Free Oscillations of Deep Nonhydrostatic Global Atmospheres: Theory and a Test of Numerical Schemes

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This report describes a method of calculating the modes of small-amplitude free oscillations of a compressible, stratified, nonhydrostatic, rotating system between two concentric spheres, including sine and cosine Coriolis terms, where \( \phi \) is the latitude. Unlike the Laplace-Taylor problem for the hydrostatic primitive equations that can be solved by the separation of variables, the present problem is non-separable. In this study, normal mode solutions are obtained numerically by setting up an eigenvalue-eigenfunction matrix problem by the combination of a spherical harmonics expansion in the horizontal direction and a finite-difference discretization in the radial direction.

A test of the numerical schemes is conducted for an isothermal basic state with a constant gravitational acceleration. In order to identify the species of solutions, a shallow atmosphere version of the same formulation is considered in parallel. Because the shallow nonhydrostatic normal mode problem can be solved also by the separation of variables, two different approaches to the shallow problem provide an aid to verify the numerical schemes of the deep normal mode problem. Numerical results are presented for the frequencies and eigen-structures of various kinds of normal modes in the deep model and compared with those from the shallow model.

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PREFACE

This report describes a method of calculating the modes of small-amplitude free oscillations of a compressible, stratified, nonhydrostatic, rotating deep atmosphere, confined between two concentric spheres, including $\sin \phi$ and $\cos \phi$ Coriolis terms, where $\phi$ is the latitude. Unlike the Laplace-Taylor problem for the hydrostatic primitive equations that can be solved by the separation of variables, the present problem is non-separable. In this study, normal mode solutions are obtained numerically by setting up an eigenvalue-eigenfunction matrix problem by the combination of a spherical harmonics expansion in the horizontal direction and a finite-difference discretization in the radial direction.

A Test of the numerical schemes is conducted for an isothermal basic state with a constant gravitational acceleration. In order to identify the species of solutions, a shallow atmosphere version of the same formulation is considered in parallel. Because the shallow nonhydrostatic normal mode problem can be solved also by the separation of variables, two different approaches to the shallow problem provide an aid to verify the numerical schemes of the deep normal mode problem. Numerical results are presented for the frequencies and eigen-structures of various kinds of normal modes in the deep model and compared with those from the shallow model.

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1. Introduction

The dynamical formulation of most of the current atmospheric models for global weather prediction and climate simulation is based on the hydrostatic primitive equation (HPE) system that adopts some traditional simplifications (Phillips 1966) in a more general form of the equations of motion. One such simplification is referred to as “shallowness (or thin-shell) approximation” based on the notion that the vertical extent of the atmosphere of our interest is rather small compared with the earth's radius.

In addition to the shallowness approximation, the HPE system adopts another simplification that the vertical acceleration is negligible in the vertical equation of motion, because hydrostatic equilibrium prevails in the atmosphere. It is important to note that the shallowness assumption alone does not justify omission of the vertical acceleration. The “hydrostatic approximation” is beneficial to eliminate the vertical propagation of acoustic waves, so that a small vertical grid increment does not overly restrict the choice of time step in explicit time integrations.

The adoption of these two simplifications may not be suitable to the dynamical formulation of the next-generation atmosphere models. On the one hand, increased computer capabilities in terms of speed and memory permit us to use finer grid resolutions thereby multi-scale phenomena can be accurately treated if the vertical acceleration is retained. On the other hand, there are many advantages if the shallowness simplification is not adopted. We can extend the top of the model atmosphere beyond the stratosphere to better utilize satellite observations, as well as to deal with the motions of the whole atmosphere. An extensive discussion is presented by White and Bromley (1995) in favor of not adopting these simplifications as a critique to the HPE formulation.

Let us write down the global atmospheric prediction system without invoking the shallowness and hydrostatic assumptions in terms of spherical coordinates \((\lambda, \phi, r)\) with \(\lambda\) being longitude, \(\phi\) latitude, and \(r\) the radial distance from the center of a sphere. The
equations of motion for velocity components \((u, v, w)\) corresponding \((\lambda, \phi, r)\) are expressed as

\[
\frac{du}{dt} - \left( f_V + \frac{u \tan \phi}{r} \right) v + \frac{1}{\rho r \cos \phi} \frac{\partial p}{\partial \lambda} - F_u = -\frac{uw}{r} - f_H w, \tag{1.1}
\]

\[
\frac{dv}{dt} + \left( f_V + \frac{u \tan \phi}{r} \right) u + \frac{1}{\rho r} \frac{\partial p}{\partial \phi} - F_v = -\frac{v w}{r}, \tag{1.2}
\]

\[
\frac{dw}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial r} + g - F_w = \frac{(u^2 + v^2)}{r} + f_H u, \tag{1.3}
\]

with \(t\) being time and

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{r \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{r} \frac{\partial}{\partial \phi} + w \frac{\partial}{\partial r}. \tag{1.4}
\]

Also,

\[
f_V = 2\Omega \sin \phi, \quad f_H = 2\Omega \cos \phi. \tag{1.5}\]

Here, \(p\) denotes the pressure and \(\rho\) the density; \((F_u, F_v, F_w)\) represents the frictional components corresponding to \((u, v, w)\). Also, \(g\) and \(\Omega\) denote, respectively, the gravitational acceleration and the angular rotation rate of the earth. Note that both the vertical and horizontal components of the Coriolis vector, \(f_V\) and \(f_H\), are present in addition to the apparent acceleration terms due to the curvature of the coordinate system.

The mass continuity equation for dry air is expressed by

\[
\frac{\partial p}{\partial t} + \frac{1}{r \cos \phi} \left[ \frac{\partial (\rho u)}{\partial \lambda} + \frac{\partial (\rho v \cos \phi)}{\partial \phi} \right] + \frac{\partial (\rho w)}{\partial r} = -\frac{2\rho w}{r}. \tag{1.6}
\]

The right-hand side of (1.6) appears as a divergence effect of radial distance \(r\) in a deep atmosphere.

The form of thermodynamic equation is unchanged as before and is expressed by

\[
\frac{dp}{dt} = \gamma RT \left( \frac{d\rho}{dt} + \frac{\rho Q}{C_p T} \right), \tag{1.7}
\]

2
where $T$ is the temperature given by the idea gas law

$$p = \rho RT$$  \hspace{1cm} (1.8)$$

with $R$ denoting the specific gas constant, $R = C_p - C_v$, in which $C_p$ and $C_v$ denote the specific heat values at constant pressure and at constant volume, respectively. In (1.7), $Q$ represents the time rate of heating/cooling per unit mass and $\gamma = C_p/C_v$.

The equations of motion and mass continuity in the HPE system are obtained by replacing $r \rightarrow a$ and $dr \rightarrow dz$ in (1.1) - (1.3) and (1.6) and neglecting the right-hand sides of (1.1) - (1.3) and (1.6) with the additional step of $dw/dt = 0$ and $F_w = 0$.

One unique aspect of the deep nonhydrostatic (DNH) prediction system (1.1) - (1.8), in addition to the divergent effect of radial distance $r$ in the continuity equation (1.6), is the inclusion of the Coriolis terms related to $f_H$ in the zonal and vertical equations of motion. White and Bromley (1995) argue through a scale analysis that the effects of $f_H$ Coriolis terms may attain magnitudes of as much as 10\% of major terms in the equations of motion for both planetary-scale and diabatically driven tropical motions. In fact, the additional terms in the DNH prediction system are now included in the operational forecast model of the United Kingdom Meteorological Office (Davies 2000). However, no evaluation on the merits of using this DNH prediction system over the traditional HPE system has been reported.

The objective of this study is to attempt to understand the basic characteristics of the DNH model by examining its normal mode solutions. The normal modes are the solutions of free small-amplitude oscillations of a dynamical system superimposed on a basic state subject to specified boundary conditions. Studies of free oscillations of the atmosphere have a long interesting history. In the next section, we review the history of research on the normal modes of the atmospheres on the sphere to place the present study of the DNH model in a historical perspective.
2. Brief history on the normal modes of the global atmospheres

2.1 Hydrostatic primitive equation (HPE) model

The first model to be considered is the traditional hydrostatic primitive equation (HPE) model with the shallowness and hydrostatic approximations. According to Taylor (1936), the oscillations of the spherical atmosphere were treated by Laplace who showed that the oscillations of an isothermal atmosphere under isothermal process are identical to those of an ocean of uniform depth. In fact, Taylor (1936) investigated the oscillations of a compressible and hydrostatic spherical atmosphere and showed that the linearized equations of the HPE model with respect to the basic state at rest can be separated into the vertical and horizontal structure equations with a separation parameter $h_e$. It turned out that the horizontal structure equations have the form, now referred to as the Laplace tidal equations, that can be derived from linearized global shallow water equations with a uniform depth $h_e$. Because of this analogy Taylor coined the terminology of "equivalent depth (or height)" for the separation parameter $h_e$.

Taylor (1936) himself did not work out the solutions of the horizontal structure equations (HSEs) and vertical structure equations (VSEs), but many investigators gradually developed the formalisms on the solutions of the HSEs and VSEs, mostly in connection with atmospheric tidal theory (e.g. Wilkes 1949; Chapman and Lindzen 1970).

Referring to atmospheric tidal theory, it is important to distinguish the problem of free oscillations, such as internal gravity waves and planetary waves, from the problem of forced oscillations, such as tides. For forced problems generated by forcings with known frequencies, the equivalent height $h_e$ is determined first as the eigenvalues of the HSEs for a given frequency. Then, the VSEs are solved with the known values of $h_e$ and a given vertical distribution of the forcing function.

In contrast, for free oscillation problems with unknown frequencies and no forcing, the VSEs are solved first to determine the equivalent height $h_e$ as the eigenvalues of VSEs under suitable boundary conditions. Then, the frequency of free oscillations is determined as the
eigenvalues of the HSEs with a given value of the equivalent height $h_e$. It is important to emphasize here that we are considering in this study the problem of free oscillations. Many authors investigated the solutions of VSEs of the HPE model. For numerical weather prediction the reader is referred to Kasahara and Puri (1981) who describe how to determine the equivalent height $h_e$ as the eigenvalues and the vertical profiles of normal modes as the eigenfunctions of VSEs under a given vertical distribution of the basic state temperature with specified boundary conditions.

Similarly, many authors have studied the solutions of the HSEs that are equivalent to those of the linearized global shallow water equations, known as Laplace tidal equations. Margules (1893) obtained the solutions of free oscillations by expanding them in a power series of trigonometric functions. A few years later, apparently unaware of Margules' work, Hough (1898) employed a series of associated Legendre functions to calculate the normal modes of free global oscillations. However, the full scope of the properties of the normal modes had not been explored until the advent of high-speed electronic computers. Extensive numerical calculations were conducted during the 1960's. In particular, Longuet-Higgins (1968) published tables of eigenfrequencies and diagrams of eigenfunctions for various values of equivalent height $h_e$. Computer codes of the eigenfrequencies and eigenfunctions for positive values of $h_e$ are now available at the National Center for Atmospheric Research (Swarztrauber and Kasahara 1985). The eigenfunctions are referred to as Hough harmonics that are expressed by the product of $\sin(s\lambda)$ or $\cos(s\lambda)$ in longitude $\lambda$, where $s$ is the zonal wavenumber, and Hough functions in latitude $\phi$.

2.2 Shallow nonhydrostatic (SNH) model

This model is identical to the HPE model except that the vertical acceleration term in the vertical equation of motion is retained. Due to the shallowness approximation, the terms related to the horizontal component of Coriolis vector $f_H$, defined in (1.5), are neglected as in the HPE model. However, this model is no longer hydrostatic.
Through the normal mode analyses by Monin and Obukhov (1959), Eckart (1960), Gill (1982), and others who adopted tangent-plane geometry, it is known that the normal modes of the SNH model consist of at least two kinds, inertio-gravity (IG) modes and acoustic (AC) modes. The frequencies of AC modes are much larger than those of IG modes. Since the AC modes are probably not important for weather prediction, a general perception is that the AC modes are a meteorological nuisance and the merit of including the vertical acceleration is questionable. However, we should note that the hydrostatic approximation markedly degrades the accuracy of internal gravity waves when the ratio of horizontal and vertical scales of motion approaches unity.

Research on the normal modes of SNH model in spherical geometry is relatively recent. Dikii (1965), who investigated the normal modes of the global SNH model, showed that the system is separable into the vertical and horizontal structure equations with a separation parameter with the dimension of equivalent height $h_e$. [Phillips (1990) also describes this step in his monograph.] One unique difference in the normal mode formulation between the SNH and HPE models is that the frequency $\sigma$ and the equivalent height $h_e$ appear in both the horizontal and vertical structure equations (HSEs and VSEs). In contrast, in the HPE model $\sigma$ does not appear in the VSEs, so that $h_e$ can be obtained by solving the VSE system first. This is not the case of the SNH model. Therefore, the HSEs and VSEs must be solved simultaneously as a coupled eigenvalue problem.

Actually, Daley (1988) proposed a slightly different approach from that of Dikii (1965) and Phillips (1990) in separating the SNH model into the HSE and VSE systems in such a way that $\sigma$ does not appear in the VSE system. [Recently, Thuburn et al. (2002a) discussed the same approach as Daley's in connection with their investigation on the normal modes of the DNH model. See the next subsection.] One drawback in doing so is that the HSE system no longer takes the form of Laplace tidal equations. Thus, the relationship between the normal modes of the HPE and SNH models becomes somewhat obscure. In fact, Kasahara and Qian (2000) have demonstrated that the coupled eigenvalue problem of
the SNH model, as formulated by Dikii (1965) and Phillips (1990), can be solved efficiently by devising an iterative method to find the values of $h_e$ and $\sigma$ in the HSE and VSE systems. We will come back to discuss this matter further in Section 7.

In the HPE model, $h_e$ depends only on the vertical scale of motion, while in the SNH model $h_e$ depends not only on the vertical scale of motion, but also on the horizontal scales in both longitude and latitude. Thus, values of $h_e$ for the three kinds (inertio-gravity, planetary, and acoustic) of normal modes are all different for a particular combination of vertical and horizontal scales of motion.

