Uncorrelated Noise in Turbulence Measurements

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ABSTRACT

We show that the error variance contributed by random, uncorrelated measurement noise can be merged with the error variance contributed by real variations in the atmosphere to obtain a single expression for the total error variance when the sampling time is much less than the integral scale of atmospheric variability. We also assume that the measured signal is a representation of a variable that is continuous on the scale of interest in the atmosphere. The characteristics of this noise are similar but not identical, to quantization noise, whose properties are briefly described. Uncorrelated noise affects the autocovariance function (or, equivalently, the structure function) only between zero and the first lag, while its effect is smeared across the entire power spectrum. For this reason, quantities such as variance dissipation may be more conveniently estimated from the structure function than from the spectrum.

The modeling results are confirmed by artificially modifying a test time series with Poisson noise and comparing the statistics from ten realizations of the modified series with the predicted error variances. We also demonstrate applications of these results to measurements of aerosol concentrations. A "figure of merit" is defined which is used to specify when instrument counting noise contributes more to measurement error than atmospheric variability. For example, for measuring the vertical flux of a trace species with a small surface resistance to deposition, the specified counting rate is about 100 counts s\(^{-1}\) for measuring flux in the surface layer and about 10\(^3\) counts s\(^{-1}\) for measuring flux throughout the convective boundary layer.
A large array of new instruments are becoming available that have the capability to resolve turbulent fluctuations of trace atmospheric constituents. Many of these instruments count discrete events, and thus their resolution and accuracy are limited by noise contributed by the counting process. Furthermore, many of the developers and users of these instruments, not being experts on atmospheric turbulence, have asked us how to use these instruments for atmospheric measurements. We could not find a satisfactory exposition in the literature that we could refer them to, which they could use to describe how this noise contributes to measurements of turbulence statistics, such as fluxes and variance. This paper was written as a response to these questions. We first submitted this paper to the Journal of Atmospheric and Oceanic Technology. The reviews suggested that such a lengthy exposition, although useful to those who want to learn and apply these concepts, was too long and pedantic for a journal article. On the other hand, Joost Businger felt that such a discussion could be very useful to workers in the field, and should be published as an NCAR Technical Note.

The resolution to this was to publish a shortened version of the paper in the Journal of Atmospheric and Oceanic Technology, and make available the entire paper as an NCAR Technical Note. We hope that this note will be useful to those who want to go through the technical details of the derivations, since the concepts presented here are a
fundamental basis for applications of instruments, whose outputs contain uncorrelated noise, to atmospheric turbulence measurements.

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1. INTRODUCTION

Measurements invariably consist of a representation of the true value of a variable or quantity being measured along with unwanted noise. By noise we mean any extraneous signal that is unavoidably included in the output of a measurement. Sources of noise are many and varied. Some examples are drifts in outputs due to changes in temperature, amplifier gain and transducer sensitivity, quantization noise introduced by digitizing an analog signal, and uncorrelated noise due to the finite number of photons, electrons, or particles that are measured as a representation of the signal. Here we consider in detail the effects on measurements in a turbulent fluid due to the contribution to the output from the latter source of noise, and also point out that quantization noise has similar (but not identical) effects on the output. Uncorrelated noise sources can be intrinsic to the measurement, as in the case of aerosol probes or instruments for measuring trace gases using photomultiplier tubes to count individual events, or they can be a by-product of the circuitry used to amplify or otherwise condition a low-level signal.

Particle-counting probes are an obvious example of instruments that have an inherent noise component due to counting. In recent years, however, other instruments have been developed whose outputs also contain a significant counting noise. In particular, instruments have been developed for measuring low concentrations of trace species in the atmosphere by chemiluminescence (Pearson and Stedman, 1980) and by laser-induced fluorescence (Bradshaw and Davis, 1982). The concentrations measured by these instruments are proportional to the number of
photons emitted by the excited molecules and counted by photomultiplier tubes. In some cases, the number of photons is small enough that the counting process is the major source of noise in the output. In the next section, we discuss how these concentration measurements are used, particularly in situations where noise is an important source of measurement error. We then discuss the effects of uncorrelated noise on measurements, particularly on higher-order moments such as the variance and the covariances of concentration with other variables. Although concentration measurements in the convective boundary layer are emphasized in subsequent discussion, the principles discussed here are applicable to any variables measured in a turbulent medium.

In the discussion that follows, we make the assumption that the measured count is a representation of a variable that is continuous. That is, the discrete elements that make up the variable on our scale of interest are so numerous that the real fluctuations in the variable due to the finite number of elements are insignificant. This is always the case for trace gas concentration measurements, and usually the case for small aerosols as well, unless the volume of interest is less than, say, a cubic centimeter. For the most part, aerosol probes count essentially all the particles in a relatively small volume, in contrast to trace gas sensors, which count a small percentage of the total molecules in a volume. Thus, aerosol probes measure real count fluctuations in a given volume, and we assume that the given volume is characteristic of a larger volume of interest in the atmosphere. In certain situations, however, the number of discrete elements in our volume of interest may be so small that assuming a continuum may be unjustified. For
precipitation particles, for example, counting statistics may contribute significant variability to the measurement even with relatively large volumes (Cornford, 1967). The same may be true of large aerosols, particularly in relatively clean air.
2. APPLICATIONS OF MEASUREMENTS

The atmosphere is a turbulent fluid. This means that mixing is accomplished mostly by turbulent eddies of a wide range of sizes and intensities. Individual time histories of these eddies are unpredictable. It has been empirically confirmed, however, that estimates of the means and higher-order moments of variables in the atmosphere often can be obtained by averaging over a time (or distance in the case of a moving platform such as an airplane) long enough to obtain a stable estimate. This property is known as (strict) ergodicity (Panofsky and Dutton, 1984). Even if the measurement is precise, real fluctuations in the atmosphere require that the variable be averaged over a period several times longer than that corresponding to the largest scales that contribute to the average in order to obtain a statistically reliable estimate. Turbulent fluctuations occur in concentrations of constituents that have sources or sinks with time scales on the order of, or less than, the time required to completely mix the atmosphere. Since this time scale is on the order of several years, turbulent fluctuations occur in essentially all constituents of the atmosphere, except perhaps for $N_2$ and the noble gases. The sources or sinks may be either internal to the fluid (e.g., chemical reactions) or external (e.g., sources and sinks at the earth's surface or concentration differences across an interface between two layers through which entrainment can occur). The fluctuations themselves, when squared and averaged, are useful in estimating magnitudes of sources and sinks (Junge, 1974). Covariances, which are products of two variables averaged over a suitable period, can
give even more precise information as to sources and sinks of trace constituents.

Perhaps the most important of these covariances is the turbulent vertical flux of a constituent $s$, i.e., the covariance between $s$ and the vertical velocity component. Near the surface, if the time scale for internal sources and sinks is greater than a few minutes (Fitzjarrald and Lenschow, 1983), the vertical turbulent flux is equal to the flux into or out of the surface. This constant flux layer, which is commonly called the surface layer, is typically of order 10 m deep. Throughout the boundary layer, the rate of change of the flux with height can be an important term in the mean concentration budget. This term can be evaluated by measuring the flux at several levels in the boundary layer, then fitting a curve to the points. Additional covariance terms involving two chemically reacting species exist in the mean concentration budget if second-order chemical reactions are significant (Fitzjarrald and Lenschow, 1983).

Similarly, equations for the budgets of variances and covariances include third-order moment terms, and so on for higher moment terms. Thus, these higher-order moment terms are relevant to problems of sources, sinks, and distributions of trace atmospheric species.

A major focus of this paper is to estimate how long an averaging period is required for observations of both mean concentrations and higher-order moment terms in a convective boundary layer, taking into account not only the variability of the atmosphere, but also random noise introduced into the variables being measured.
3. THEORETICAL CONSIDERATIONS

Before discussing the effects of uncorrelated noise on estimating means and higher-order moments of random signals, we first discuss some basic concepts that apply to random signals that are uncontaminated by uncorrelated noise. A good reference for this discussion is Papoulis (1965).

