIFIED

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