To facilitate understanding of the SNH global normal modes, Qian and Kasahara (2003) investigated the normal modes of SNH model using Cartesian coordinates on midlatitude and equatorial beta-planes and discussed the correspondence of normal modes between spherical and beta-plane configurations.

2.3 Deep nonhydrostatic (DNH) model

This is the case of DNH model described in Section 1. Only a few studies have ever been made on the normal modes of this model in spherical geometry and our understanding on this problem is far from complete. This situation is partly due to the lack of immediate urgency to investigate this problem, but also due to mathematical difficulties in getting the normal mode solutions. To indicate how complex it is to analyze the normal modes of this model, we cite a few past studies. One is by Jones (1971 a,b) who developed a general theory of oscillations of a deep atmosphere for application to tidal oscillations of the combined atmospheric and ionospheric system including the complete form of Coriolis force. Needler and LeBlond (1973) examined the oscillations of a stratified and incompressible ocean model in spherical geometry including the horizontal component of the earth's rotation. However, they dealt primarily with long period waves on which the influence of the $f_H$-terms is found to be small. Also, Unno et al. (1989) discussed several studies of mathematical analyses of the oscillations of rotating stars with and
without the traditional approximations referred to in the Introduction. Recently, Thuburn et al. (2002a) investigated the normal modes of the DNH model in spherical geometry and calculated the eigenfunctions and wave frequencies by using a finite-difference method. We will come back to the discussion of their work later.

A mathematical difficulty in obtaining the normal modes of the DNH model arises from the fact that the linearized form of the basic equations is not separable into a simultaneous system of VSEs and HSEs unlike the SNH model. This non-separability, however, applies to the case of spherical geometry. In the case of tangent-plane geometry without consideration of the meridional variations of Coriolis parameters $f_V$ and $f_H$, we can transform the system of basic linearized equations into the system of VSEs, corresponding to plane wave solutions in the horizontal direction, as shown earlier by Eckart (1960), who discussed only the solutions of Lamb waves as the external mode. Recently, Thuburn et al. (2002b) and Kasahara (2003a,b) investigated the normal mode solutions of the tangent-plane geometry DNH model in details.

In the spherical version of the DNH model, because of the presence of $f_H$-terms, the normal mode problem is no longer separable. Thus, it must be solved as a two-dimensional eigenvalue problem in the vertical and meridional directions as done, for example, by using a finite-difference method by Thuburn et al. (2002a).

In what follows, we will attempt to solve the DNH normal mode problem in a close association with the traditional approach to solve the normal mode problems of HPE and SNH models as initiated by Hough (1898) and Taylor (1936). Such an approach is desirable in order to interpret the solutions of a nonseparable eigenvalue problem by a direct numerical method.
3. Linearized equations of deep nonhydrostatic (DNH) global model

Here we describe a linearized version of the system of DNH model, expressed by (1.1) - (1.3) for the momentum, (1.6) for the mass continuity, and (1.7) for the law of thermodynamics with the equation of state (1.8). The basic state on which perturbation motions are superimposed are assumed to be at rest with temperature $T_o(r)$, pressure $p_o(r)$, and density $\rho_o(r)$, in hydrostatic equilibrium $dp_o/dr = -\rho_o g$, where $g$ denotes the gravitational acceleration. And, $T_o$ is defined through the equation of state, $p_o = \rho_o R T_o$. The subscript zero referred to the basic state quantities.

The perturbation velocity components are denoted by $(u', v', w')$ in $(\lambda, \phi, r)$ and the perturbation pressure and density are denoted by $p'$ and $\rho'$, respectively. However, in writing the basic linearized equations, we use a new variable $q'$ instead of $\rho'$ through the relation

$$ q' = \frac{g}{C^2} p' - g \rho' \quad (3.1) $$

The variable $q'$ is related to perturbation of the logarithm of potential temperature (Gill 1982) and the use of $q'$ helps the derivation of perturbation energy equation easier (Eckart 1960).

In (3.1), $C$ is defined by

$$ C = \left( \frac{RT_o}{1 - \kappa} \right)^{\frac{1}{2}} \quad (3.2) $$

and denotes the speed of sound in the basic state. Here, $\kappa = R/C_p$, with $C_p$ being the specific heat at constant pressure.

We now follow the procedure of Eckart (1960, p.52) and introduce the "field" variables $(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{q})$ defined by

$$(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{q}) = (\rho_{\frac{1}{3}} u', \rho_{\frac{1}{3}} v', \rho_{\frac{1}{3}} w', \rho_{\frac{1}{3}} p', \rho_{\frac{1}{3}} q'). \quad (3.3)$$

In terms of these new variables, the linearized equations of the DNH model are expressed by

$$ \frac{\partial \hat{u}}{\partial t} - 2\Omega \sin \phi \, \hat{v} + 2\Omega \cos \phi \, \hat{w} + \frac{1}{r \cos \phi} \frac{\partial \hat{p}}{\partial \lambda} = 0, \quad (3.4) $$
\[ \frac{\partial \phi}{\partial t} + 2\Omega \sin \phi \dot{u} + \frac{2}{r} \frac{\partial \phi}{\partial \phi} = 0, \quad (3.5) \]
\[ \frac{\partial \psi}{\partial t} - 2\Omega \cos \phi \dot{u} - \dot{q} + \frac{\partial \phi}{\partial r} + \Gamma \dot{v} = 0, \quad (3.6) \]
\[ \frac{1}{C^2} \frac{\partial \theta}{\partial t} + \frac{2}{r \cos \phi} \left[ \frac{\partial \dot{u}}{\partial \lambda} + \frac{\partial}{\partial \phi} (\dot{\phi} \cos \phi) \right] + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \dot{r} \right) - \Gamma \psi = 0, \quad (3.7) \]
\[ \frac{1}{N^2} \frac{\partial \psi}{\partial t} + \ddot{\psi} = 0, \quad (3.8) \]

where
\[ N^2 = -g \left( \frac{1}{\rho_o} \frac{d \rho_o}{dr} + \frac{g}{C^2} \right) \quad (3.9) \]
and
\[ \Gamma = \frac{1}{2\rho_o} \frac{d \rho_o}{dr} + \frac{g}{C^2}. \quad (3.10) \]

Here, \( N \) denotes the Brunt-Väisälä frequency. The quantity \( \Gamma \) is the third important parameter of the oscillating system and is referred to as Eckart parameter by Gossard and Hooke (1975).

The system of equations (3.4) - (3.8) can be expressed compactly as
\[ \frac{\partial \mathbf{M}}{\partial t} + \mathbf{L} \cdot \mathbf{M} = 0 \quad (3.11) \]

by defining the vector field variables \( \mathbf{M} \) in the form
\[ \mathbf{M} = \left[ \dot{u}, \dot{v}, \dot{\psi}, \frac{\dot{\phi}}{C}, \frac{\dot{\psi}}{N} \right]^T \quad (3.12) \]

and the operator \( \mathbf{L} \) in the form
\[ \mathbf{L} = \begin{pmatrix}
0 & -2\Omega \sin \phi & 2\Omega \cos \phi & \frac{C}{r \cos \phi} \frac{\partial}{\partial \lambda} & 0 \\
2\Omega \sin \phi & 0 & 0 & \frac{C}{r \cos \phi} \frac{\partial}{\partial \phi} & 0 \\
-2\Omega \cos \phi & 0 & 0 & C \left( \frac{\partial}{\partial r} + \Gamma \right) & -N \\
\frac{C}{r \cos \phi} \frac{\partial}{\partial \lambda} & \frac{C}{r \cos \phi} \frac{\partial}{\partial \phi} \left[ (\cos \phi) \right] & C \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left[ (\cos \phi)^2 \right] - \Gamma \right\} & 0 & 0 \\
0 & 0 & 0 & N & 0 & 0
\end{pmatrix} \quad (3.13) \]
Our task is to find the solution $M$ of Eq. (3.11) under the following boundary conditions to conserve the total perturbation energy of the system in the three-dimensional shell space:

$$
\lambda : 0 \text{ to } 2\pi, \text{ periodic,} \quad (3.14a)
$$

$$
\phi : \hat{\phi} = 0 \text{ at } \phi = \frac{\pi}{2} \text{ and } -\frac{\pi}{2}, \quad (3.14b)
$$

$$
\rho : \hat{\rho} = 0 \text{ at } r = a \text{ and } r^2 \hat{\rho} = 0 \text{ at } r = r_T, \quad (3.14c)
$$

where $a$ denotes the radius of the bottom and $r_T$ denotes the top of the shell.

By applying the scalar multiplication of (3.11) by $M^*$, where the asterisk (*) denotes complex conjugate, and integrating the product with respect to $\lambda, \phi$, and $r$ in the three-dimensional shell space with the boundary conditions (3.14), we obtain the conservation equation of the total energy in the form

$$
\frac{\partial}{\partial t} \int_{-\pi/2}^{\pi/2} \int_{a}^{r_T} \int (TE) r^2 \cos \phi \ dr \ d\phi \ d\lambda = 0, \quad (3.15)
$$

where $TE$ denotes the energy density

$$
TE = \frac{1}{2} \left( |\hat{u}|^2 + |\hat{\phi}|^2 + |\hat{w}|^2 + \frac{1}{C^2} |\hat{p}|^2 + \frac{1}{N^2} |\hat{q}|^2 \right). \quad (3.16)
$$
4. Alternate form of the DNH basic equations

The form of the total energy equation (3.15) suggests that it is more convenient to rewrite the basic equations (3.4) - (3.8) by introducing the new variables

\[ (U,V,W,P,Q) = \left( r\hat{u}, r\hat{v}, r\hat{w}, \frac{r\hat{P}}{C}, \frac{r\hat{q}}{N} \right). \]  (4.1)

The result is the following system:

\[
\frac{\partial U}{\partial t} - 2\Omega \sin \phi V + 2\Omega \cos \phi W + \frac{C}{r \cos \phi} \frac{\partial P}{\partial \lambda} = 0, \quad (4.2)
\]

\[
\frac{\partial V}{\partial t} + 2\Omega \sin \phi U + \frac{C}{r} \frac{\partial P}{\partial \phi} = 0, \quad (4.3)
\]

\[
\frac{\partial W}{\partial t} - 2\Omega \cos \phi U - NQ + C \left[ r \frac{\partial}{\partial r} \left( \frac{P}{r} \right) + \Gamma P \right] = 0, \quad (4.4)
\]

\[
\frac{\partial P}{\partial t} + \frac{C}{r \cos \phi} \left[ \frac{\partial U}{\partial \lambda} + \frac{\partial}{\partial \phi} (V \cos \phi) \right] + C \left[ \frac{1}{r} \frac{\partial}{\partial r} (rW) - \Gamma W \right] = 0, \quad (4.5)
\]

\[
\frac{\partial Q}{\partial t} + NW = 0. \quad (4.6)
\]

Because \( U \) and \( V \) are vector components and become singular at the poles, it is convenient to introduce the velocity potential \( \Phi \) and the stream function \( \Psi \) through the relationships

\[
U = \frac{1}{\cos \phi} \frac{\partial \Phi}{\partial \lambda} - \frac{\partial \Psi}{\partial \phi}, \quad (4.7)
\]

\[
V = \frac{\partial \Phi}{\partial \phi} + \frac{1}{\cos \phi} \frac{\partial \Psi}{\partial \lambda}. \quad (4.8)
\]

Hence,

\[
\nabla^2 \Phi = \frac{1}{\cos \phi} \left[ \frac{\partial U}{\partial \lambda} + \frac{\partial}{\partial \phi} (V \cos \phi) \right], \quad (4.9)
\]

\[
\nabla^2 \Psi = \frac{1}{\cos \phi} \left[ \frac{\partial V}{\partial \lambda} - \frac{\partial}{\partial \phi} (U \cos \phi) \right], \quad (4.10)
\]
where

\[ \nabla^2 = \frac{1}{\cos \phi} \left[ \frac{1}{\cos \phi} \frac{\partial^2}{\partial \lambda^2} + \frac{\partial}{\partial \phi} \left( \cos \phi \frac{\partial}{\partial \phi} \right) \right]. \]  \hspace{1cm} (4.11)

With the use of cross-differentiation between (4.2) and (4.3) and by replacing \( U \) and \( V \) through (4.7) and (4.8), we obtain the following alternate system

\[ \left( \frac{\partial}{\partial t} \nabla^2 + 2\Omega \frac{\partial}{\partial \lambda} \right) \Phi - 2\Omega \left( \sin \phi \nabla^2 + \cos \phi \frac{\partial}{\partial \phi} \right) \Psi \]
\[ + 2\Omega \frac{\partial W}{\partial \lambda} + \frac{C}{r} \nabla^2 P = 0, \] \hspace{1cm} (4.12)

\[ \left( \frac{\partial}{\partial t} \nabla^2 + 2\Omega \frac{\partial}{\partial \lambda} \right) \Psi + 2\Omega \left( \sin \phi \nabla^2 + \cos \phi \frac{\partial}{\partial \phi} \right) \Phi \]
\[ + 2\Omega \left( 2\sin \phi - \cos \phi \frac{\partial}{\partial \phi} \right) W = 0, \] \hspace{1cm} (4.13)

\[ \frac{\partial W}{\partial t} - 2\Omega \frac{\partial \Phi}{\partial \lambda} + 2\Omega \cos \phi \frac{\partial \Psi}{\partial \phi} \]
\[ - NQ + C \left[ r \frac{\partial}{\partial r} \left( \frac{P}{r} \right) + \Gamma P \right] = 0, \] \hspace{1cm} (4.14)

\[ \frac{\partial P}{\partial t} + \frac{C}{r} \nabla^2 \Phi + C \left[ \frac{1}{r} \frac{\partial}{\partial r} (rW) - \Gamma W \right] = 0, \] \hspace{1cm} (4.15)

\[ \frac{\partial Q}{\partial t} + NW = 0. \] \hspace{1cm} (4.16)

The fifth equation above is unchanged from (4.6), but we included it in the above system for completeness.
5. Spectral form of the DNH equations

We seek the solutions of (4.12) - (4.16) in the form

\[
\begin{pmatrix}
\Phi \\
\Psi \\
W \\
\it P \\
\it Q
\end{pmatrix} =
\begin{pmatrix}
\dot{\Phi}(\phi,r) \\
\dot{\Psi}(\phi,r) \\
\dot{W}(\phi,r) \\
\dot{P}(\phi,r) \\
\dot{Q}(\phi,r)
\end{pmatrix} e^{i(s\lambda-\sigma t)}, \tag{5.1}
\]

where \( s \) denotes the zonal wavenumber, \( \sigma \) denotes the frequency, and \( i = \sqrt{-1} \). The frequency of this system can be shown to be real from the property of the operator \( L \) of (3.13) with the boundary conditions (3.14).