3.1. Basic concepts. Let $\alpha(t)$ be a strictly ergodic time series. This implies that the time average

$$\alpha_T = \frac{1}{T} \int_0^T \alpha(t) \, dt$$

(1)

has a finite limit for $T \to \infty$, and that this limit is equal to the time-independent ensemble average $\langle \alpha \rangle$ of $\alpha(t)$, i.e.,

$$\lim_{T \to \infty} \alpha_T = \alpha_\infty = \langle \alpha \rangle.$$  

(2)

Panofsky and Dutton (1984) have given a brief and instructive discussion of various forms of ergodicity. The ensemble average is the average obtained from an infinite number of realizations of the same experiment. For a particular value of $T$, the different realizations of $\alpha_T$ fluctuate about $\langle \alpha \rangle$, with an ensemble mean square deviation $\sigma^2(\alpha; T)$, which we call the error variance. Observations in the atmosphere normally can provide only one realization; the variability of the atmosphere does not often allow us to repeat an experiment. Therefore, the concept of an ensemble of infinitely many realizations is a mathematical abstraction.
However, it does allow us to address the question of how large $T$ must be in order to estimate $\alpha$ with a specified accuracy defined by $\sigma(\alpha; T)$, where

$$\sigma^2(\alpha; T) = \langle (\alpha - \langle \alpha \rangle)^2 \rangle. \tag{3}$$

Substituting (1) into (3) we obtain

$$\sigma^2(\alpha; T) = \langle \left[ \frac{1}{T} \int_0^T \tilde{\alpha}(t) dt \right]^2 \rangle, \tag{4}$$

where

$$\tilde{\alpha}(t) = \alpha(t) - \langle \alpha \rangle \tag{5}$$

is the fluctuating part of $\alpha(t)$. The error variance can be further reduced by noting that ensemble averaging commutes with integration (Lumley and Panofsky, 1964), and that the autocovariance function,

$$R_\alpha(t', t'') = \langle \tilde{\alpha}(t')\tilde{\alpha}(t'') \rangle, \tag{6}$$

for a stationary time series is a function only of the time difference $t'' - t'$. Writing $R_\alpha(t'' - t')$ instead of $R_\alpha(t', t'')$, we get

$$\sigma^2(\alpha; T) = \langle \frac{1}{T^2} \int_0^T \int_0^T \tilde{\alpha}(t')\tilde{\alpha}(t'') \rangle$$

$$= \frac{1}{T^2} \int_0^T \int_0^T \tilde{\alpha}(t')\tilde{\alpha}(t'') \rangle \tag{7}$$

$$= \frac{1}{T^2} \int_0^T \int_0^T R_\alpha(t'' - t') \tag{7}$$

Equation (7) can be reduced to a single integral by the transformation
\[ t = \frac{(t' + t'')}{2} \]
\[ \tau = t'' - t' \]  \hspace{1cm} (8)

The transformation is shown graphically in Fig. 1; we note that the area of the rhombus on the right side is identical to the area of the square on the left.

We can now easily transform (7) by inspection of Fig. 1:

\[
\sigma^2(\alpha; T) = \frac{1}{T^2} \int_{-T}^{T} d\tau \int_{-T}^{T} d\tau' R_\alpha(\tau) \frac{|\tau|}{2} \\
= \frac{1}{T} \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) R_\alpha(\tau) d\tau 
\]  \hspace{1cm} (9)

Fig. 1. Schematic diagram of the area of integration before the area-conserving transformation (left) and after the transformation (right).
We note from (6) that $R_\alpha(0)$ is equal to the ensemble variance $\sigma_\alpha^2$ of $\alpha(t)$. Defining the autocorrelation function by

$$\rho_\alpha(\tau) = \frac{R_\alpha(\tau)}{\sigma_\alpha^2},$$

(10)

(9) can be written as

$$\sigma^2(\alpha; T) = 2 \frac{\sigma_\alpha^2}{T} \int_0^T (1 - \frac{\tau}{T}) \rho_\alpha(\tau) d\tau,$$

(11)

since according to (6), $\rho_\alpha(\tau)$ is an even function of $\tau$. We see that $\rho_\alpha(0) = 1$, and that for large $\tau$, $\alpha(t + \tau)$ is essentially uncorrelated with $\alpha(t)$ so that $\rho_\alpha(\tau) \to 0$ for $\tau \to \infty$. Because of the assumption of ergodicity, this convergence is strong enough that the integral time scale, defined by

$$\Gamma_\alpha = \int_0^\infty \rho_\alpha(\tau) d\tau,$$

(12)

exists. This time scale is a quantitative measure of the "memory" of $\alpha(t)$; if $\Gamma_\alpha$ is large, $\alpha(t)$ will be correlated with itself over large values of $\tau$.

From (11) we see that if the averaging time $T$ is large enough, the error variance can be well approximated by

$$\sigma^2(\alpha; T) = 2 \frac{\sigma_\alpha^2}{T} \int_0^{\Gamma_\alpha} \rho_\alpha(\tau) d\tau = 2\sigma_\alpha^2 \frac{\Gamma_\alpha}{T}.$$

(13)

Thus, if the ensemble variance $\sigma_\alpha^2$ is large and the "memory" $\Gamma_\alpha$ is long, then we must average for a long time $T$ in order to make $\sigma^2(\alpha; T)$ small and thereby accurately measure $<\alpha>$ by means of (1).
Although the preceding derivation can be found elsewhere (e.g., Panofsky and Dutton, 1984), we have chosen to discuss it in some detail in order to establish a foundation for deriving the error variance for second-order moments that includes the effects of uncorrelated noise.

We first consider the covariance $F$ between $\alpha(t)$ and another strictly ergodic time series $w(t)$. If $w(t)$ is the vertical velocity, this covariance is the vertical flux of $\alpha$, which itself can be a velocity component or a scalar such as temperature or a species concentration.

Analogous to (1), we introduce the time averages

$$w_T = \frac{1}{T} \int_0^T w(t) dt$$

and

$$F_T = \frac{1}{T} \int_0^T [\alpha(t) - \langle\alpha_T\rangle] [w(t) - w_T] dt .$$

Again we assume strong ergodicity so that the ensemble-average covariance is equal to $\lim_{T \to \infty} F_T$:

$$F_\infty = \lim_{T \to \infty} F_T = \langle[\alpha(t) - \langle\alpha\rangle] [w(t) - \langle w\rangle]\rangle .$$

We note that for the first-order quantities $\alpha_T$ and $w_T$ we have relations of the form

$$\langle\alpha_T\rangle = \langle\alpha\rangle = \alpha_\infty ,$$

$$\langle w_T \rangle = \langle w \rangle = w_\infty .$$

(17a)  

(17b)
but that this is in general not the case for $F_T$, or any other second-order quantity. In fact, it can easily be shown that

$$\langle F_T \rangle = F_\infty - \frac{1}{T} \int_T^{-T} (1 - \frac{|t|}{T}) R_{\alpha w}(\tau) d\tau ,$$

(18)

where

$$R_{\alpha w}(\tau) = \langle [\alpha(t) - \langle \alpha \rangle] [w(t + \tau) - \langle w \rangle] \rangle .$$

(19)

is the covariance function between $\alpha$ and $w$. Again, because we assume stationarity, $R_{\alpha w}$ is not a function of absolute time, but only of time difference. We see from (18), by the assumption of ergodicity, that $\langle F_T \rangle$ approaches $F_\infty$ for $T \to \infty$.

The error variance for

$$F(t) = [\alpha(t) - \alpha_T] [w(t) - w_T]$$

(20)

is defined as

$$\sigma^2(F;T) = \langle (F_T - F_\infty)^2 \rangle .$$

(21)

With the aid of (18), (21) can be written as

$$\sigma^2(F;T) = \langle F^2 \rangle + 2F_\infty \frac{1}{T} \int_T^{-T} (1 - \frac{|\tau|}{T}) R_{\alpha w}(\tau) d\tau - F_\infty^2 .$$

(22)

The first term, which is quite complicated since it contains fourth-order moments of $\alpha$ and $w$, can be written as
\[
\langle F^2 \rangle = \frac{1}{T^2} \int_0^T \int_0^T \langle \tilde{\alpha}(t') \tilde{w}(t') \tilde{\alpha}(t'') \tilde{w}(t'') \rangle \\
- \frac{2}{T^3} \int_0^T \int_0^T \int_0^T \langle \tilde{\alpha}(t') \tilde{w}(t') \tilde{\alpha}(t'') \tilde{w}(t'') \rangle \\
+ \frac{1}{T^4} \int_0^T \int_0^T \int_0^T \int_0^T \langle \tilde{\alpha}(t') \tilde{w}(t') \tilde{\alpha}(t'') \tilde{w}(t'') \rangle \\
(23)
\]

where the tilde over \( \alpha \) and \( w \) indicates that the ensemble means have been subtracted.

To reduce (23) beyond this point requires information about how fourth-order moments can, if possible, be related to lower order moments. We adopt the commonly used assumption that the distributions of \( \alpha \) and \( w \) are approximately joint Gaussian. After considerable manipulation, in particular of the second term of (23), the error variance (21) becomes, in the limit of large \( T \),

\[
\sigma^2(F;T) = \frac{2}{T} \int_0^\infty [R_{\alpha}(\tau)R_w(\tau) + R_{\alpha w}(\tau)R_{\alpha w}(-\tau)]d\tau \\
- 4\sigma^2_\alpha \sigma^2_w \frac{\Gamma_{\alpha w}}{T^2} \\
(24)
\]

where \( R_w(\tau) \) and \( \Gamma_w \) are the autocovariance function and the time integral scale for \( w(t) \), respectively. We see immediately that the second term is of higher order in \( 1/T \) than the first, and therefore can be neglected. The first term can also be expressed in terms of the power spectra \( S_\alpha(\omega) \) and \( S_w(\omega) \), and the cross-spectrum \( S_{\alpha w}(\omega) \). We use the following general definition of a spectrum:
\[ S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-i\omega \tau} d\tau, \quad (25) \]

where
\[ R(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{i\omega \tau} d\omega \quad (26) \]

can be any of the three covariance functions in the first term of (24).

The spectral formulation of (24) is then
\[ \sigma^2(F;T) = \frac{2\pi}{T} \int_{-\infty}^{\infty} [S(\omega)S_w(\omega) + S_{\alpha w}(\omega)S^*_{\alpha w}(\omega)] d\omega, \quad (27) \]

where the asterisk denotes complex conjugation. Defining the spectral coherence between \( \alpha \) and \( w \) as
\[ \text{coh}_{\alpha w}(\omega) = \frac{S_{\alpha w}(\omega)S^*_{\alpha w}(\omega)}{[S_{\alpha w}(\omega)S_{\alpha w}(\omega)]}, \quad (28) \]

the error variance becomes
\[ \sigma^2(F;T) = \frac{2\pi}{T} \int_{-\infty}^{\infty} S_{\alpha w}(\omega)S_w(\omega)[1 + \text{coh}_{\alpha w}(\omega)] d\omega. \quad (29) \]

Since \( 0 < \text{coh}_{\alpha w}(\omega) \leq 1 \) and both \( S_{\alpha}(\omega) \) and \( S_w(\omega) \) are even, (29) can be written as
\[ \sigma^2(F;T) = \frac{2\pi}{T} c \int_{-\infty}^{\infty} S_{\alpha w}(\omega)S_w(\omega) d\omega, \]

\[ = \frac{4\pi}{T} c \int_{0}^{\infty} S_{\alpha w}(\omega)S_w(\omega) d\omega \quad (30) \]

where \( 1 < c < 2 \) is a constant. The last three equations for \( \sigma^2(F;T) \)
show that the accuracy with which we can estimate the error variance
depends upon how well we know the power spectra and cross-spectra. The
mean value theorem (Gradshteyn and Ryzhik, 1980) can be applied to (30)
in two different ways:

$$\sigma^2(F; T) = \frac{2\pi}{T} c \begin{cases} \sigma^2_w S_\alpha(\omega_\alpha) \\ \sigma^2_w S_\alpha(\omega_w) \end{cases} ,$$

(31)

where $\omega_\alpha$ and $\omega_w$ are frequencies in the interval $(0, \infty)$. A recent
review by Panofsky and Dutton (1984) shows that it is reasonable to
assume that $S_\alpha(\omega)$ and $S_w(\omega)$ are decreasing functions of $\omega$ in this
interval. We now see that both of the following conditions,

$$\begin{align*}
\sigma^2(F; T) &\leq \frac{2c}{T} \frac{\sigma^2_\alpha \sigma^2_w \Gamma_\alpha}{T} , \\
\sigma^2(F; T) &\leq \frac{2c}{T} \frac{\sigma^2_\alpha \sigma^2_w \Gamma_w}{T},
\end{align*}$$

(32)

are fulfilled, since the integral scale, according to (10), (12), and
(25), is related to the spectrum by

$$\Gamma_x = \pi \frac{S_x(0)}{\sigma^2_x} , \quad x = \alpha, w.$$  

(33)

From these considerations, we can set an upper limit by

$$\sigma^2(F; T) \leq 4\sigma^2_\alpha \sigma^2_w \min \left( \frac{\Gamma_\alpha', \Gamma_w'}{T} \right) .$$

(34)
If we knew more about the spectra, it would probably be possible to set an even lower limit for $a^2(F;T)$.

The upper limit for the error variance of the variance of $\alpha$ is easily found under the same conditions as those that were used to derive (34). We simply replace $w$ with $\alpha$ in (34) and obtain

$$\sigma^2(A;T) \leq 4\sigma^4 \frac{\Gamma}{\alpha} \frac{\alpha}{\Gamma}, \quad (35)$$

where $A(t) = [\alpha(t) - \alpha_T]^2$ in analogy to (20). It should be pointed out that this approach of determining the error variance of second-order variables is different from, but not inconsistent with, that suggested by Lumley and Panofsky (1964) and later applied by Wyngaard (1973). They do not make the approximation that the signals are joint Gaussian, but express the error variances in terms of fourth-order moments directly. They then must estimate the integral scales of the variance of the variables. For second-order moment variables, this means that the integral scales of fourth-order moment variables must be estimated.

From a practical point of view, there is probably not any significant difference between calculations using the two approaches.

3.2. Uncorrelated noise from a constant source. Uncorrelated noise can be thought of as a random series of unit pulses emitted independently of each other. A simple example is radioactive decay. If the mean lifetime of $N$ identical radioactive atoms is $\tau_0$, the average number of decays per unit time is
\[ \eta = \frac{N}{T_0}. \]  \hspace{1cm} (36)

Since the decays are independent of each other, the number of decays in a particular time interval \( \Delta t \) is Poisson-distributed about the mean
\[ \alpha = \eta \Delta t. \]  \hspace{1cm} (37)

The probability for \( n \) decays is
\[ P_n(\alpha) = \frac{\alpha^n}{n!} e^{-\alpha}. \]  \hspace{1cm} (38)

Similar considerations apply to photons or electrons emitted by instruments or electronic devices, or to atmospheric aerosols sensed by counting devices.