We also introduce the following operators and symbols:

\[
\begin{align*}
\mu &= \sin \phi, \\
\mathcal{L} &= \cos \phi \frac{\partial}{\partial \phi} = (1 - \mu^2) \frac{\partial}{\partial \mu}, \\
\nabla_s^2 &= \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial}{\partial \mu}\right] - \frac{s^2}{1 - \mu^2}, \\
\nu &= \frac{\sigma}{2\Omega} \quad \text{(dimensionless frequency)}, \\
\bar{r} &= \frac{r}{a} \quad \text{(dimensionless radial distance)}, \\
\bar{C} &= \frac{C}{2a\Omega} \quad \text{(dimensionless sound speed)}, \\
\bar{\Gamma} &= a\Gamma \quad \text{(dimensionless Eckart parameter)}, \\
\bar{N} &= \frac{N}{2\Omega} \quad \text{(dimensionless Brunt – Vaisala frequency)}. \\
\end{align*}
\]

Using (5.1) and (5.2), we rewrite Eqs. (4.12) - (4.16) as follows:

\[
(\nu \nabla_s^2 - s)(i\dot{\Phi}) + (\mu \nabla_s^2 + \mathcal{L})\dot{\Psi} - is\dot{W} - \bar{C} \frac{1}{\bar{r}} \nabla_s^2 \dot{P} = 0, \tag{5.3}
\]

\[
(\nu \nabla_s^2 - s)\ddot{\Phi} + (\mu \nabla_s^2 + \mathcal{L})(i\ddot{\Psi}) + (2\mu - \mathcal{L})(i\ddot{W}) = 0, \tag{5.4}
\]
\[-\nu(i\ddot{W}) - s(i\ddot{\phi}) + L\ddot{\psi} - \ddot{N}\dot{\phi} + \ddot{C} \left[ \ddot{\tau} \frac{\partial}{\partial \ddot{\tau}} \left( \frac{\ddot{P}}{\ddot{\tau}} \right) + \ddot{\Gamma}\ddot{P} \right] = 0, \quad (5.5)\]

\[\nu\ddot{P} + \ddot{C} \frac{1}{\ddot{\tau}} \nabla_s^2 (i\ddot{\phi}) + \ddot{C} \left[ \frac{1}{\ddot{\tau}} \frac{\partial}{\partial \ddot{\tau}} (\ddot{\phi}i\ddot{W}) - \ddot{\Gamma}(i\ddot{W}) \right] = 0, \quad (5.6)\]

\[\nu\dot{Q} + \ddot{N}(i\ddot{W}) = 0. \quad (5.7)\]

Now, we express the variables \(\ddot{\phi}, \ddot{\psi}, \ddot{W}, \ddot{P},\) and \(\dot{Q}\) by the series of the products involving the associated Legendre function \(P^s_n(\mu)\) with order \(n\) and rank \(s\), where \(n \geq s\), in the form

\[
\begin{align*}
\begin{array}{c|c|c|c}
\hat{\phi} & iA^s_n(\ddot{\tau}) \\
\hat{\psi} & B^s_n(\ddot{\tau}) \\
\hat{W} & \sum_{n=s}^{\infty} iC^s_n(\ddot{\tau}) P^s_n(\mu) \\
\dot{P} & D^s_n(\ddot{\tau}) \\
\dot{Q} & E^s_n(\ddot{\tau}) \\
\end{array}
\end{align*}
\]

Note that the coefficients \(A^s_n, B^s_n, \ldots\) are the functions of \(\ddot{\tau}\) and depend on zonal wavenumber \(s\) and meridional index \(n\).

Before proceeding further, it is necessary to state some properties of \(P^s_n(\mu)\) that is normalized by

\[
\int_{-1}^{1} P^s_n(\mu) P^s_{n'}(\mu) d\mu = \begin{cases} 
0 & \text{for } n \neq n' \\
\frac{2}{2n+1} \frac{(n+s)!}{(n-s)!} & \text{for } n = n'.
\end{cases} \quad (5.9)
\]

They are:

\[
\nabla_s^2 P^s_n = -n(n+1)P^s_n \quad \text{for } n \geq s > 0,
\]

\[
\mu P^s_n = \frac{n+s}{2n+1} P^s_{n-1} + \frac{n-s+1}{2n+1} P^s_{n+1},
\]

\[
L P^s_n = \frac{(n+1)(n+s)}{2n+1} P^s_{n-1} - \frac{n(n-s+1)}{2n+1} P^s_{n+1}, \quad (5.10)
\]

\[
\mu \nabla_s^2 P^s_n = -n(n+1)\mu P^s_n = -n(n+1)(n+s) P^s_{n-1} - \frac{n(n+1)(n-s+1)}{2n+1} P^s_{n+1}.
\]

Therefore,

\[
(\mu \nabla_s^2 + L) P^s_n = \frac{-(n-1)(n+1)(n+s)}{2n+1} P^s_{n-1} - \frac{n(n+2)(n-s+1)}{2n+1} P^s_{n+1}, \quad (5.11)
\]
\[(2\mu - \mathcal{L})P_n^s = \frac{-(n-1)(n+s)}{2n+1}P_{n-1}^s + \frac{(n+2)(n-s+1)}{2n+1}P_{n+1}^s. \quad (5.12)\]

By substituting the series representation of \((5.8)\), using the relationships \((5.10) - (5.12)\), and collecting the coefficients of \(P_n^s\), we can find the following equations that express relationships among the coefficients, \(A_n^s, B_n^s, C_n^s, D_n^s,\) and \(E_n^s\) which are functions of radial distance \(\tilde{r}\):

\[\left[\nu(n+1) + s\right]A_n^s - \frac{n(n+2)(n+s+1)}{2n+3}B_{n+1}^s\]
\[- \frac{(n-1)(n+1)(n-s)}{2n-1}B_{n-1}^s\]
\[+ s C_n^s + \frac{\tilde{C}}{\tilde{r}} n(n+1)D_n^s = 0, \quad (5.13)\]

\[-\left[\nu(n+1) + s\right]B_n^s + \frac{n(n+2)(n+s+1)}{2n+3}A_{n+1}^s\]
\[+ \frac{(n-1)(n+1)(n-s)}{2n-1}A_{n-1}^s\]
\[+ \frac{n(n+s+1)}{2n+3}C_{n+1}^s\]
\[- \frac{(n+1)(n-s)}{2n-1}C_{n-1}^s = 0, \quad (5.14)\]

\[\nu C_n^s + sA_n^s + \frac{(n+2)(n+s+1)}{2n+3}B_{n+1}^s\]
\[- \frac{(n-1)(n-s)}{2n-1}B_{n-1}^s\]
\[-\tilde{N}E_n^s + \tilde{C}\left[\frac{d}{\tilde{r}} \frac{D_n^s}{\tilde{r}}\right] + \tilde{\Gamma}D_n^s\] \[= 0, \quad (5.15)\]

\[\nu D_n^s + \frac{\tilde{C}}{\tilde{r}} n(n+1)A_n^s - \tilde{C}\left[\frac{1}{\tilde{r}} \frac{d}{d\tilde{r}}(\tilde{r}C_n^s) - \tilde{\Gamma}C_n^s\right] = 0, \quad (5.16)\]
\[ \nu E_n^s - \tilde{N} C_n^s = 0. \] (5.17)

To simplify the forms of Eqs. (5.13) - (5.17), we introduce the functional expressions:

\[ K_n = \frac{-s}{n(n+1)}, \] (5.18)

\[ p_n = \frac{(n+1)(n+s)}{n(2n+1)}, \] (5.19)

\[ q_n = \frac{n(n-s+1)}{(n+1)(2n+1)}, \] (5.20)

\[ g_n = \frac{(n+s)}{n(2n+1)}, \] (5.21)

\[ h_n = \frac{-(n-s+1)}{(n+1)(2n+1)}, \] (5.22)

\[ x_n = \frac{-(n+1)(n+s)}{2n+1}, \] (5.23)

\[ y_n = \frac{n(n-s+1)}{2n+1}, \] (5.24)

\[ z_n = -n(n+1). \] (5.25)

Using the definitions (5.18) - (5.25), Eqs. (5.13) - (5.17) are written in the form:

\[ (-\nu + K_n) A_n^s (\bar{r}) + p_{n+1} B_{n+1}^s (\bar{r}) + q_{n-1} B_{n-1}^s (\bar{r}) \]
\[ + K_n C_n^s (\bar{r}) - \frac{C}{\bar{r}} D_n^s (\bar{r}) = 0, \] (5.26)

\[ (-\nu + K_n) B_n^s (\bar{r}) + p_{n+1} A_{n+1}^s (\bar{r}) + q_{n-1} A_{n-1}^s (\bar{r}) \]
\[ + g_{n+1} C_{n+1}^s (\bar{r}) + h_{n-1} C_{n-1}^s (\bar{r}) = 0, \] (5.27)

\[ -\nu C_n^s (\bar{r}) - s A_n^s (\bar{r}) + x_{n+1} B_{n+1}^s (\bar{r}) + y_{n-1} B_{n-1}^s (\bar{r}) \]
\[ + \tilde{N} E_n^s (\bar{r}) - \tilde{C} \left[ \bar{r} \frac{d}{d\bar{r}} \left( \frac{D_n^s (\bar{r})}{\bar{r}} \right) \right] + \bar{\Gamma} D_n^s (\bar{r}) = 0, \] (5.28)
\[-\nu D_n^s(\tilde{r}) + \tilde{C} \frac{C_n}{\tilde{r}} A_n^s(\tilde{r}) \]
\[+ \tilde{C} \left[ \frac{1}{\tilde{r}} \frac{d}{d\tilde{r}} \left( \tilde{r} C_n^s(\tilde{r}) \right) - \tilde{r} C_n^s(\tilde{r}) \right] = 0, \tag{5.29} \]
\[-\nu E_n^s(\tilde{r}) + \tilde{N} C_n^s(\tilde{r}) = 0. \tag{5.30} \]

Eqs. (5.26) - (5.30) constitute a system of simultaneous homogeneous equations of the expansion coefficients $A_n^s, B_n^s, C_n^s, D_n^s$ and $E_n^s$, which are all functions of the radial distance $\tilde{r}$ for particular values of $\nu$ that are obtained as the eigenvalues of this system. The expansion coefficients are determined as the eigenfunctions of $\tilde{r}$ from this system.

The system (5.26) - (5.30) possesses two independent solutions. One consists of $A_n^s, C_n^s, D_n^s$, and $E_n^s$ for $n = s, s + 2, \ldots$, and $B_n^s$ for $n = s + 1, s + 3, \ldots$. In this case, the velocity potential, vertical velocity, pressure, and log-potential temperature are symmetric relative to the equator and the stream function is antisymmetric.

The other consists of $A_n^s, C_n^s, D_n^s$, and $E_n^s$ for $n = s + 1, s + 2, \ldots$, and $B_n^s$ for $n = s, s + 2, \ldots$. In this case, the velocity potential, vertical velocity, pressure, and log-potential temperature are antisymmetric relative to the equator and the stream function is symmetric.

The system of differential equations with respect to the radial distance $\tilde{r}$ is solved with the boundary conditions that the vertical velocity vanishes at the bottom ($r = a$) and the top ($r = r_T$). This means that

\[C_n^s = E_n^s = 0 \text{ at } \tilde{r} = 1 \text{ and } \tilde{r}_T, \tag{5.31} \]

where $\tilde{r}_T = r_T/a$. 

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6. Numerical procedure to solve the eigenvalue problem of (5.26) - (5.30)

6.1 Vertical discretization

In order to solve the system of differential equations with respect to $\tilde{\tau}$, we use the finite-difference method. Figure 1 shows the layout of vertical discretization of variables. We place $\hat{W}$ and $\hat{Q}$, i.e., $C_n^s$ and $E_n^s$ at the integer index levels, and $\hat{\Phi}$, $\hat{\Psi}$, and $\hat{P}$, i.e., $A_n^s$, $B_n^s$, and $D_n^s$ at the half-integer index levels.

Before writing down the specific difference equations of (5.26) - (5.30), we should mention that there are three dimensionless parameters $\tilde{C}$, $\tilde{\Gamma}$, and $\tilde{N}$ defined in (5.2) which are functions of $\tilde{\tau}$. The difference system that we will describe in this section can be made to handle the case of these parameters being functions of $\tilde{\tau}$. However, for the sake of simplification hereafter we will consider the case of an isothermal basic state atmosphere. In this case, these parameters are expressed by

\[
\tilde{C} = \frac{1}{2a\Omega} \left[ \frac{RT_o}{1 - \kappa} \right]^{\frac{1}{2}},
\tag{6.1}
\]

\[
\tilde{N} = \frac{1}{2\Omega} \left[ \frac{\kappa g^2}{RT_o} \right]^{\frac{1}{2}},
\tag{6.2}
\]

\[
\tilde{\Gamma} = \frac{ag(1 - 2\kappa)}{2RT_o},
\tag{6.3}
\]

where $T_o$ is a constant temperature. If we further assume that the acceleration due to gravity is treated as constant, then all three parameters can be treated as constant.