We first assume that we have a constant source of uncorrelated noise. We want to determine how long a time \( T \) we must count in order to obtain a specified error variance on the average count in the time interval \( \Delta t \). We let \( n_i \) (\( i = 1, 2, \ldots, I \)) be the count in the \( i \)th interval of length \( \Delta t \). The average number of counts in the interval \( T = I \Delta t \) is
\[ \langle \sum_{i=1}^{I} n_i \rangle = I \alpha \]  \hspace{1cm} (39)

This means that the average count in a time interval \( \Delta t \) is
\[ \langle \frac{1}{I} \sum_{i=1}^{I} n_i \rangle = \alpha. \]  \hspace{1cm} (40)

The variance of \( \sum_{i=1}^{I} n_i \) is, according to well-known results for Poisson-distributed variables, equal to \( I \alpha \). This means that the error variance,
i.e., the variance of \( \frac{1}{I} \sum_{i=1}^{I} n_i \), is

\[
\sigma^2(n;T) = \frac{1}{I} \sum_{i=1}^{I} n_i - \alpha^2
\]

\[
= \frac{1}{I^2} I \alpha = \frac{\alpha}{I} = \alpha \frac{\Delta t}{I}
\]

(41)

We have arrived at a result that is very similar to (13) in that the error variance is inversely proportional to the averaging time and directly proportional to the variance (\( = \alpha \)). Similarly, the error variance of the variance of \( \alpha \) becomes

\[
\sigma^2(n^2;T) = \alpha(2\alpha + 1) \frac{\Delta t}{I} - \alpha (\alpha + 2) \left( \frac{\Delta t}{I} \right)^2 + \alpha \left( \frac{\Delta t}{I} \right)^3
\]

(42)

which is quite similar to (35).

We arrived at (41) and (42) by methods that seem quite unlike those used to derive (13) and (35). We will now show that (41) can be derived in a way that uses the same concepts as those that led to (13), in particular using the concept of autocovariance and autocorrelation functions. First, we point out that the counts in two nonoverlapping time intervals are uncorrelated, but if the time intervals overlap, they are correlated. This can be seen graphically in Fig. 2. In order to calculate the correlation between the counts \( n \) and \( m \) in the two overlapping time intervals we must first derive the probability that the count is \( n \) in the interval \((t, t + \Delta t)\) and \( m \) in \((t + \tau, t + \tau + \Delta t)\).

Since the three time intervals \((t, t + \tau), (t + \tau, t + \Delta t),\) and \((t + \Delta t, t + \tau + \Delta t),\) marked I, II, and III in Fig. 2, are not overlapping, the
counts in these three intervals are uncorrelated and Poisson-distributed around the means $\eta_T$, $\eta(\Delta t - \tau)$, and $\eta_T$, respectively. The joint probability we seek is

$$P_{n,m}(\alpha) = \sum_{\lambda=0}^{\min(n,m)} P_\lambda(\eta(\Delta t - \tau)) P_{n-\lambda}(\eta_T) P_{m-\lambda}(\eta_T)$$

$$= e^{-\eta(\Delta t + \tau)} \sum_{\lambda=0}^{\min(n,m)} \frac{(\eta(\Delta t - \tau))^\lambda}{\lambda!} \frac{(\eta_T)^{n+m-2\lambda}}{(n-\lambda)! (m-\lambda)!} . \quad (43)$$

Fig. 2. Schematic illustration of the overlapping time intervals used to calculate the autocorrelation function of uncorrelated noise.
This formula is valid for $\tau < \Delta t$. For $\tau > \Delta t$ the counts $n$ and $m$ are statistically independent, and we have

$$P_{n,m}(\alpha) = P_n(\alpha) P_m(\alpha) \quad . \quad (44)$$

It is not quite obvious, looking at (43), that the sum over $n$ and $m$ of $P_{n,m}(\alpha)$ is unity, as it should be; but reordering the summations as follows gives the correct result:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \min(n,m) \sum_{\ell=0}^{\infty} P_{\ell} \eta(\Delta t - \tau) P_{n-\ell}(\eta \tau) P_{m-\ell}(\eta \tau)$$

$$= \sum_{\ell=0}^{\infty} P_{\ell} \eta(\Delta t - \tau) \sum_{n'=0}^{\infty} P_{n'}(\eta \tau) \sum_{m'=0}^{\infty} P_{m'}(\eta \tau)$$

$$= \sum_{\ell=0}^{\infty} P_{\ell} \eta(\Delta t - \tau) \sum_{n'=0}^{\infty} P_{n'}(\eta \tau) \sum_{m'=0}^{\infty} P_{m'}(\eta \tau) = 1 \quad . \quad (45)$$

Using a similar approach for the means and second-order moments, we obtain

$$\langle n \rangle = \langle m \rangle = \eta \Delta t = \alpha \quad \quad (46)$$

and

$$\langle nm \rangle = (\eta \tau)^2 + \eta(\Delta t - \tau) = \alpha^2 + \alpha(1 - \frac{\tau}{\Delta t}) \quad . \quad (47)$$

Equation (46) is true for all $\tau$, but (47) is true only for $0 < \tau < \Delta t$.

For $\tau > \Delta t$ we get

$$\langle nm \rangle = \langle n \rangle \langle m \rangle = \alpha^2 , \quad \tau > \Delta t \quad . \quad (48)$$
We have tacitly assumed that $\tau$ is positive; $\tau$ can also be negative, so that more generally we can write $|\tau|$ instead of $\tau$ in (43), (45), and (47).

The autocovariance function for the number of counts is

$$R_n(\tau) = \langle nm \rangle - \langle n \rangle \langle m \rangle$$

$$= \begin{cases} 
\alpha (1 - \frac{|\tau|}{\Delta t}) & \text{for } |\tau| \leq \Delta t \\
0 & \text{for } |\tau| \geq \Delta t 
\end{cases} \quad (49)$$

For $\tau = 0$, we have $\sigma_n^2 = \alpha$; normalizing $R_n(\tau)$ by this quantity, we have the autocorrelation function

$$\rho_n(\tau) = \begin{cases} 
1 - \frac{|\tau|}{\Delta t} & \text{for } |\tau| \leq \Delta t \\
0 & \text{for } |\tau| \geq \Delta t 
\end{cases} \quad (50)$$

Substituting (50) into (13), the error variance becomes

$$\sigma^2 \langle \tau \rangle = 2 \alpha \int_{-\infty}^{\infty} \rho_n(\tau) d\tau = 2 \alpha \int_{-\Delta t}^{\Delta t} (1 - \frac{\tau}{\Delta t}) d\tau = \alpha \Delta t \quad (51)$$

in agreement with (41).

Thus, we have shown that the two methods of deriving the error variance are equivalent. We now calculate the error variance in the more complicated case where the source of the uncorrelated noise $\alpha$ is itself a random function of time, as is the case in most geophysical applications.
3.3. Uncorrelated noise from a time-varying source. We assume that \( \alpha \) is a strictly ergodic time series with an integral time scale \( T_\alpha \) much larger than \( \Delta t \). Initially we consider the case of a signal being represented by a finite number of events. As an example, we can think of \( \alpha = \alpha(t) \) as the average number of decays in time \( \Delta t \) of a radioactive gas such as radon in a test volume \( V \). By virtue of (36) and (37), \( \alpha \) is a dimensionless measure of a time-varying concentration of radon atoms, averaged over time \( \Delta t \) and volume \( V \).

Let \( \phi(\alpha) da \) be the probability for this concentration measure to be between \( \alpha \) and \( \alpha + da \) in a particular time interval of length \( \Delta t \). Then the ensemble mean of \( \alpha \) is

\[
\langle \alpha \rangle = \int_{0}^{\infty} \alpha \phi(\alpha) da.
\]  

(52)

If \( \alpha \) is given, then the (conditional) probability for \( n \) counts in the same time interval is \( P_n(\alpha) \) and the unconditional probability \( P_n \) for \( n \) counts becomes

\[
P_n = \int_{0}^{\infty} P_n(\alpha) \phi(\alpha) da.
\]  

(53)

The mean number of counts in the interval is therefore

\[
\langle n \rangle = \sum_{n=0}^{\infty} n P_n = \int_{0}^{\infty} \phi(\alpha) da \sum_{n=0}^{\infty} n P_n(\alpha)
\[
= \int_{0}^{\infty} \alpha \phi(\alpha) da = \langle \alpha \rangle.
\]  

(54)

We now proceed to the joint distribution and second-order moments.
Let the joint probability for a concentration in the interval $(a', a' + da')$ at time $t'$ and a concentration in the interval $(a'', a'' + da'')$ at time $t''$ be $\phi(a', a'')d\alpha' d\alpha''$ and let the conditional probability, $\alpha'$ and $\alpha''$ given, for $n'$ and $n''$ counts at the same times, be $P_{n', n''}(\alpha', \alpha'')$. Then the unconditional probability for $n'$ and $n''$ counts at the times $t'$ and $t''$ is

$$P_{n', n''} = \int_0^\infty \int_0^\infty P_{n', n''}(\alpha', \alpha'') \phi(\alpha', \alpha'') d\alpha' d\alpha''.$$  \hspace{1cm} (55)

The ensemble mean of $n'n''$ is

$$\langle n'n'' \rangle = \int_0^\infty \int_0^\infty \phi(\alpha', \alpha'') \sum_{n'=0}^\infty P_{n'}(\alpha') \sum_{n''=0}^\infty P_{n''}(\alpha'') \langle n'n'' \rangle.$$ \hspace{1cm} (56)

This quantity is easily evaluated by realizing that the two cases $|t'' - t'| < \Delta t$ and $|t'' - t'| > \Delta t$ each can be simplified, the first under the assumption that

$$\Gamma_\alpha \gg \Delta t,$$ \hspace{1cm} (57)

where $\Gamma_\alpha$ is given by (12).

**Case 1, $|t'' - t'| > \Delta t$:**

The two counting intervals are not overlapping and the two counts $n'$ and $n''$ are statistically independent. This means that

$$P_{n', n''}(\alpha', \alpha'') = P_{n'}(\alpha') P_{n''}(\alpha'')$$ \hspace{1cm} (58)

and (56) reduces to
\[ \langle n'n'' \rangle = \int_0^\infty \int_0^\infty \alpha' \alpha'' \phi(\alpha', \alpha'') = \langle \alpha' \alpha'' \rangle . \] (59)

Case 2, \( |t'' - t'| < \Delta t \):

Since \( \alpha \) changes very little over time intervals less than \( \Delta t \), from (57), \( \phi(\alpha', \alpha'') \) can be approximated by

\[ \phi(\alpha', \alpha'') = \frac{\phi(\frac{\alpha' + \alpha''}{2})}{2} \delta(\alpha'' - \alpha') , \] (60)

where \( \phi(\alpha) \) is the marginal distribution

\[ \phi(\alpha) = \int_0^\infty \phi(\alpha', \alpha) d\alpha' , \] (61)

used in (52), (53), and (54), and \( \delta(\alpha) \) is Dirac's delta function.

In this case we obtain by substituting in (56)

\[ \langle n'n'' \rangle = \int_0^\infty \phi(\alpha) d\alpha \sum_{n'=0}^{\infty} \sum_{n''=0}^{\infty} n'n'' P_{n', n''}(\alpha) , \] (62)

where, of course,

\[ P_{n', n''}(\alpha) = P_{n', n''}(\alpha, \alpha) , \] (63)

given by (43). Using (47), we obtain

\[ \langle n'n'' \rangle = \int_0^\infty \phi(\alpha) \left[ \alpha^2 + \alpha (1 - \frac{|t'' - t'|}{\Delta t}) \right] d\alpha \]
\[ = \langle \alpha^2 \rangle + \langle \alpha \rangle \left( 1 - \frac{|t'' - t'|}{\Delta t} \right) , \] (64)

so that the covariance becomes
\[ R_n(t'' - t') = \langle n'n'' \rangle - \langle \alpha \rangle^2 = \sigma_a^2 + \langle \alpha \rangle(1 - \frac{|t'' - t'|}{\Delta t}) \]  \hspace{1cm} (65)

Since
\[ \sigma_a^2 = R_\alpha(\tau) \text{ for } |\tau| \ll \Gamma_\alpha \]  \hspace{1cm} (66)

we can reformulate (64) and write for all values of \( \tau \), including the result (59):

\[ R_n(\tau) = \begin{cases} 
R_\alpha(\tau) + \langle \alpha \rangle(1 - \frac{|\tau|}{\Delta t}) \text{ for } |\tau| \leq \Delta t \\
R_\alpha(\tau) \text{ for } |\tau| > \Delta t 
\end{cases} \]  \hspace{1cm} (67)

This autocovariance function is shown in Fig. 3.

Fig. 3. Schematic of an autocovariance function for a signal containing random uncorrelated noise.
The noise contribution \( \langle \alpha \rangle (1 - \frac{|\tau|}{\Delta t}) \) for \( |\tau| < \Delta t \) can, of course, in digital data analysis of finite time series be observed only for \( |\tau| = i \cdot t \quad (i = 0, 1, 2, \ldots, I-1) \), where \( I = T / \Delta t \). From this point of view, the noise autocovariance function is just \( \langle \alpha \rangle \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta. The spectrum of the noise, obtained by means of a finite, discrete Fourier transformation (FDFT) (Bendat and Piersol, 1971) is constant over frequency \( \omega \) from \(-\omega_n \) to \( \omega_n \), where \( \omega_n \) is the maximum observable frequency \( \pi / \Delta t \). This frequency is the so-called Nyquist or folding frequency.

Kristensen (1974) has considered the analogous autocovariance function for quantization noise. An example of a signal containing this noise is the output of a digital voltmeter continuously measuring an analog voltage. Its output is \( \xi \Delta \), where \( \xi \) is an integer and \( \Delta \) corresponds to the least significant digit. This quantization increases the variance of the analog signal \( \sigma_\alpha^2 \) by \( \Delta^2 / 12 \) if \( \Delta^2 / \sigma_\alpha^2 \leq 5 \). (This is the so-called Sheppard's correction [Sheppard, 1898; Wold, 1934].) Assuming that the variables \( \alpha(t) \) and \( \alpha(t + \tau) \) are joint Gaussian for all \( \tau \), Kristensen (1974) derived an expression for the quantization noise autocovariance function \( r(\tau) \), which is added to \( R_\alpha(\tau) \),

\[
r(\tau) = \frac{\Delta^2}{12} + \Delta^2 \sum_{\ell=1}^{\infty} \frac{\ell^2}{\sqrt{2\pi} \Delta^2} \text{erfc}(-\frac{\ell \Delta}{\sqrt{2D_\alpha(\tau)}}) - \frac{\Delta^2}{\sqrt{2\pi} \Delta^2} \sum_{\ell=-\infty}^{\infty} \exp(-\frac{\ell^2 \Delta^2}{2D_\alpha(\tau)})
\]

where

(68)
\[ D_\alpha(\tau) = <[a(t) - a(t + \tau)]^2> = 2[\sigma_\alpha^2 - R_\alpha(\tau)] \]

\[ = 2\sigma_\alpha^2[1 - \rho_\alpha(\tau)] \quad (69) \]

is the structure function, and

\[ \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \text{e}^{-t^2} dt \quad (70) \]

is the complementary error integral.

Kristensen (1974) showed that (68) is very nearly a triangle, and thus closely resembles (67). He also showed that a good working approximation to (68) is given by

\[ r(\tau) = \begin{cases} \frac{\Delta^2}{12} (1 - \frac{\tau}{\tau_*}) & \text{for } |\tau| < \tau_* \\ 0 & \text{for } |\tau| > \tau_* \end{cases}, \quad (71) \]

where \( \tau_* \) is obtained from the solution to the equation

\[ (1 - \rho_\alpha(\tau_*))^{1/2} = 0.4 \frac{\Delta}{\sigma_\alpha} \quad (72) \]

If, for example, \( \rho_\alpha(\tau) \) has the form (Panofsky and Dutton, 1984)

\[ \rho_\alpha(\tau) = 1 - \frac{(2\tau)^{2/3}}{5\Gamma_\alpha} \quad (73) \]

here characterized by the integral scale \( \Gamma_\alpha \), rather than the rates of dissipation, in a way that satisfies (12) identically, we obtain

\[ \tau_* = 0.16 \Gamma_\alpha (\Delta/\sigma_\alpha)^3 \quad (74) \]
Equation (71) can be substituted into a FDFT to calculate the spectral contribution of quantization noise. If the sampling period $\Delta t > \tau_*$, the spectral contribution is flat from zero to the Nyquist frequency, as in the case of Poisson noise. For $\Delta t < \tau_*$, the spectral contribution decreases with the absolute value of the frequency. Therefore, we see that only when $\Delta t = \tau_*$ is quantization noise the same as Poisson noise; in general, they are different.