6.2 Vertical structure equations in difference form for symmetric modes

The eigenvector $X$ of the symmetric system is defined by

\[
X = \text{col} \left( A_s^\frac{1}{2}, B_{s+1}^\frac{1}{2}, D_s^\frac{1}{2}, C_s^1, E_s^1, \right). 
\]
\[
A_s^s\left(\ell + \frac{1}{2}\right), B_{s+1}^s\left(\ell + \frac{1}{2}\right), D_s^s\left(\ell + \frac{1}{2}\right), C_s^s(\ell + 1), E_s^s(\ell + 1),
\]

\[
A_s^s\left(L - \frac{1}{2}\right), B_{s+1}^s\left(L - \frac{1}{2}\right), D_s^s\left(L - \frac{1}{2}\right),
\]

\[
A_{s+2}^s\left(\frac{1}{2}\right), B_{s+3}^s\left(\frac{1}{2}\right), D_{s+2}^s\left(\frac{1}{2}\right), C_{s+2}^s(1), E_{s+2}^s(1),
\]

\[
A_{s+2}^s\left(\ell + \frac{1}{2}\right), B_{s+3}^s\left(\ell + \frac{1}{2}\right), D_{s+2}^s\left(\ell + \frac{1}{2}\right), C_{s+2}^s(\ell + 1), E_{s+2}^s(\ell + 1),
\]

\[
A_{s+2}^s\left(L - \frac{1}{2}\right), B_{s+3}^s\left(L - \frac{1}{2}\right), D_{s+2}^s\left(L - \frac{1}{2}\right),
\]

\[
A_{s+4}^s\left(\frac{1}{2}\right), B_{s+5}^s\left(\frac{1}{2}\right), D_{s+4}^s\left(\frac{1}{2}\right), C_{s+4}^s(1), E_{s+4}^s(1),
\]

where \(A_n^s(\ell + \frac{1}{2})\) and \(C_n^s(\ell + 1)\) denote the value of \(A_n^s\) at a half-integer level (Fig. 1) and that of \(C_n^s\) at an integer level, respectively. The boundary conditions (5.31) are already incorporated, i.e., \(C_n^s(0) = E_n^s(0) = C_n^s(L) = E_n^s(L) = 0\), so that they are not included in (6.4).

The eigenvector \(X\) of (6.4) may be expressed compactly as

\[
X = \text{col} \left( X_1, X_2, X_3, \ldots \right),
\]

where

\[
X_1 = \text{col} \left( A_s^s\left(\frac{1}{2}\right), B_{s+1}^s\left(\frac{1}{2}\right), \ldots, D_s^s\left(L - \frac{1}{2}\right) \right),
\]

\[
X_2 = \text{col} \left( A_{s+2}^s\left(\frac{1}{2}\right), B_{s+3}^s\left(\frac{1}{2}\right), \ldots, D_{s+2}^s\left(L - \frac{1}{2}\right) \right),
\]

and so on.
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<th>Coefficients</th>
<th>Variables</th>
</tr>
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<td>$\hat{W}, \hat{Q}$</td>
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</tbody>
</table>

Figure 1: Layout of the equally-spaced vertical grid for the deep NH model. Index $\ell = 0$ corresponds to $\tilde{r} = 1$ and $\ell = L$ to $\tilde{r} = r_T/a$. 
The difference form of the simultaneous vertical structure equations for the symmetric modes is expressed by the following eigenvalue problem.

\[(A - \nu I)\mathbf{X} = 0, \quad (6.7)\]

where \(I\) is the unit matrix and \(A\) denotes a tri-diagonal block matrix in the form

\[
A = \begin{bmatrix}
A_{11} & A_{12} & 0 & 0 & - & - \\
A_{21} & A_{22} & A_{23} & 0 & - & - \\
0 & A_{32} & A_{33} & A_{34} & - & - \\
- & - & - & - & - & - \\
- & - & - & - & A_{NN-1} & A_{NN}
\end{bmatrix}. \quad (6.8)
\]

Here, the elements of \(A\), denoted by \(A_{ij}\)'s, are rather complicated but sparse matrices that can be constructed from the following centered-difference equations derived from the system (5.26) - (5.30) with \(\Delta \tilde{r} = |\tilde{r}(L) - \tilde{r}(0)|/L\), where \(L\) denotes the number of equally-spaced layers.

\[
(-\nu + K_n) A^s_n \left( \ell + \frac{1}{2} \right) + p_{n+1} B^s_{n+1} \left( \ell + \frac{1}{2} \right) + q_{n-1} B^s_{n-1} \left( \ell + \frac{1}{2} \right) \\
+ K_n \left[ \frac{C^s_n(\ell) + C^s_n(\ell + 1)}{2} \right] - \frac{\tilde{C}}{\tilde{r}(\ell + \frac{1}{2})} D^s_n \left( \ell + \frac{1}{2} \right) = 0, \quad (6.9)
\]

\[
(-\nu + K_{n+1}) B^s_{n+1} \left( \ell + \frac{1}{2} \right) + p_{n+2} A^s_{n+2} \left( \ell + \frac{1}{2} \right) + q_n A^s_n \left( \ell + \frac{1}{2} \right) \\
+ g_{n+2} \left[ \frac{C^s_{n+2}(\ell) + C^s_{n+2}(\ell + 1)}{2} \right] + h_n \left[ \frac{C^s_n(\ell) + C^s_n(\ell + 1)}{2} \right] = 0, \quad (6.10)
\]

\[
- \nu D^s_n \left( \ell + \frac{1}{2} \right) + \frac{\tilde{C} z_n}{\tilde{r}(\ell + \frac{1}{2})} A^s_n \left( \ell + \frac{1}{2} \right) - \tilde{C} \left[ \frac{C^s_n(\ell) + C^s_n(\ell + 1)}{2} \right] \\
+ \frac{\tilde{C}}{\tilde{r}(\ell + \frac{1}{2})} \left[ \frac{\tilde{r}(\ell + 1) C^s_n(\ell + 1) - \tilde{r}(\ell) C^s_n(\ell)}{\Delta \tilde{r}} \right] = 0, \quad (6.11)
\]
\[-\nu C_n^s (\ell + 1) - s \left[ \frac{A_n^s (\ell + \frac{1}{2}) + A_n^s (\ell + \frac{1}{2})}{2} \right] + x_{n+1} \left[ \frac{B_{n+1}^s (\ell + \frac{1}{2}) + B_{n+1}^s (\ell + \frac{1}{2})}{2} \right] \]

\[+ y_{n-1} \left[ \frac{B_{n-1}^s (\ell + \frac{1}{2}) + B_{n-1}^s (\ell + \frac{1}{2})}{2} \right] + \tilde{N} E_n^s (\ell + 1) \]

\[-\tilde{C} \tilde{r}(\ell + 1) \left[ \frac{D_n^s (\ell + \frac{1}{2}) + D_n^s (\ell + \frac{1}{2})}{2} \right] \]

\[-\tilde{C} \frac{\tilde{r}(\ell + 1)}{\Delta \tilde{r}} \left[ \frac{D_n^s (\ell + \frac{1}{2}) - D_n^s (\ell + \frac{1}{2})}{\tilde{r}(\ell + \frac{1}{2})} \right] = 0 , \quad (6.12) \]

\[-\nu E_n^s (\ell + 1) + \tilde{N} C_n^s (\ell + 1) = 0. \quad (6.13) \]

The specific forms of $A_{ij}$'s are not shown here to save space. The subscript $i$ or $j$ refers to the subscript 1, 2, ... and so on as shown in (6.8) to indicate the subscript $n = s, n = s + 2, \text{ and so on for the expansion coefficient vectors.}$ The reason that the matrix $A$ becomes tri-diagonal is that, as seen from (6.9), (6.10), and (6.12), the subscripts of $A_n^s$ and $B_{n+1}^s$ span $n - 1, n, \text{ and } n + 1.$

The order of the square sub-matrix $A_{ij}$ is determined by the number of the vertical layer $L$ (see, Fig. 1) and becomes $3 \times L + 2 \times (L - 1),$ because there are three variables $(A_n^s, B_n^s, D_n^s)$ at a half-integer level and two variables $(C_n^s, E_n^s)$ at an integer level by considering the fact that $C_n^s$ and $E_n^s$ vanish at the top and bottom of the system.

By choosing the maximum number of the meridional modes to be $N_m,$ the order of the matrix $A$ becomes $N_m \times [3 \times L + 2 \times (L - 1)].$

The frequency $\nu$ and the expansion coefficient vector $X$ are determined by solving the eigenvalue problem (6.7).
6.3 Vertical structure equations in difference form for antisymmetric modes

The eigenvector $Y$ of the antisymmetric system is defined by

$$Y = \text{col} \left( B_s^s \left( \frac{1}{2} \right), A_{s+1}^s \left( \frac{1}{2} \right), D_{s+1}^s \left( \frac{1}{2} \right), C_{s+1}^s (1), E_{s+1}^s (1), \right)$$

$$B_s^s \left( \ell + \frac{1}{2} \right), A_{s+1}^s \left( \ell + \frac{1}{2} \right), D_{s+1}^s \left( \ell + \frac{1}{2} \right), C_{s+1}^s (\ell + 1), E_{s+1}^s (\ell + 1),$$

$$B_s^s \left( L - \frac{1}{2} \right), A_{s+1}^s \left( L - \frac{1}{2} \right), D_{s+1}^s \left( L - \frac{1}{2} \right),$$

$$B_{s+2}^s \left( \frac{1}{2} \right), A_{s+3}^s \left( \frac{1}{2} \right), D_{s+3}^s \left( \frac{1}{2} \right), C_{s+3}^s (1), E_{s+3}^s (1),$$

$$B_{s+2}^s \left( \ell + \frac{1}{2} \right), A_{s+3}^s \left( \ell + \frac{1}{2} \right), D_{s+3}^s \left( \ell + \frac{1}{2} \right), C_{s+3}^s (\ell + 1), E_{s+3}^s (\ell + 1),$$

$$B_{s+2}^s \left( L - \frac{1}{2} \right), A_{s+3}^s \left( L - \frac{1}{2} \right), D_{s+3}^s \left( L - \frac{1}{2} \right),$$

$$B_{s+4}^s \left( \frac{1}{2} \right), A_{s+5}^s \left( \frac{1}{2} \right), D_{s+5}^s \left( \frac{1}{2} \right), C_{s+5}^s (1), E_{s+5}^s (1),$$

using the same notation used in (6.4).

The eigenvector $Y$ of (6.14) may be expressed compactly as

$$Y = \text{col}(Y_1, Y_2, Y_3, \ldots)$$

where

$$Y_1 = \text{col} \left( B_s^s \left( \frac{1}{2} \right), A_{s+1}^s \left( \frac{1}{2} \right), \ldots, D_{s+1}^s \left( L - \frac{1}{2} \right) \right),$$

$$Y_2 = \text{col} \left( B_{s+2}^s \left( \frac{1}{2} \right), A_{s+3}^s \left( \frac{1}{2} \right), \ldots, D_{s+3}^s \left( L - \frac{1}{2} \right) \right),$$

(6.16)
and so on.

The difference form of the simultaneous vertical structure equations for the antisymmetric modes is expressed by the following eigenvalue problem.

\[(B - \nu I)Y = 0, \quad (6.17)\]

where \(B\) denotes a tri-diagonal block matrix in the form

\[
B = \begin{bmatrix}
B_{11} & B_{12} & 0 & 0 & - & - \\
B_{21} & B_{22} & B_{23} & 0 & - & - \\
0 & B_{32} & B_{33} & B_{34} & - & - \\
- & - & - & - & - & - \\
- & - & - & - & B_{NN-1} & B_{NN}
\end{bmatrix}. \quad (6.18)
\]

Again, the elements of \(B\), denoted by \(B_{ij}\)'s, are complicated but sparse matrices that can be constructed from the following difference equations derived from (5.26) - (5.30).

\[
(-\nu + K_n)B_n^s(\ell + \frac{1}{2}) + p_{n+1} A_{n+1}^s(\ell + \frac{1}{2}) + q_{n-1} A_{n-1}^s(\ell + \frac{1}{2})
+ g_{n+1} \left[ \frac{C_{n+1}^s(\ell) + C_{n+1}^s(\ell + 1)}{2} \right] + h_{n-1} \left[ \frac{C_{n-1}^s(\ell) + C_{n-1}^s(\ell + 1)}{2} \right] = 0, \quad (6.19)
\]

\[
(-\nu + K_{n+1}) A_{n+1}^s \left( \ell + \frac{1}{2} \right) + p_{n+2} B_{n+2}^s \left( \ell + \frac{1}{2} \right) + q_n B_n^s \left( \ell + \frac{1}{2} \right)
+ K_{n+1} \left[ \frac{C_{n+1}^s(\ell) + C_{n+1}^s(\ell + 1)}{2} \right] - \frac{\bar{C}}{\bar{\tau}(\ell + \frac{1}{2})} D_n^s \left( \ell + \frac{1}{2} \right) = 0, \quad (6.20)
\]

\[
-\nu D_{n+1}^s \left( \ell + \frac{1}{2} \right) + \frac{\bar{C} z_{n+1}}{\bar{\tau}(\ell + \frac{1}{2})} A_{n+1}^s \left( \ell + \frac{1}{2} \right) - \frac{\bar{C}}{\bar{\tau}(\ell + \frac{1}{2})} \left[ \frac{C_{n+1}^s(\ell) + C_{n+1}^s(\ell + 1)}{2} \right]
+ \frac{\bar{C}}{\bar{\tau}(\ell + \frac{1}{2})} \left[ \frac{\bar{\tau}(\ell + 1) C_{n+1}^s(\ell + 1) - \bar{\tau}(\ell) C_{n+1}^s(\ell)}{\Delta \bar{\tau}} \right] = 0, \quad (6.21)
\]
\[-\nu C_{n+1}^s(\ell + 1) - s \left[ \frac{A_{n+1}^s(\ell + \frac{1}{2}) + A_{n+1}^s(\ell + 1\frac{1}{2})}{2} \right] + \tilde{N} E_{n+1}^s(\ell + 1) \]

\[+ x_{n+2} \left[ \frac{B_{n+2}^s(\ell + \frac{1}{2}) + B_{n+2}^s(\ell + 1\frac{1}{2})}{2} \right] + y_n \left[ \frac{B_n^s(\ell + \frac{1}{2}) + B_n^s(\ell + 1\frac{1}{2})}{2} \right] \]

\[-\tilde{C} \tilde{\Gamma} \left[ \frac{D_{n+1}^s(\ell + \frac{1}{2}) + D_{n+1}^s(\ell + 1\frac{1}{2})}{2} \right] \]

\[-\frac{\tilde{C} \tilde{\Gamma}(\ell + 1)}{\Delta \tilde{r}} \left[ \frac{D_{n+1}^s(\ell + 1\frac{1}{2})}{\tilde{r}(\ell + 1\frac{1}{2})} - \frac{D_{n+1}^s(\ell + \frac{1}{2})}{\tilde{r}(\ell + \frac{1}{2})} \right] = 0, \quad (6.22)\]

\[-\nu E_{n+1}^s(\ell + 1) + \tilde{N} C_{n+1}^s(\ell + 1) = 0. \quad (6.23)\]

Again, the specific forms of \(B_{ij}\)'s are not shown here to save space. The subscript \(i\) or \(j\) refers to the subscript 1, 2, \ldots and so on as shown in (6.18) to indicate the subscript \(n = s, n = s + 2\) and so on for the expansion coefficient vectors. The order of the matrix \(B\) is the same as that of \(A\).