The error variance for Poisson noise $\sigma^2(n; T)$ can now be calculated using the covariance function (67)

\[
\sigma^2(n; T) = \frac{2}{T} \int_0^\infty R(n; \tau) d\tau
\]

\[
= \frac{2}{T} \int_0^\infty R(n; \tau) d\tau + \frac{2}{T} \int_0^{\Delta t} <\omega> (1 - \frac{\tau}{\Delta t}) d\tau
\]

\[
= \sigma^2(\alpha; T) + <\omega> \frac{\Delta t}{T} .
\]  

(75)

We see that an extra contribution exists in the error variance due to the counting noise. This term becomes significant when both $<\alpha>$ and $\sigma_\alpha/<\alpha>$ are small.

We have carried through a similar calculation for the vertical flux $n_w$, and assuming that $\alpha$ and $w$ are mutually joint Gaussian we obtain an additional error variance $\delta \sigma^2(F; T)$ to $\sigma^2(F; T)$ due entirely to the counting noise,

\[
\delta \sigma^2(F; T) = \sigma^2_w <\omega> \frac{\Delta t}{T} \left(1 - 2 \frac{T_w}{T}\right) \approx \sigma^2_w <\omega> \frac{\Delta t}{T} ,
\]  

(76)
where $\sigma_w^2$ and $\Gamma_w$ are the variance and the integral time scale, respectively, of the vertical velocity component.

We believe on the basis of analogy, without having rigorously proved it, that the extra contribution to the error variance of the variance of $\alpha$ (35) is given by (42); i.e.,

$$\delta \sigma^2(A, T) = \langle \alpha \rangle \left\{ 2\langle \alpha \rangle + 1 \right\} \frac{\Delta t}{T}. \quad (77)$$

We also point out that instruments for counting radioactivity, photons, electrons, aerosols, etc., usually have uncorrelated noise from two sources: the signal itself and the background noise. In order to determine the true signal, we must determine the average background $\alpha_0$ by shutting off the true signal. The error variance of the signal plus background noise must be small compared to the difference between the signal and the background noise. The error variance that determines the necessary measuring time is given by (75), (76), and (77) with $\langle \alpha \rangle + \alpha_0$ substituted for $\langle \alpha \rangle$. 
4. NUMERICAL NOISE SIMULATION

In order to test the theoretical developments of the previous section, we use an actual time series that is nominally noise-free, then modify it by adding varying amounts of artificially generated Poisson-distributed noise which is randomized with a random number generator. By using different "seeds" in the random number generator, we can produce a series of realizations of the same signal, each with different randomly distributed noise added to the signal. In each realization, however, the random noise contribution would follow the same distribution in the limit of an infinitely long series.

We selected a 540-s humidity time series for these calculations that was obtained from the NCAR King Air aircraft at a level of about 90 m above the ground in a convective boundary layer. The sensor was a Lyman-alpha hygrometer (Buck, 1976) sampled at 50 s$^{-1}$ at an aircraft speed of ~80 m s$^{-1}$, filtered with a four-pole low-pass Butterworth filter with a cutoff at ~10 Hz, then reduced to a sample rate of 20 s$^{-1}$ by using the standard NCAR Research Aviation Facility interpolation algorithm. We then added Poisson-distributed noise to simulate an output obtained from an instrument which senses a signal by counting discrete events. Thus, the number of counts is proportional to the signal being sensed, with the addition of random uncorrelated noise contributed by the predicted random fluctuations introduced by the process of counting a limited number of events.

The Poisson-distributed noise was generated by multiplying the time series by a factor that converted its mean value to the mean value of...
the number of counts desired, \( \alpha \). A table was generated such that for each entry \( \alpha \) in the table, a cumulative Poisson distribution about that value was calculated. [The cumulative distribution is the sum of the distribution itself (38) from zero to the specified integer value.] A resolution of 0.01 was used in the table for \( \alpha \). Then, for each value of \( \alpha \) in the time series, the entry in the table closest to but less than that value was selected. A random number which has an equal probability of being anywhere in the interval from zero to one was then obtained from a random number generator. The value of the cumulative probability distribution in the table that was closest to but less than this random number was selected, and the integer value \( n \) corresponding to this random number then was used as the "noisy signal." For values of \( n \) greater than 20, it was not possible to directly compute the Poisson distribution on the computer because of overflow in computing the factorial. Therefore we used an asymptotic formula for the gamma function (6.1.41 in Davis, 1972). Using the first two terms of this expansion, we have

\[
P_n(\alpha) = \frac{1}{\sqrt{2\pi n}} \left( \frac{\alpha}{n} \right)^n \exp \left( -\alpha + n - \frac{1}{12n} \right).
\]

Neglecting the second term in this expansion \([-1/(12n)\) in (78)] has negligible effect on the mean value, but the skewness was significantly different from its theoretical value.

Fig. 4 shows the original time series for vertical velocity and humidity as well as a rescaled test time series \( \alpha(t) \) for humidity with a
mean value of 10, and a modified time series with \( a(t) \) replaced by a Poisson-distributed variable \( n(t) \) whose mean value is \( a(t) \). As expected, the added noise increases the variance of the time series by an amount equal to \( a \).

Fig. 4. From the top: vertical velocity and absolute humidity measured from the NCAR King Air aircraft at 90 m above the ground in a convective boundary layer; our test time series which is humidity rescaled to obtain \( a = 10 \); and the test time series with Poisson noise added.
Fig. 5 shows the spectra of $a(t)$ and $n(t)$. As discussed in Section 3, $n(t)$ has a nearly white noise spectrum (i.e., a slope of +1 on a plot of wave number times the logarithm of amplitude versus the logarithm of wave number) in contrast to the original signal which increases with wave number to a peak at 0.006 $r$ m$^{-1}$, then decreases above this with a -2/3 slope characteristic of the inertial subrange of atmospheric turbulence.

![Graph showing spectra of $a(t)$ and $n(t)$](image)

Fig. 5. Spectra of the test time series plotted versus wave number $k$, using Taylor's hypothesis and a mean wind speed of -80 m s$^{-1}$, and same time series with added Poisson noise for $<n> = 10$. 
We now investigate the effects of this uncorrelated random noise on mean and higher-order moment quantities by examining the statistics obtained from several sets of realizations of this noise-modified time series. This was done by selecting several mean values of \( \alpha(t) \) and calculating ten realizations for each choice. In order to isolate the variability due solely to the Poisson noise, we calculated

\[
n_T = \frac{1}{T} \int_0^T n(t) dt \quad \text{[in analogy to (1)]},
\]

\[
[(n - \alpha_T)^2]_T
\]

and

\[
[(w - w_T)(n - \alpha_T)]_T
\]

as functions of \( T \), the averaging time. The value of \( T \) was varied from 20s to \( T_0 = 540 \) s in increments of 20 s. This was done by selecting several values of \( \alpha_T \) and calculating ten realizations for each choice. The results are shown in Fig. 6 for \( \alpha_T = 10 \). The additional quantities of the unmodified test series, \( \alpha_T \), \( [(\alpha - \alpha_T)^2]_T + \alpha_T \), and

\[
[(w - w_T)(\alpha - \alpha_T)]_T
\]

are shown as heavy dashed lines for comparison. The term \( \alpha_T \) is added to the variance so that the unmodified test-series variance can be compared to the noise-modified variance; the two variances are additive for \( \Delta t / T_0 << 1 \). The 5% and 95% confidence limits, which are calculated assuming the variables are normally distributed with error variances given by (75), (76), and (77), are also plotted. The predicted confidence limits are in good agreement with the
envelope of the ten realizations for each quantity. We have also calculated the 90% confidence limits for the average of the measured error variances for $T/\Delta t = 400$ and $10,800$, which are assumed to follow a chi-squared distribution. We found that in all cases the theoretical error variances are within the confidence limits.