The eigenvalue problems (6.7) and (6.17) are solved by the eigenvalue routines ORTHES, ORTRAN, and HQR2 in the EISPACK to obtain the frequencies \(\nu\) as the eigenvalues and the expansion coefficient vectors \(X\) and \(Y\) as the eigenvectors. It can be shown that the eigenvalues are all real. There are as many eigenvalues as the order of matrices \(A\) and \(B\). And, the classification of frequencies in terms of the spices of the eigenmodes is not straightforward unless we have knowledge to identify them. Thus, it is helpful to treat a problem simpler than this one that has known solutions using essentially the same algorithms discussed here. For this purpose, we now treat the normal mode problem of shallow nonhydrostatic (SNH) model that has known solutions discussed by Kasahara and Qian (2000) who adopted a semi-analytical method to solve.
7. Normal modes of shallow nonhydrostatic (SNH) model

7.1 Basic equations

The basic equations of shallow nonhydrostatic (SNH) model can be reduced from the system (5.3) - (5.7) by the following procedure. We replace the scaled radial distance \( \tilde{r} \) by unity. Then, we neglect the Coriolis terms involving the vertical velocity \( \tilde{W} \) in (5.3) and (5.4), and the horizontal motions \( \tilde{\Phi} \) and \( \tilde{\Psi} \) in (5.5). Note that the vertical acceleration term in (5.5) is retained as a nonhydrostatic system.

Thus, we get the following system:

\[
(\nu \nabla^2_s - s)(i\tilde{\Phi}) + (\mu \nabla^2_s + \mathcal{L})(\tilde{\Psi} - \tilde{C} \nabla^2_s \tilde{P}) = 0, \quad (7.1)
\]

\[
(\nu \nabla^2_s - s)\tilde{\Psi} + (\mu \nabla^2_s + \mathcal{L})(i\tilde{\Phi}) = 0, \quad (7.2)
\]

\[
-\nu(i\tilde{W}) - \tilde{N} \tilde{Q} + \tilde{C} \left[ \frac{\partial \tilde{P}}{\partial \tilde{Z}} + \tilde{\Gamma} \tilde{P} \right] = 0, \quad (7.3)
\]

\[
\nu \tilde{P} + \tilde{C} \nabla^2_s (i\tilde{\Phi}) + \tilde{C} \left[ \frac{\partial (i\tilde{W})}{\partial \tilde{Z}} - \tilde{\Gamma}(i\tilde{W}) \right] = 0, \quad (7.4)
\]

\[
\nu \tilde{Q} + \tilde{N}(i\tilde{W}) = 0. \quad (7.5)
\]

Note that we use the same symbols for the dependent variables, but the radial increment \( d\tilde{r} \) is replaced by \( d\tilde{Z} \) where \( \tilde{Z} = Z/a \) with \( Z \) denoting the altitude above the earth’s surface. We also treat the case of isothermal basic state.

One merit of this system is that this is a separable problem. We can express the dependent variables as the products of the horizontal and vertical structure functions:

\[
\begin{align*}
\tilde{\Phi} & = i\hat{A}(\phi)\xi(\tilde{Z}) \\
\tilde{\Psi} & = \hat{B}(\phi)\xi(\tilde{Z}) \\
\tilde{W} & = i\hat{D}(\phi)\eta(\tilde{Z}) \\
\tilde{P} & = \hat{D}(\phi)\xi(\tilde{Z}) \\
\tilde{Q} & = \hat{D}(\phi)\Theta(\tilde{Z})
\end{align*}
\]

(7.6)
Moreover, we can assume that
\[ \nabla_s^2 \hat{A} = \frac{\nu}{H_e} \hat{D}, \] (7.7)
where \( H_e \) is the separation constant (dimensionless) and is referred to as the dimensionless equivalent height.

By substituting (7.6) into (7.1) - (7.5) and using (7.7), we find the system (7.1) - (7.5) can be separated into two systems of horizontal and vertical structure equations. The system of vertical structure equations is:
\[
\nu \eta - \tilde{N} \Theta + \tilde{C} \left( \frac{d \xi}{dZ} + \tilde{\Gamma} \xi \right) = 0, \tag{7.8}
\]
\[
\nu \left( 1 - \frac{\tilde{C}}{H_e} \right) \xi - \tilde{C} \left( \frac{d \eta}{dZ} - \tilde{\Gamma} \eta \right) = 0, \tag{7.9}
\]
\[
\nu \Theta - \tilde{N} \eta = 0. \tag{7.10}
\]
The system of horizontal structure equations is
\[
(\nu \nabla_s^2 - s) \hat{A} - (\mu \nabla_s^2 + \mathcal{L}) \hat{B} + \tilde{C} \nabla_s^2 \tilde{D} = 0, \tag{7.11}
\]
\[
(\nu \nabla_s^2 - s) \hat{B} - (\mu \nabla_s^2 + \mathcal{L}) \hat{A} = 0, \tag{7.12}
\]
\[
\nu \tilde{D} - H_e \nabla_s^2 \hat{A} = 0. \tag{7.13}
\]
The system of Eqs. (7.11) - (7.13) has the same form of the horizontal structure equations of the hydrostatic primitive equation (HPE) model as given by Eqs. (4.11) - (4.13) of Kasahara (1976). Thus, the frequency \( \nu \) can be determined, if the equivalent height \( H_e \) is known. However, in the SNH model there are two eigenvalue problems for two unknowns of \( \nu \) and \( H_e \). Therefore, for the SNH model we must solve two simultaneous eigenvalue problems as a coupled system. Kasahara and Qian (2000) formulated an iteration method to solve this coupled system. However, we can determine \( \nu \) in the system (7.1) - (7.5) directly using the difference formulation presented in Section 6. Therefore, we now have two methods of solving the SNH problem and the solutions from the two methods must
agree. Thus, we can use the method presented in this section to check the numerical algorithm of direct approach presented in Section 6.

7.2 Numerical solutions of the vertical structure equations (7.8) - (7.10)

In the case of an isothermal basic state, the system of (7.8) - (7.10) can be solved analytically. However, here we use a vertical discretization as shown in Fig. 2. By eliminating $\Theta$ in (7.8) by using (7.10), the eigenvalue problem of vertical structure equations becomes to solve the following equations.

\[(\bar{N}^2 - \nu^2)\eta = \tilde{C}\nu \left( \frac{d\xi}{dZ} + \tilde{\Gamma}\xi \right), \quad (7.14)\]

\[\nu \left( 1 - \frac{\tilde{C}}{H_e} \right) \xi = \tilde{C} \left( \frac{d\eta}{dZ} - \tilde{\Gamma}\eta \right). \quad (7.15)\]

Here, the boundary conditions are

\[\eta = 0 \text{ at } \tilde{Z} = 0 \text{ and } \tilde{Z} = \tilde{Z}_T. \quad (7.16)\]

7.2.1 Internal modes

The central difference form of Eq. (7.14) at level $\ell$ is expressed by

\[\begin{equation}
(\bar{N}^2 - \nu^2)\eta_\ell = \tilde{C}\nu \left[ \frac{\xi_{\ell+\frac{1}{2}} - \xi_{\ell-\frac{1}{2}}}{\Delta \tilde{Z}} + \tilde{\Gamma} \left( \frac{\xi_{\ell+\frac{1}{2}} + \xi_{\ell-\frac{1}{2}}}{2} \right) \right].
\end{equation}\]

Similarly, the difference forms of Eq. (7.15) at $\ell + \frac{1}{2}$ and $\ell - \frac{1}{2}$ are written as

\[\begin{equation}
\nu \left( 1 - \frac{\tilde{C}}{H_e} \right) \xi_{\ell+\frac{1}{2}} = \tilde{C} \left[ \frac{\eta_{\ell+1} - \eta_\ell}{\Delta \tilde{Z}} - \tilde{\Gamma} \left( \frac{\eta_{\ell+1} + \eta_\ell}{2} \right) \right],
\end{equation}\]

and

\[\begin{equation}
\nu \left( 1 - \frac{\tilde{C}}{H_e} \right) \xi_{\ell-\frac{1}{2}} = \tilde{C} \left[ \frac{\eta_\ell - \eta_{\ell-1}}{\Delta \tilde{Z}} - \tilde{\Gamma} \left( \frac{\eta_\ell + \eta_{\ell-1}}{2} \right) \right].
\end{equation}\]
Figure 2: Layout of the vertical grid for the shallow NH model. The grid increment is equally spaced with $\Delta \bar{Z} = [\bar{Z}(L) - \bar{Z}(0)] / L$. 
By eliminating $\xi_{l+\frac{1}{2}}$ and $\xi_{l-\frac{1}{2}}$ in (7.17) by using (7.18) and (7.19), we get

$$\tilde{C}^2 \left[ \frac{\eta_{l+1} - 2\eta_l + \eta_{l-1}}{(\Delta \tilde{Z})^2} \right]$$

$$+ \left[ \left( 1 - \frac{\tilde{C}}{H_e} \right)(\nu^2 - \tilde{N}^2) - \frac{\tilde{C}^2 \tilde{\Gamma}^2}{2} \right] \eta_l$$

$$- \frac{\tilde{C}^2 \tilde{\Gamma}^2}{2} \left( \frac{\eta_{l+1} + \eta_{l-1}}{2} \right) = 0. \quad (7.20)$$

The solution of the difference equation (7.20) that satisfies the boundary conditions (7.16) is given by

$$\eta_l = \eta_0 \sin \left( \frac{k \pi}{Z_T} \ell \Delta \tilde{Z} \right), \quad (7.21)$$

where $\eta_0$ is a constant and $k$ denotes the vertical modal index $1, 2, \ldots$ and so on.

By substituting (7.21) into (7.20), we have the eigenvalue equation

$$\frac{2\tilde{C}^2}{(\Delta \tilde{Z})^2} \left[ \cos \left( \frac{k \pi}{Z_T} \Delta \tilde{Z} \right) - 1 \right] + \left( 1 - \frac{\tilde{C}}{H_e} \right)(\nu^2 - \tilde{N}^2) - \frac{\tilde{C}^2 \tilde{\Gamma}^2}{2} \left[ 1 + \cos \left( \frac{k \pi}{Z_T} \Delta \tilde{Z} \right) \right] = 0, \quad (7.22)$$

for $k = 1, 2, 3, \ldots$ Alternatively, (7.22) can be rewritten as

$$(\nu^2 - \tilde{N}^2) = \frac{\frac{4}{(\Delta \tilde{Z})^2} \sin^2 \left( \frac{k \pi}{Z_T} \Delta \tilde{Z} \right) + \tilde{\Gamma}^2 \cos^2 \left( \frac{k \pi}{Z_T} \Delta \tilde{Z} \right)}{\left( \frac{1}{\tilde{C}^2} - \frac{1}{\tilde{C}H_e} \right)}. \quad (7.23)$$

It is instructive to show the limit of (7.23) for $\Delta \tilde{Z} \to 0$. The limit becomes

$$(\nu^2 - \tilde{N}^2) = \frac{k^2 \pi^2}{Z_T^2} + \tilde{\Gamma}^2 \left( \frac{1}{\tilde{C}^2} - \frac{1}{\tilde{C}H_e} \right). \quad (7.24)$$

It can be shown that (7.24) is identical to (6.5) of Kasahara and Qian (2000) who investigated the same vertical structure equations in continuum form.

For programming purpose, it is convenient to rewrite (7.23) in the form

$$\tilde{C}H_e = \frac{\tilde{C}^2 (\tilde{N}^2 - \nu^2)}{\tilde{N}^2 - \nu^2 + \tilde{C}^2 \lambda_k}, \quad (7.25)$$

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where
\[ \lambda_k = \frac{4}{(\Delta Z)^2} \sin^2 \left( \frac{k\pi}{2\Delta Z} \Delta \tilde{Z} \right) + \tilde{\Gamma}^2 \cos^2 \left( \frac{k\pi}{2\Delta Z} \Delta \tilde{Z} \right), \quad (7.26) \]
which is the difference form of (6.9) of Kasahara and Qian (2000) and both agree when \( \Delta \tilde{Z} \to 0 \). Moreover, we now know that in the case of hydrostatic primitive equation model, Eq. (7.25) with \( \nu^2 = 0 \) provides the formula to calculate the hydrostatic equivalent height in finite difference form.

The vertical structure function \( \xi_{\ell-\frac{1}{2}} \) can be derived by substituting (7.21) into (7.19). The result is

\[ \nu \left( 1 - \frac{\tilde{C}}{H_e} \right) \xi_{\ell-\frac{1}{2}} = \tilde{C} \eta_0 \left\{ \sin \left( \frac{k\pi}{\Delta Z} \ell \Delta \tilde{Z} \right) \left[ 2 \sin^2 \left( \frac{k\pi}{2\Delta Z} \Delta \tilde{Z} \right) - \tilde{\Gamma} \cos^2 \left( \frac{k\pi}{2\Delta Z} \Delta \tilde{Z} \right) \right] + \cos \left( \frac{k\pi}{\Delta Z} \ell \Delta \tilde{Z} \right) \left[ \frac{\sin \left( \frac{k\pi}{\Delta Z} \ell \Delta \tilde{Z} \right)}{\Delta \tilde{Z}} + \frac{\tilde{\Gamma}}{2} \sin \left( \frac{k\pi}{\Delta Z} \ell \Delta \tilde{Z} \right) \right] \right\}. \tag{7.27} \]

Thus, at the limit of \( \Delta \tilde{Z} \to 0 \), we get
\[ \nu \left( 1 - \frac{\tilde{C}}{H_e} \right) \xi = \tilde{C} \eta_0 \left[ \left( \frac{k\pi}{\Delta Z} \right) \cos \left( \frac{k\pi}{\Delta Z} \tilde{Z} \right) - \tilde{\Gamma} \sin \left( \frac{k\pi}{\Delta Z} \tilde{Z} \right) \right]. \tag{7.28} \]

The above result agrees with (6.10) of Kasahara and Qian (2000) in continuum form.