Fig. 6. Estimates (see text for details) of the mean, variance, and vertical flux of the test time series (heavy dashed line), ten realizations of these variables for the Poisson-noise-modified test time series (thin solid lines), and the 5% - 95% confidence limits for the noise-modified variables (dotted lines).
5. MEASUREMENT OF AEROSOLS

Counting aerosols in the atmosphere is a good example of an inherently discrete measurement which has great practical and research interest, and has become commonplace. The technology has now advanced to the point that measurements of aerosol turbulent fluxes, and co-spectra of aerosol concentration and vertical velocity have also been reported (e.g., Wesely et al., 1983; Katen and Hubbe, 1983; Sievering, 1983). Wesely and Hart (1984) and Fairall (1984) have also addressed the question of the contribution of counting noise to measurements of aerosol turbulence fluxes, but their approaches are different from ours.

To illustrate the application of our results to actual atmospheric measurements, we show here some results obtained from an Active Scattering Aerosol Spectrometer Probe Model ASAP-100-X (Particle Measuring Systems, Inc.). The data were obtained for the same flight leg as discussed previously, except that the time series is limited to 320 s. The instrument measures particles in 15 different size ranges from 0.12 to 3.12 μm. Here we present results from the first five size channels, whose widths (in μm) are given in Table 1. Fig. 7 shows the time series for these five channels; the spectra for channels 2 and 5, which contain the highest and lowest number of counts, respectively, are shown in Fig. 8. We see that at high frequencies both spectra are essentially white noise. However, at low frequencies channel 2 shows some indication of a real signal. This is consistent with the comparison presented in Table 1 between the measured variance and the variance expected if the measured counts were entirely noise.
Estimates of error variances for the first five channels of an aerosol counting probe using a 320-s sample time series obtained from the NCAR King Air aircraft at 90 m above the ground at an airspeed of -80 m s\(^{-1}\) in a convective boundary layer. The vertical velocity variance is \(\sigma_w^2 = 0.52 \text{ m s}^{-2}\). The integral length scale, estimated by the integral of the autocorrelation function from zero lag to the first zero crossing, was calculated to be -89 m. At an aircraft speed of 80 m s\(^{-1}\), \(\Gamma_w = 1.1 \text{ s}\). In a similar calculation, the integral scale of the aerosol count was estimated to be \(\Gamma_w^*\).

We define \(\Sigma^2(k;T) \equiv \sigma^2(k;T) + \delta\sigma^2(k;T)\), where \(k = F\) or \(A\).

<table>
<thead>
<tr>
<th>Channel No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size range ((\mu m))</td>
<td>0.120 - 0.145</td>
<td>0.145 - 0.195</td>
<td>0.195 - 0.270</td>
<td>0.270 - 0.370</td>
<td>0.370 - 0.495</td>
</tr>
<tr>
<td>(F_n) (m s(^{-1}))</td>
<td>0.264</td>
<td>0.088</td>
<td>0.092</td>
<td>0.0015</td>
<td>-0.0129</td>
</tr>
<tr>
<td>(\sigma_n^2)</td>
<td>35.5</td>
<td>51.2</td>
<td>24.1</td>
<td>3.70</td>
<td>0.97</td>
</tr>
<tr>
<td>(\langle n \rangle = \langle n^* \rangle)</td>
<td>29.3</td>
<td>47.0</td>
<td>22.7</td>
<td>3.63</td>
<td>0.95</td>
</tr>
<tr>
<td>(\sigma_a^2 = \sigma_a^2 - \langle n \rangle)</td>
<td>6.3</td>
<td>4.3</td>
<td>1.41</td>
<td>0.072</td>
<td>0.0147</td>
</tr>
<tr>
<td>(F_n/(\sigma_w\sigma_a))</td>
<td>0.015</td>
<td>0.059</td>
<td>0.107</td>
<td>0.008</td>
<td>-0.147</td>
</tr>
<tr>
<td>(\sigma^2(F;T)) [Eq. (8)]</td>
<td>0.046</td>
<td>0.031</td>
<td>0.0103</td>
<td>0.00052</td>
<td>0.000107</td>
</tr>
<tr>
<td>(\delta\sigma^2(F;T)) [Eq. (23)]</td>
<td>0.0048</td>
<td>0.0077</td>
<td>0.0037</td>
<td>0.00060</td>
<td>0.000156</td>
</tr>
<tr>
<td>(F_n/\Sigma(F;T))</td>
<td>0.117</td>
<td>0.45</td>
<td>0.78</td>
<td>0.045</td>
<td>-0.80</td>
</tr>
<tr>
<td>(\sigma^2(A;T))</td>
<td>0.49</td>
<td>0.23</td>
<td>0.25</td>
<td>6.5\times10^{-5}</td>
<td>2.8\times10^{-6}</td>
</tr>
<tr>
<td>(\delta\sigma^2(A;T))</td>
<td>0.54</td>
<td>1.40</td>
<td>0.33</td>
<td>0.0094</td>
<td>8.6\times10^{-4}</td>
</tr>
<tr>
<td>(\sigma_a^2/\Sigma(A;T))</td>
<td>6.17</td>
<td>3.35</td>
<td>2.37</td>
<td>0.74</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Table 1
Fig. 7. Time series of vertical velocity and the first five channels of the PMS (Particle Measuring Systems, Inc.) probe sampled at 10 s$^{-1}$, obtained from the NCAR King Air aircraft at 90 m above the ground in a convective boundary layer.
Fig. 8. Spectra of vertical velocity, and PMS probe channels 2 and 5, and cospectra of vertical velocity combined with channels 2 and 5. In these and subsequent spectra, frequency has been converted to wave-number $k$ by the transformation $k = \frac{2\pi f}{U}$, where $U$ is the mean aircraft airspeed.

We also show in Fig. 9 the fluxes for each of the channels as a function of averaging time $T$. We note that the fluxes for four of the channels are very close to zero; only channel 5 has a negative flux for the whole time period. From Table 1, we see that the square root of the error variance for the flux is larger than the flux in all channels and that the Poisson-noise term is less than the integral scale term in the first three channels and greater than the integral scale term in the
last two channels. The cospectra between \( w \) and \( n \) are shown in Fig. 8 for channels 2 and 5; the other channels are very similar to channel 2, in that they all have a negative peak at 0.05 \( r \, m^{-1} \) and a positive peak at about 0.5 \( r \, m^{-1} \). For channels 1 through 4, the positive peak dominates slightly (but is still not significantly different from zero), while for channel 5, the negative peak contributes more area than the positive peak. Although it is beyond the scope of our consideration here, it is likely that the dominant positive and negative peaks in the cospectra are significant, since they are repeated on all five channels, while the total flux is not significant since the areas very nearly cancel out.
Fig. 9. Vertical flux of particle counts normalized by the time-averaged estimate of $\sigma_w \sigma_n$ for all the five PMS probe channels shown in Fig. 6 as a function of time.
6. EFFECTS ON SPECTRA AND STRUCTURE FUNCTIONS

The effects of Poisson-distributed noise on the spectrum can be obtained by substituting the autocovariance function (67) into a FDFT. The result is a white-noise contribution to the spectrum. On the other hand, the noise contribution to the structure function,

\[ D_n(\tau) = 2[\sigma^2 + \langle \alpha \rangle - R_n(\tau)] , \tag{79} \]

is

\[ \delta D_\alpha(\tau) = D_n(\tau) - D_\alpha(\tau) = \begin{cases} 2\langle \alpha \rangle \frac{|\tau|}{\Delta t} & \text{for } |\tau| \leq \Delta t \\ 2\langle \alpha \rangle & \text{for } |\tau| > \Delta t \end{cases} \tag{80} \]

This has important implications for estimating parameters that are characterized by small-scale turbulent fluctuations. Variance dissipation, for example, is often estimated by use of Kolmogorov's inertial subrange hypothesis (Panofsky and Dutton, 1984), which asserts that the variance dissipation is related to the magnitude of the -5/3 region of the spectra, or equivalently, to the +2/3 region of the structure function. Since Poisson noise contributes to the structure function only between 0 and \( \Delta t \), in principle the structure function may be negligibly affected by noise over the rest of its domain. As an example of this effect, Fig. 10 shows a spectrum of ozone measured over the same time period as the previous examples. The ozone is measured with an instrument that counts photons that are emitted by the chemiluminescent reaction of ozone in the ambient air mixed together with nitric oxide in a reaction chamber (Pearson and Stedman, 1980). We see that the -2/3
inertial subrange region is contaminated with noise to such an extent that the inertial subrange constant cannot be estimated; there is no region of -2/3 slope, and, without knowing precisely the noise characteristics, it is difficult to remove the noise contribution from the spectrum.