### 7.2.2. External modes

In addition to the internal modes discussed in the previous subsection, there are external modes known as the Lamb waves. In the isothermal case, the external modes are characterized by no vertical motion, i.e., \( \eta = \Theta = 0 \). In this case, (7.15) gives

\[ H_e = \tilde{C}, \quad (7.29) \]

and the difference form of (7.14), namely (7.17) yields

\[ \xi_{\ell+\frac{1}{2}} = \frac{1 - \frac{\tilde{\Gamma} \Delta \tilde{Z}}{2}}{1 + \frac{\tilde{\Gamma} \Delta \tilde{Z}}{2}} \xi_{\ell-\frac{1}{2}}, \quad (7.30) \]
assuming that $\nu$ does not vanish. Thus, $\xi_{\ell+\frac{1}{2}}$ is a linearly decreasing function of index $\ell$.

Because $H_e$ is already known by (7.29), the frequency $\nu$ can be uniquely determined from the horizontal structure equations (7.11) - (7.13).

### 7.3 Numerical solutions of the horizontal structure equations (7.11) - (7.13).

To solve Eqs. (7.11) - (7.13), we express

$$
\begin{vmatrix}
\hat{A} \\
\hat{B} \\
\hat{D}
\end{vmatrix}
= \sum_{n=s}^{\infty}
\begin{vmatrix}
A_n^s \\
B_n^s \\
D_n^s
\end{vmatrix}
P_n^s(\mu).
\tag{7.31}
$$

Note that we use the same symbols $A_n^s, B_n^s,$ and $D_n^s$ as in (5.8), but they are just the coefficients of real numbers and not functions of $\bar{r}$. With this caution in mind, we obtain the following equations after substitution of (7.31) into (7.11) - (7.13) and equating the coefficients of $P_n^s$ to zero:

$$
(-\nu + K_n)A_n^s + p_{n+1} B_{n+1}^s + q_{n-1} B_{n-1}^s - \bar{C} D_n^s = 0,
\tag{7.32}
$$

$$
(-\nu + K_n)B_n^s + p_{n+1} A_{n+1}^s + q_{n-1} A_{n-1}^s = 0,
\tag{7.33}
$$

$$
-\nu D_n^s + H_e z_n A_n^s = 0,
\tag{7.34}
$$

where the coefficients $p_n, q_n, \text{ and } z_n$ are defined by (5.19), (5.20), and (5.25), respectively.

### 7.3.1 Symmetric modes

Equations (7.32) - (7.34) contain two independent systems, describing symmetric and antisymmetric modes. Symmetric modes are represented by the sets of $A_n^s$ and $D_n^s$ for $n = s, s+2, \ldots$ and $B_n^s$ for $n = s+1, s+3, \ldots$

Let $X_c$ be the column vector

$$
X_c = \text{col} (A_s^s, B_{s+1}^s, D_s^s, A_{s+2}^s, B_{s+3}^s, D_{s+2}^s, \ldots),
\tag{7.35}
$$
and $A_c$ be the matrix

$$
A_c = \begin{bmatrix}
K_s & p_{s+1} & -\bar{C} & 0 & 0 & 0 & 0 & \cdots \\
q_s & K_{s+1} & 0 & p_{s+2} & 0 & 0 & 0 & \cdots \\
H_e z_s & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & q_{s+1} & 0 & K_{s+2} & p_{s+3} & -\bar{C} & 0 & \cdots \\
0 & 0 & 0 & q_{s+2} & K_{s+3} & 0 & p_{s+4} & \cdots \\
0 & 0 & 0 & H_e z_{s+2} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad (7.36)
$$

then the frequency $\nu$ and the vector $X_c$ are determined by solving the following eigenvalue problem:

$$
(A_c - \nu I)X_c = 0. \quad (7.37)
$$

Obviously, the system (7.37) is much easier to solve than the system (6.7), if the equivalent height $H_e$ in (7.34) is known. However, that is not the case here, because the solutions of the vertical structure equations discussed in Section 7.2 contain also the two unknowns of $\nu$ and $H_e$. We will come back later to discuss how to solve the simultaneous eigenvalue problem of this SNH model.

### 7.3.2 Antisymmetric modes

The antisymmetric system is obtained from (7.32) - (7.34) by choosing the sets of $A^s_n$ and $D^s_n$ for $n = s + 1, s + 3, \ldots$ and $B^s_n$ for $n = s, s + 2, \ldots$

Let $Y_c$ be the column vector

$$
Y_c = \text{col} \left( B^s_s, A^s_{s+1}, D^s_{s+1}, B^s_{s+2}, A^s_{s+3}, D^s_{s+3}, \ldots \right), \quad (7.38)
$$
and $B_c$ be the matrix

$$B_c = \begin{bmatrix}
K_s & p_{s+1} & 0 & 0 & 0 & 0 & \cdots \\
q_s & K_{s+1} & \tilde{C} & p_{s+2} & 0 & 0 & 0 & \cdots \\
0 & H_e z_{s+1} & 0 & 0 & 0 & 0 & \cdots \\
0 & q_{s+1} & 0 & K_{s+2} & p_{s+3} & 0 & 0 & \cdots \\
0 & 0 & 0 & q_{s+2} & K_{s+3} & \tilde{C} & p_{s+4} & \cdots \\
0 & 0 & 0 & 0 & H_e z_{s+3} & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad (7.39)$$

then $\nu$ and $Y_c$ are determined from:

$$(B_c - \nu I) Y_c = 0. \quad (7.40)$$

Before discussing how to solve the two simultaneous eigenvalue problems involving both $\nu$ and $H_e$ as unknowns, it is useful to derive approximate solutions in the case that $\tilde{C}H_e$ is large in (7.32) - (7.34).

### 7.3.3 Approximate solutions of (7.32) - (7.34) when $\tilde{C}H_e$ is large

If we eliminate $D_n$ in (7.32) using (7.34), Eqs. (7.32) - (7.34) are reduced to

$$(-\nu + K_n - \frac{\tilde{C}H_e}{\nu} z_n) A_n^s + p_{n+1} B_{n+1}^s + q_{n-1} B_{n-1}^s = 0, \quad (7.41)$$

$$(-\nu + K_n) B_n^s + p_{n+1} A_{n+1}^s + q_{n-1} A_{n-1}^s = 0. \quad (7.42)$$

The symmetric modes are represented by $A_n^s$ for $n = s, s + 2, \ldots$ and $B_n^s$ for $n = s + 1, s + 3, \ldots$. Therefore, the symmetric system can be written as

$$\begin{bmatrix}
(-\nu + K_s - \frac{\tilde{C}H_e}{\nu} z_s) & p_{s+1} & 0 & 0 & \cdots \\
q_s & (-\nu + K_{s+1}) & p_{s+2} & 0 & \cdots \\
0 & q_{s+1} & (-\nu + K_{s+2} - \frac{\tilde{C}H_e}{\nu} z_{s+2}) & p_{s+3} & \cdots \\
0 & 0 & q_{s+2} & (-\nu + K_{s+3}) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \begin{bmatrix}
A_n^s \\
B_n^s
\end{bmatrix} = 0. \quad (7.43)$$
Thus, if \( \tilde{C}H_e \) is large, we get approximately the product

\[
\prod_n \left( \nu - K_n + \frac{\tilde{C}H_e}{\nu} z_n \right) \left( \nu - K_{n+1} \right) \simeq 0,
\]

for \( n = s, s + 2, s + 4, \ldots \)

From this product we expect that there are three different kinds of oscillations in (7.43).

7.3.3.1 Oscillations of the first and third kinds

These kinds are obtained by equating the expression in the first parentheses of the product (7.44) to be zero. Then with reference to (5.18) and (5.25), we have, assuming that \( \nu \neq 0 \), that

\[
\nu^2 + \frac{s\nu}{n(n + 1)} - \tilde{C}H_e n(n + 1) = 0,
\]

for \( n = s, s + 2, s + 4, \ldots \)

This gives

\[
\nu = \frac{-s}{2n(n + 1)} \pm \left[ \frac{s^2}{4n^2(n + 1)^2} + \tilde{C}H_e n(n + 1) \right]^\frac{1}{2},
\]

for \( n = s, s + 2, s + 4, \ldots \)

If \( \tilde{C}H_e \) is sufficiently large, (7.46) is further simplified by

\[
\nu = \pm \left[ \tilde{C}H_e n(n + 1) \right]^\frac{1}{2},
\]

for \( n = s, s + 2, s + 4, \ldots \)

We can repeat the same procedure of getting approximate solutions of (7.41) and (7.42) for the antisymmetric modes. It turns out that the same approximate formulas (7.46) and (7.47) are obtained for \( n = s + 1, s + 3, s + 5, \ldots \) Therefore, (7.46) and (7.47) are valid for symmetric modes with \( n = s, s + 2, s + 4, \ldots \) and antisymmetric modes with \( n = s + 1, s + 3, s + 5, \ldots \).
The expression of frequency represented by (7.46) or (7.47) has the form identical to the approximate frequency of oscillations that correspond to the wave motions of the first kind by Margules (1893) and Hough (1898) and have been discussed extensively by Longuet-Higgins (1968) and others based on the system of global shallow water equations, i.e., Laplace tidal equations.

In the hydrostatic primitive equation model, there is only one kind of equivalent height $H_e$ and the oscillations represented by the frequency (7.46) or (7.47) are identified as the inertio-gravity waves, called the oscillations of the first kind. However, in the SNH model there exist not only the inertio-gravity waves, but also acoustic waves modified by the earth's rotation which we call the oscillations of the third kind. In other words, the approximate frequency (7.46) or (7.47) also represents that of acoustic oscillations. Only distinction between the first and third kinds of oscillations can be made through different values of $H_e$ as we will describe in Section 7.4.

These kinds of oscillations, propagating both eastward and westward directions, are irrotational in nature and are approximated by

$$
\dot{\Phi} \propto i A_n^s P_n^s (\sin \phi) \exp \left[ i (s \lambda - \sigma t) \right] \quad \text{and} \quad \dot{\Psi} \approx 0.
$$

\(7.48\)

\[7.3.3.2\] Oscillations of the second kind

The other kind of oscillations is obtained by equating the expression in the second parentheses of (7.44) to be zero for the symmetric modes. We can repeat the same procedure for the antisymmetric modes of (7.41) - (7.42). We then obtain, with reference to (5.18), that

$$
\nu = \frac{-s}{n(n + 1)}
$$

(7.49)

with $n = s, s + 2, s + 4, \ldots$ for antisymmetric modes and $n = s + 1, s + 3, \ldots$ for symmetric modes.

Remembering that the dimensionless frequency $\nu$ is scaled by $2\Omega$, it is clear that (7.49) represents a well-known wave formula derived by Haurwitz (1940) based on the
nondivergent two-dimensional vorticity equation on the spherical earth. Actually, Hough (1898) already derived even a higher-order approximation for the frequency of this kind of oscillation that has been referred to by Haurwitz (1937) and Dikii and Golitsyn (1968).

This type of westward propagating waves is called oscillations of the second kind by Margules (1893) and Hough (1898). They are rotational in nature with

\[ \Phi \approx 0 \quad \text{and} \quad \Psi \propto B_n^s P_n^s(\sin \phi) \exp [i(s \lambda - \sigma t)]. \quad (7.50) \]

It is rather interesting that the meteorological significance of this type of wave motions was first noted by Rossby and collaborators (1939) through the analyses of upper-air data. A historical review of the second kind of oscillations is given by Platzman (1968).

### 7.4 Approximate frequencies of the first and third kinds in the SNH model

For the first and third kinds of oscillations in this model, \( \nu \) can be calculated approximately from (7.45) if the value of \( \tilde{C}H_e \) is known. Luckily, the formula to calculate \( \tilde{C}H_e \) is given by (7.25) as determined from the vertical structure equations. Therefore, by substitution of (7.25) into (7.45), we get

\[ \nu^2 + \frac{s \nu}{n(n+1)} - \frac{\tilde{C}^2(N^2 - \nu^2) n(n+1)}{(N^2 - \nu^2 + \tilde{C}^2 \lambda_k)} = 0, \quad (7.51) \]

with \( n = s, s + 2, \ldots \) for symmetric modes and \( n = s + 1, s + 3, \ldots \) for antisymmetric modes, and \( \lambda_k \) is defined by (7.26).

Equation (7.51) can be expressed in the form

\[ \nu^4 + A \nu^3 - (\lambda_k \tilde{C}^2 + \tilde{N}^2 + B) \nu^2 - A(\lambda_k \tilde{C}^2 + \tilde{N}^2) \nu + B \tilde{N}^2 = 0, \quad (7.52) \]

where

\[ A = \frac{s}{n(n+1)} \quad \text{and} \quad B = \tilde{C}^2 n(n+1), \quad (7.53) \]

with \( n = s, s + 2, \ldots \) for symmetric modes, and \( n = s + 1, s + 3, \ldots \) for antisymmetric modes.

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It is important to remark here on the case of $s = 0$, because (7.52) with (7.53) is not valid for $n = 0$. Therefore, if $s = 0$ the first meridional mode is antisymmetric with $n = 1$ and the second meridional mode is symmetric and so on.

Now, there are four real roots of Eq. (7.52), consisting of two pairs of plus and minus $\nu$'s. One pair correspond to eastward and westward propagating acoustic waves, modified by the earth's rotation, with very high frequencies, referred to as the third kind of oscillations (Kasahara and Qian 2000).

The other pair correspond to eastward and westward propagating inertio-gravity waves, referred to as the first kind of oscillations. Their frequencies are much smaller than those of the acoustic waves. The nature of these modes is similar to that of the first kind of modes in the hydrostatic primitive equation (HPE) model. However, the similarity of the first kind of oscillations in the HPE model with those in the SNH model applies only to planetary to large-scale waves.

Once the values of $\nu$ are calculated from the quotic equation (7.52), the values of $\hat{C}H_e$ can be obtained from (7.25) with (7.26).

The approximate frequency formula of the second kind of oscillations is presented by (7.49). Actually, this formula is not very useful as it does not include its dependence on the equivalent height $H_e$. Although higher order formulas can be derived as mentioned earlier, there is a better way to derive much higher order solutions as we will discuss in the next section.

7.5 Iterative method to solve numerically the SNH problem

We consider two approaches to solve the SNH problem numerically. One is to use the same numerical algorithm discussed in Section 6, without resort to the separation of the variables, by setting up a block matrix through the discretization of both radial and meridional directions. Some simplifications can be made, such as the replacement of the scaled radial distance $\tilde{r}$ by unity and the omission of Coriolis terms involving $2\Omega \cos \phi$. 

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However, the numerical procedure of solving the matrix is essentially the same as in the DNH problem. In fact, this approach is used to check the numerical code of solving the DNH problem.

The other approach is to use the method of separation of variables as presented in this section. One problem is that this approach leads to a coupled eigenvalue problem involving both $\nu$ and $H_e$ as unknowns. However, we can apply iterative methods to solve the coupled problem as described by Kasahara and Qian (2000).

The iteration methods are successful, because the variations of the values of $\nu$ with respect to an increase/decrease of the values of $H_e$ calculated from the horizontal structure equations (Laplace tidal equations) and the vertical structure equations are opposite. Therefore, the two values of $\nu$ obtained from the horizontal and vertical problems can converge provided that the two problems are solved iteratively using a sufficiently small increment of $H_e$ starting from good initial guesses of $H_e$.

The initial guesses $\nu$'s of the acoustic and inertio-gravity modes are obtained from two pairs of plus and minus roots of (7.52) for a specified vertical mode index $k$ and given values of $s$ and $n$. Then, the corresponding values of $H_e$ are determined from (7.25) for a given value of $\nu^2$. The values of $H_e$ thus obtained for the acoustic modes are generally very large and provide very good initial guesses. The same can be said about the initial guess values $H_e$ for the inertio-gravity modes, though the values of $H_e$ of the inertio-gravity modes are much smaller than those of the acoustic modes.

For the second kind of modes, the values of $H_e$ based on the HPE model provide good initial guess values of $H_e$ to solve the SNH problem by the iteration method, involving the horizontal eigenvalue equation (7.37) or (7.40) and the vertical eigenvalue equation (7.25).

The values of $H_e$ for the HPE model can be calculated from

$$\tilde{H}_e = \frac{\tilde{C} \tilde{N}^2}{\tilde{N}^2 + \tilde{C}^2 \lambda_k},$$

which is obtained from (7.25) by assuming $\nu^2 = 0$ as shown by Kasahara and Qian (2000). The omission of the vertical acceleration in the vertical equation of motion in the HPE
model simplifies the vertical structure equation in such a way that the eigenvalue problem of the HPE model is not a coupled eigenvalue problem.

Before leaving this section, we should note that $H_e$ is dimensionless. The reason that $H_e$ is referred to as the equivalent height is that the horizontal structure equations (7.11) - (7.13) are equivalent to the global shallow water equations, known as Laplace tidal equations, for the depth of undisturbed water $h_e$, provided that

$$\tilde{C}H_e = \frac{C H_e}{2\Omega a} = \frac{g h_e}{4a^2 \Omega^2},$$

(7.55)

where $g$ denote the acceleration due to gravity in the SNH model which is a constant. Therefore, the relationship

$$g h_e = 4a^2 \Omega^2 \tilde{C}H_e$$

(7.56)

is used to interpret the values of dimensionless $H_e$ in terms of the equivalent height $h_e$ with the dimension of height.

8. Test of algorithms to solve the eigenvalue problem of the DNH model

In this section, we describe the results of test calculations to solve the eigenvalue problems of (6.7) and (6.17). Because the system is coupled in both meridional and radial directions, the number of simultaneous equations involved can be very large, depending on the discretization resolution in the two directions. Accordingly, the number of wave frequencies obtained from these systems can be very large and the identification of oscillations in terms of different modes becomes difficult. Therefore, we should use as small as possible resolutions in both directions for the sake of checking the numerical algorithms. Thus, our objectives here are to check the correctness of numerical codes and to identify the species of modes, rather than obtaining high resolution solutions. However, the use of a large
number of decimal places in tabulation is necessary to show the accuracy of numerical solutions for given discretizations.

The order of the matrices $\mathbf{A}$ of (6.8) and $\mathbf{B}$ of (6.18) becomes

$$N_m \times [3 \times L + 2 \times (L - 1)] ,$$

where $N_m$ denotes the number of meridional modes and $L$ the number of vertical levels. Here, we choose $L$ to be 4 and $N_m$ to be 2, so that the order of matrices $\mathbf{A}$ and $\mathbf{B}$ becomes 36, an easily manageable size for testing the codes. The numerical codes are written for arbitrary numbers of integer $N_m > 1$ and integer $L > 3$. The numerical values used for testing the codes are shown in Table 1.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Numerical values of constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>Earth’s radius</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>Earth’s rotation rate</td>
</tr>
<tr>
<td>$g$</td>
<td>Earth’s gravity</td>
</tr>
<tr>
<td>$r_T$</td>
<td>$a + Z_T$</td>
</tr>
<tr>
<td>$T_o$</td>
<td>Basic state temperature</td>
</tr>
<tr>
<td>$R$</td>
<td>Gas constant for dry air</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$R/C_p$</td>
</tr>
</tbody>
</table>

Thuburn et al. (2002a) investigated the numerical solutions of the DNH and SNH models identical to ours using a finite-difference method that is considerably different from the present spectral approach. Although we used different numerical values for model resolutions, constants, and parameters, a comparison to their results with ours provides valuable reference to verification of our test calculations.

8.1 Wave frequencies

Now we present the dimensionless frequency $\nu$ (scaled by $2 \, \Omega$) obtained from Eqs. (6.7) and (6.17) of the DNH model.
Table 2 shows the symmetric mode frequencies of the DNH model for wavenumber \( s = 1 \). There are 36 roots of Eq. (6.7) and the frequency values are tabulated from the largest positive value to the smallest negative value in sequence with a serial number in the first column.

The first six roots represent positive (eastward propagation) acoustic modes. Because \( N_m \) is two, there are meridional modes with index \( j \) of 2 or 0 as shown in the last column. This index \( j \) is defined by \( j = n - s \). Thus, \( j = 0 \) corresponds to \( n = s \) which is the first term in the expansion (5.8) and \( j = 2 \) corresponds to \( n = s + 2 \) as the third term in the expansion. The \( n - s \) index is used to identify the meridional modes in the solutions of Laplace tidal equations adopted by Longuet-Higgins (1968). The same procedure for indicating the meridional modes is used by Kasahara (1976, 1977), and Swarztrauber and Kasahara (1985) for Hough functions.

Because \( L \) is chosen to be 4, there are three internal modes indicated by positive integer index \( k \). Thus, there are six frequency values in total in the positive acoustic modes (AC\(^+\)). The difference of frequency values among them is more noticeable among three vertical modes than between two meridional modes. In order to identify a particular \((j, k)\) mode among the six frequency values, it is necessary to examine the vertical profiles of the eigenvector \( A \) or \( B \), corresponding to a particular value of frequency. We will discuss the matter related to eigenvectors in Section 8.2.

Similarly, the last six roots (#31-36) represent the negative (westward propagation) acoustic modes (AC\(^-\)) corresponding to the first six roots of AC\(^+\) modes.

The group of 8 roots (#7-14) represent the positive inertio-gravity modes (IG\(^+\)) with much smaller magnitudes of frequency than those of AC\(^+\) modes. The reason for having two additional roots with vertical mode \( k = 0 \) is that they are the external modes in contrast to remaining six internal modes. In the case of SNH model, the solutions of (7.37) show that those external modes have the properties as discussed in Section 7.2.2, namely the vertical motion is zero and the velocity potential, stream function, and pressure...
Table 2. Wavenumber $s=1$ DNH symmetric mode frequencies and their differences from the SNH counterparts for specific vertical mode($k$) and meridional mode($j$).

<table>
<thead>
<tr>
<th>Ser. #</th>
<th>DNH frequency</th>
<th>DNH-SNH freq. diff.</th>
<th>Vert. mode($k$)</th>
<th>Merid. mode($j$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eastward Acoustic Symmetric Modes</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>210.8156245580911</td>
<td>-0.0146545518899</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>210.8141561507080</td>
<td>-0.0145422085459</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>188.2005731292829</td>
<td>-0.0566473244588</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>188.1993637676079</td>
<td>-0.0564081121418</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>162.4673637776584</td>
<td>-0.1120213932787</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>162.4661920563670</td>
<td>-0.1121422991149</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eastward Inertio-Gravity Symmetric Modes</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1.2896584324244</td>
<td>-0.0030366836622</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>1.1212413954228</td>
<td>-0.0071833891329</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>1.0128166625700</td>
<td>-0.005538947814</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>0.9421539510647</td>
<td>-0.0049031602074</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>0.3694258822264</td>
<td>-0.001194433046</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
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<td>-0.0025157119307</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>0.2592748888386</td>
<td>-0.002120920512</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>0.2298374804306</td>
<td>-0.0011214229911</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Westward Rotational Symmetric Modes</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>-0.0341833683607</td>
<td>-0.002162239484</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>16</td>
<td>-0.0367782740604</td>
<td>-0.001066952344</td>
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<td>3</td>
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<tr>
<td>17</td>
<td>-0.0396358856267</td>
<td>-0.002168210298</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>18</td>
<td>-0.0425628182819</td>
<td>-0.000390715714</td>
<td>0</td>
<td>3</td>
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<tr>
<td>19</td>
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<td>-0.000245933354</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>-0.0774444210546</td>
<td>-0.0005987711434</td>
<td>2</td>
<td>1</td>
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<tr>
<td>21</td>
<td>-0.0869346773192</td>
<td>-0.0007455569213</td>
<td>1</td>
<td>1</td>
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(Φ, Ψ, P) have all decreasing vertical profiles with altitude. Actually, these external modes are called Lamb modes and should be treated separately from either group of AC or IG modes.

In the DNH model, the properties of the external modes, k = 0, are very similar to those of SNH model. However, there exists very weak vertical motion, but Φ, Ψ, and P all decrease monotonically with altitude. Thus, it is still appropriate to classify them as Lamb modes.

Likewise, the group of 8 roots (#23-30) represent the negative (westward propagation) inertio-gravity modes (IG−). Although the magnitudes of frequency of IG− modes appear comparable to those of IG+ modes, there are marked differences in the magnitudes between frequencies of IG+ modes and those of IG− modes for meridional index j = 0. In fact, the group of 4 roots (#11-14) for j = 0 correspond to so-called Kelvin modes (Matsuno, 1966; Holton and Lindzen, 1968). One notable difference exists in an increasing or decreasing trend in the magnitudes of AC and IG frequencies depending on vertical mode index k. Namely, the magnitudes of AC frequencies increase as k increases, while those of IG decrease as k increases.

The remaining 8 roots (#15-22) are westward propagating rotational (RO) modes. Those are equivalent to the so-called oscillations of the second kind (Margules 1893; Hough 1898). The two roots with k = 0 correspond to the external modes and the remaining roots to the internal modes for j = 0 and 2.

As mentioned earlier, the type of each frequency in the DNH model is identified by the one-to-one correspondence between the roots of deep and shallow NH models calculated from (7.37) and (7.40), respectively. The third column in Table 2 shows the difference of each corresponding root, deep NH - shallow NH. Those differences are negative for positive frequencies and positive for negative frequencies, namely the magnitudes of DNH frequencies are consistently smaller than those of corresponding SNH frequencies. This finding is in agreement with that of Thuburn et al. (2002a).
Table 3. Wavenumber $s=500$ DNH symmetric mode frequencies and their differences from the SNH counter parts for specific vertical mode (k) and meridional mode (j).

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<td>0.7772527444925</td>
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Table 3 shows the same as Table 2, except for $s=500$. The classification of frequencies are the same, but many differences exist numerically between values in Tables 2 and 3. For example, the magnitudes of AC and IG frequencies of $s=500$ are larger than those of $s=1$, particularly for the IG modes. Note that the magnitudes of the external modes (#7, 8, 29, and 30) in the IG groups exceed the Brunt-Väisälä frequency ($N=136.1278$) and close to those of AC groups. In fact, Thuburn, et al. (2002a) refer to the deep Lamb modes as the external acoustic modes. It is well known that the Lamb modes behave like IG external modes for large-scale motions and like AC external modes for small-scale motions. In contrast, rotational frequencies become very small as expected. Moreover, the differences of values among different vertical modes are extremely small for this low vertical resolution case.

The differences of frequencies between two models (DNH - SNH) are listed in the third column of Table 3 as well as Table 2. We find that the magnitudes of DNH frequencies are smaller than the SNH counterparts for both $s=1$ and 500. The frequency differences between the DNH and SNH models are small and less than 1% in agreement with Thuburn et al. (2002a).

Concerning the frequency differences between DNH and SNH models, Thuburn et al. (2002a) noted the following observations in their numerical results. For IG and RO modes, the DNH frequencies with surface at $r=a$ and top at $r=a+Z_T$ were found to be smaller in magnitude than the SNH frequencies of radius $a$, but greater than the SNH frequencies of radius $a+Z_T$. However, the internal AC modes behave rather differently. Their DNH frequencies with surface at $r=a$ and top $r=a+Z_T$ were found to be smaller in magnitude than the SNH frequencies of radius either $a$ or $a+Z_T$.

Table 4 shows the differences in frequency between DNH run and modified SNH run with radius $a+Z_T$ for $s=1$ (left) and $s=500$ (right). By comparing the values on the column of (DNH - SNH freq. diff.) in Table 2 for $s=1$ and those in Table 3 for $s=500$ with the values on the left column of (DNH-modSNH diff.) in Table 4 for $s=1$ and those
Table 4. Differences in frequency between DNH run and modified SNH run with radius $a + ZT$ for $s=1$ (left) and $s=500$ (right). VM stands for vertical mode ($k$) and MM for meridional mode ($j$).