Fig. 10. Spectrum of ozone concentration (parts per billion by volume) measured from the NCAR King Air aircraft at 90 m above ground in a convective boundary layer with a chemiluminescence sensor at a sample rate of 20 s⁻¹. The dashed line is obtained from the Fourier transform of the autocovariance function corresponding to the noise-corrected structure function \(D_{03}(r)\) shown in Fig. 12.
On the other hand, the autocorrelation function, shown in Fig. 11, is affected only for $t < \Delta t$, or equivalently, $r < \Delta r$, where $r = t/U$ and $U$ is the airplane airspeed. By fitting a least squares line $a + br^{2/3}$ to the structure function $D_n(r)$ between $r = 2\Delta r$ and $85\Delta r$, as shown in Fig. 12, and subtracting the constant $a$ from $D_n(r)$, we obtain the modified structure function $D_{03}(r)$ shown in Fig. 12. The dashed line in Fig. 10 is the transform of the covariance function corresponding to the dashed line through $D_{03}(r)$ in Fig. 12. We see that because of the noise, which is approximately constant across the spectrum, there is no region of $-2/3$ slope; it appears that removing the noise from the spectrum is less convenient and probably less accurate than removing it from the structure function (Lenschow, 1966).

![Graph](image)

Fig. 11. Autocorrelation function for the ozone time series plotted as a function of longitudinal displacement, using Taylor's hypothesis and an airplane speed of $U = 80$ m s$^{-1}$ ($\Delta r = U\Delta t = 4$m).
Fig. 12. Structure function of the ozone time series (parts per billion by volume) with random noise included [D_n(r)], and with noise removed by substituting the r = 0 intercept of a least squares fit of the function $y = a + br^{2/3}$ from the second to the 85th lag (vertical dashed lines) to $D_n(r)$. 
7. CONCLUDING REMARKS

The error variances calculated from the ten realizations of the test series confirm the theoretical results. Applications of the results to aerosol measurements illustrate how the theoretical results can be used in practice. We are now in a position to summarize the results for practical applications by specifying a "figure of merit" \( Q \) that can be used in designing systems for atmospheric measurements or specifying system limitations. We can do this by taking the ratio of the error variance contributed by uncorrelated noise to the variance contributed by atmospheric variability. If the ratio is of order one or greater, uncorrelated noise is significant. This can be done for the mean and variance of \( a \), and the covariance of \( a \) and \( w \). For the mean and covariance, we take the ratios of the last term in (75) to (13), and (76) to the right hand side of (34), respectively, and obtain

\[
\frac{\langle a \rangle \Delta t / I}{2 \sigma^2(a; T)} = \frac{\delta^2(F; T)}{\sigma^2(F; T)} = \frac{\langle a \rangle \Delta t}{4 \sigma^2 T},
\]

where \( T \) represents \( T_a \) or \( \min(T_a, T_w) \). The ensemble-averaged number of counts in the time interval \( \Delta t \) can be written as

\[
\langle a \rangle = nV\Delta t <q>,
\]

where \( q \) is the concentration (e.g., molecules m\(^{-3}\)), \( V \) is the reference volume, and \( n \) is the number of counts per molecules in \( V \) per unit time. Similarly, the variance of \( a \) is given by

\[
\sigma_a^2 = n^2V^2(\Delta t)^2\sigma_q^2.
\]
Therefore, applying our criteria for uncorrelated noise to be insignificant, we obtain, by substituting (82) and (83) in (81),

\[ Q = \langle q \rangle \zeta \Gamma > \frac{\langle q^2 \rangle}{4\sigma^2} \quad . \]  

(84)

Here \( \zeta = V_n \) (counts s\(^{-1}\) concentration\(^{-1}\)) is the relevant design parameter that determines the contribution of counting noise to the measurement that needs to be specified for a particular system.

The figure of merit \( Q \) is the number of counts per unit concentration during one integral time scale \( \Gamma \).

As an example, we can apply this relationship to the particular situation where a trace species in a convective boundary layer is being deposited at the ground surface. We assume that this deposition flux is the only source of variance in the boundary layer, and that the trace species follows the same scaling laws as temperature. Observations indicate that the maximum variance occurs very near the surface and decreases to a minimum near the top of the boundary layer. We consider two cases for which we calculate the minimum value of \( Q \) necessary to measure flux such that the counting noise contribution to the error is less than the error due to atmospheric variability. The first is the minimum value of \( Q \) such that the counting noise does not contribute significantly to the error in the flux measured near the surface, and the second is the minimum value of \( Q \) such that the counting noise does not contribute significantly to the error in the flux measured throughout the boundary layer. Wyngaard et al. (1971) and Wesely et
al. (1970) have found that, close to neutral stability,

\[ \sigma_q = \frac{2F_q}{u_*} , \quad (85) \]

where \( u_* \) is the friction velocity. Substituting (85) into (84), we have

\[ Q > 0.06u_*^2/v_d^2 , \quad (86) \]

where \( v_d = -F_q/\langle q \rangle \) is the surface deposition velocity. For species efficiently deposited at the surface, \( v_d \) is of order 0.01 m s\(^{-1}\); \( u_* \) is typically of order 0.1 m s\(^{-1}\). Therefore, (86) can be approximated by \( Q > 10 \). From Kaimal et al. (1972) we find that close to neutral, \( \Gamma = \Gamma_w = 0.3 z/U \), where \( z \) is the height above the surface and \( U \) is the mean wind speed. Therefore, from (84) we find that for typical surface layer experiments, \( \langle q \rangle \xi \geq 100 \) counts s\(^{-1}\).

Next we consider the case in which flux is measured by aircraft somewhere in the middle of a convective boundary layer, with the deposition at the surface still being the major source of variance. Lenschow et al. (1980) have found that

\[ \min(\sigma_q) = 1.3 F_q/w_* , \quad (87) \]

where \( w_* \) is the convective velocity scale, typically of order 1 m s\(^{-1}\). Therefore,

\[ Q > 0.15 w_*^2/v_d^2 , \quad (88) \]
Thus, if we require measurement of flux everywhere in a convective boundary layer with a surface deposition velocity $v_d = 0.01\, \text{m s}^{-1}$, $Q > 1000$. Since the integral length scale in the middle of the convective boundary is typically of order 100 m, then for an aircraft speed of $\sim 80\, \text{m s}^{-1}$, $\Gamma_w \sim 1.25\, \text{s}$ and $\langle q \zeta \rangle > 10^3$ counts s$^{-1}$.

We can also apply (81) to the turbulence aerosol flux measurements presented in Table 1, and we find that the ratio increases with channel number from $-0.1$ for channel 1 to $-1.6$ for channel 5; thus, the error variance in channel 1 is due mainly to atmospheric variability, while in channel 5 counting noise dominates. At first glance, it may seem surprising that the ratio of the flux to the square root of the error variance is largest for channel 5. However, this is due to the larger magnitude of the correlation coefficient $F_n/(\sigma_w \sigma_\alpha)$ for this channel. We can calculate a similar ratio for the variances in Table 1, by taking the ratio of (77) to (35), and we find that the ratio increases from about one for channel 1 to about 300 for channel 5; thus counting noise is the major source of variance in all the channels. The techniques discussed here are applicable to other problems of noisy data as well. For example, noise contributions to higher-order moments can be calculated. We have also not directly addressed the question of how much noise can be tolerated before extraction of a usable signal becomes impossible.
REFERENCES


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