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Eastward Inertio-Gravity Symmetric Modes

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Westward Rotational Symmetric Modes

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<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>-0.0119358061852</td>
<td>0</td>
<td>2</td>
<td>-2.129283074872</td>
<td>0</td>
<td>2</td>
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</tr>
</tbody>
</table>

Westward Inertio-Gravity Symmetric Modes

<table>
<thead>
<tr>
<th>Ser.</th>
<th>$s=1$</th>
<th>DNH-modSNH diff.</th>
<th>VM</th>
<th>MM</th>
<th>DNH-modSNH diff.</th>
<th>VM</th>
<th>MM</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>0.1107463648352</td>
<td>1</td>
<td>0</td>
<td>-0.723983841952</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>0.1105933015882</td>
<td>1</td>
<td>2</td>
<td>-0.7277731666457</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>0.0557939991453</td>
<td>2</td>
<td>0</td>
<td>-0.6919974764408</td>
<td>2</td>
<td>0</td>
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</tr>
<tr>
<td>34</td>
<td>0.0559887511235</td>
<td>2</td>
<td>2</td>
<td>-0.6969515621832</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>0.0143930234789</td>
<td>3</td>
<td>0</td>
<td>-0.6388433537802</td>
<td>3</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>0.014568515544</td>
<td>3</td>
<td>2</td>
<td>-0.6438605603522</td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
on the right column of (DNH-modSNH diff.) in Table 3 for \( s = 500 \), respectively, we find that the above observations noted by Thuburn et al. (2002a) are applicable also to our results examined, except for the AC modes for \( s = 500 \) in which case the DNH frequencies are smaller than the SNH frequencies with radius \( a \), but are greater than the modSNH frequencies with radius \( a + Z_T \). Thuburn et al. (2002a) also pointed out that the AC mode frequencies for the DNH model extending \( r = a \) to \( r = a + Z_T \) are smaller in magnitude than those for the SNH model of either radius \( a \) or radius \( a + Z_T \) as long as the vertical wavelength is much smaller than the horizontal wavelength. Therefore, our results of AC modes for \( s = 500 \) seems not inconsistent with their analysis.

So far, we have been concerned with the symmetric modes. The general properties of the symmetric modes that we found in our results are common in those of the antisymmetric modes. We, therefore, present only Table 5 as an example of antisymmetric modes for \( s = 1 \) that corresponds to Table 2. We should make a comment on the meridional mode index \( j(= n - s) \) listed in the last column. For AC and IG modes, the antisymmetric modes are associated with an odd index, while the symmetric modes are counted by an even index, starting from \( j = 0 \). In contrast, for RO modes, the antisymmetric (symmetric) modes are associated with an even (odd) index.

Lastly, we present the DNH frequencies for \( s = 0 \) in Table 6, in which both symmetric (left half) and antisymmetric (right half) modes with corresponding vertical mode index \( k \) and meridional mode index \( j(= n - s) \) are shown. The magnitudes of positive and negative frequencies of the AC and IG modes are the same. The RO modes are absent for \( s = 0 \). Each mode of the DNH model has a counterpart in the SNH model. And, the magnitudes of both AC and IG frequencies of the DNH model are slightly smaller than those of the SNH model with radius \( a \).
### Table 5. Wavenumber s=1 DNH antisymmetric mode frequencies and their differences from the SNH counter parts for specific vertical mode(k) and meridional mode(j).

<table>
<thead>
<tr>
<th>Ser.</th>
<th>DNH frequency</th>
<th>DNH-SNH freq. diff.</th>
<th>Vert. mode(k)</th>
<th>Merid. mode(j)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Eastward Acoustic Antisymmetric Modes</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>210.8167612209652</td>
<td>-0.0147824905313</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>210.8146986669112</td>
<td>-0.0146319890067</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>188.2012962310375</td>
<td>-0.0570830918862</td>
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<tr>
<td>4</td>
<td>188.1996391473035</td>
<td>-0.0567121563882</td>
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<td>1</td>
</tr>
<tr>
<td>5</td>
<td>162.4673648908897</td>
<td>-0.1128610127104</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>162.4660232198152</td>
<td>-0.1127314119175</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td><strong>Eastward Inertio-Gravity Antisymmetric Modes</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1.5590626405673</td>
<td>-0.0041956133612</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>1.3227213629724</td>
<td>-0.0110903254070</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>1.1695306006889</td>
<td>-0.0083604940647</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>1.0688775489521</td>
<td>-0.0071435086567</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>11</td>
<td>0.8900365834857</td>
<td>-0.0022020448109</td>
<td>0</td>
<td>1</td>
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<tr>
<td>12</td>
<td>0.7663184680729</td>
<td>-0.0050953687580</td>
<td>1</td>
<td>1</td>
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<tr>
<td>13</td>
<td>0.6824256706529</td>
<td>-0.0042705665100</td>
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<td>1</td>
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<tr>
<td>14</td>
<td>0.6245703851795</td>
<td>-0.0040995481519</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td><strong>Westward Rotational Antisymmetric Modes</strong></td>
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<td></td>
<td></td>
<td></td>
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<tr>
<td>15</td>
<td>-0.0420031187512</td>
<td>0.0003672474860</td>
<td>3</td>
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<tr>
<td>16</td>
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<td>0.0003799217464</td>
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<td>17</td>
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<td>0.0004194448490</td>
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<tr>
<td>18</td>
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<td>0.000100953687580</td>
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<tr>
<td>19</td>
<td>-0.3645843939388</td>
<td>0.0011799648200</td>
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<tr>
<td>20</td>
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<td>0.0010417396252</td>
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<td>0</td>
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<tr>
<td>21</td>
<td>-0.3993588964363</td>
<td>0.0009625235418</td>
<td>1</td>
<td>0</td>
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<tr>
<td>22</td>
<td>-0.4209392471874</td>
<td>0.0003286129437</td>
<td>0</td>
<td>0</td>
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<td><strong>Westward Inertio-Gravity Antisymmetric Modes</strong></td>
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<tr>
<td>23</td>
<td>-0.9185861992718</td>
<td>0.0040163822251</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>24</td>
<td>-0.9704456482752</td>
<td>0.0041696237975</td>
<td>2</td>
<td>1</td>
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<tr>
<td>25</td>
<td>-1.0413486807355</td>
<td>0.0057239182069</td>
<td>1</td>
<td>1</td>
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<tr>
<td>26</td>
<td>-1.1471664232279</td>
<td>0.0018582425951</td>
<td>0</td>
<td>1</td>
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<tr>
<td>27</td>
<td>-1.1679756577970</td>
<td>0.0059750274898</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>28</td>
<td>-1.2532865957307</td>
<td>0.0080233702397</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>29</td>
<td>-1.3296580478626</td>
<td>0.0018662978149</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>30</td>
<td>-1.62169025300224</td>
<td>0.0040703030682</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td><strong>Westward Acoustic Antisymmetric Modes</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>-162.4674149643081</td>
<td>0.1113409602300</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>32</td>
<td>-162.4687580001316</td>
<td>0.1114691951527</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>33</td>
<td>-188.200245944151</td>
<td>0.0561068982471</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>34</td>
<td>-188.201944222323</td>
<td>0.0564764396812</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>35</td>
<td>-210.8148396267952</td>
<td>0.0144925287040</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>36</td>
<td>-210.8169025300224</td>
<td>0.0146426810532</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>
Table 6. Frequencies of $s = 0$ DNH symmetric (left side) and antisymmetric (right side) modes for specific vertical mode $(k)$ and meridional mode $(j)$.

| Ser. # | Symmetric frequency | Antisymmetric frequency | | |
|-------|---------------------|-------------------------| | |
|       | Ser. frequency Ver. Mer. frequency Ver. Mer. | | |
|       | # (k) (j) | (k) (j) | | |
| 1     | 210.8147514617775 3 2 | 210.8155714503299 3 3 | | |
| 2     | 210.8138610621466 3 0 | 210.8140856672847 3 1 | | |
| 3     | 188.2000735965611 2 2 | 188.2004256171799 2 3 | | |
| 4     | 188.1990789515523 2 0 | 188.1991063758417 2 1 | | |
| 5     | 162.4675990981150 1 2 | 162.4671955277424 1 3 | | |
| 6     | 162.4660247889019 1 0 | 162.4657507874651 1 1 | | |

Inertio-Gravity Modes (positive and negative)

| Ser. # | Acoustic Modes (positive and negative) | | |
|-------|-----------------------------------------| | |
| 7     | 1.0537495922372 0 2 | 1.2768067628795 0 3 | | |
| 8     | 0.9489179757491 1 2 | 1.1060298551420 1 3 | | |
| 9     | 0.8838046796304 2 2 | 0.9999224217246 2 3 | | |
| 10    | 0.8423974512296 3 2 | 0.9326542535134 3 3 | | |
| 11    | 0.0000000000000 0 0 | 0.6239108015437 0 1 | | |
| 12    | 0.0000000000000 1 0 | 0.5522475791739 1 1 | | |
| 13    | 0.0000000000000 2 0 | 0.5007278996065 2 1 | | |
| 14    | 0.0000000000000 3 0 | 0.4630302628181 3 1 | | |
8.2 Eigenfunctions of normal modes

The magnitudes of the eigenvectors $X$ and $Y$ are arbitrary and they should be normalized for each mode. Here we apply a normalization based on the condition that the total energy (TE) defined by (3.16) is invariant for each mode corresponding to a specific frequency. The global integral $I_m$ of TE is expressed by

$$I_m = \int \int \int a+Z \frac{1}{2} \left\{ |U|^2 + |V|^2 + |W|^2 + |P|^2 + |Q|^2 \right\} dr \, d\mu \, d\lambda$$

$$= \int \int \int a+Z \frac{1}{2} \{-\Phi^* \nabla^2 \Phi - \Psi^* \nabla^2 \Psi + |W|^2 + |P|^2 + |Q|^2\}dr \, d\mu \, d\lambda , \quad (8.1)$$

where the expression of the second integral is derived by using (4.7) and (4.8) and carrying out the integrations with respect to $\phi$ and $\lambda$.

By substituting (5.1) and (5.8) into the second expression of $I_m$ in (8.1), and using (5.9) and (5.10), we obtain

$$\frac{1}{L} \sum_{\ell=1}^{L} \sum_{n=s}^{\infty} \left[ n(n+1) \left\{ A_n^s(\ell - \frac{1}{2}) \right\}^2 + B_n^s(\ell - \frac{1}{2}) \right]$$

$$+ \left[ C_n^s(\ell) \right]^2 + \left[ D_n^s(\ell - \frac{1}{2}) \right]^2 + \left[ E_n^s(\ell) \right]^2$$

$$\times \frac{2}{2n+1} \frac{(n+s)!}{(n-s)!} = K_m , \quad (8.2)$$

where

$$K_m = I_m(\pi Z_T)^{-1} . \quad (8.3)$$

We select $K_m = 1$ to normalize the eigenvector $X$ or $Y$ for each mode associated with each frequency for a specific mode $m = (s,j,k)$ with the vertical index $k$, meridional index $j(=n-s)$, and zonal wavenumber $s$. 

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Once the normalized eigenvector $X$ or $Y$ is obtained, the wind components $U$ and $V$ are calculated from the eigenvectors using (4.7) and (4.8).

As an explicit form of the eigenfunctions, we express the field variables (4.1) as

$$
\begin{align*}
U & = \tilde{U}_{j,k}^s(\phi, \ell - \frac{1}{2}) \\
V & = i\tilde{V}_{j,k}^s(\phi, \ell - \frac{1}{2}) \\
W & = K_s \left[ i\tilde{W}_{j,k}^s(\phi, \ell) + \tilde{Q}_{j,k}^s(\phi, \ell) \right] e^{i(s\lambda - \nu \ell)},
\end{align*}
$$

where $K_s$ denotes the scaling constant, $\rho_s^\frac{1}{2} aC$, with $\rho_s$ being a scaling density (constant). The argument $(\phi, \ell)$ or $(\phi, \ell - \frac{1}{2})$ indicates that each variable is a function of the latitude $\phi$ and the discrete radial level $\ell$ or $\ell - \frac{1}{2}$. Also, each variable depends on three indices of $(s, j, k)$.

The field variables are calculated from

$$
\begin{align*}
\tilde{U}_{j,k}^s &= - \left[ s \sum_{n=s}^\infty A_n^s(\ell - \frac{1}{2}) P_n^s(\mu) + \sum_{n=0}^\infty B_n^s(\ell - \frac{1}{2}) \mathcal{L} P_n^s(\mu) \right] (\cos \phi)^{-1}, \\
\tilde{V}_{j,k}^s &= \left[ \sum_{n=s}^\infty A_n^s(\ell - \frac{1}{2}) \mathcal{L} P_n^s(\mu) + s \sum_{n=0}^\infty B_n^s(\ell - \frac{1}{2}) \mathcal{L} P_n^s(\mu) \right] (\cos \phi)^{-1}, \\
\tilde{W}_{j,k}^s &= \sum_{n=s}^\infty C_n^s(\ell) P_n^s(\mu), \\
\tilde{P}_{j,k}^s &= \sum_{n=s}^\infty D_n^s(\ell - \frac{1}{2}) P_n^s(\mu), \\
\tilde{Q}_{j,k}^s &= \sum_{n=s}^\infty E_n^s(\ell) P_n^s(\mu).
\end{align*}
$$

Since the normal modes are normalized and orthogonal, the calculations of the field variables can be verified by examining the following conditions.
\[
\frac{1}{L} \sum_{\ell=1}^{L} \left[ \int_{-1}^{1} \left( \tilde{U}_{j,k}^s \tilde{U}_{j',k'}^s + \tilde{V}_{j,k}^s \tilde{V}_{j',k'}^s + \tilde{W}_{j,k}^s \tilde{W}_{j',k'}^s 
+ \tilde{P}_{j,k}^s \tilde{P}_{j',k'}^s + \tilde{Q}_{j,k}^s \tilde{Q}_{j',k'}^s \right) d\mu \right] = 1 \quad \text{if} \quad j = j' \quad \text{and} \quad k = k'
\]
\[
= 0 \quad \text{if} \quad i \neq j' \quad \text{or} \quad k \neq k'.
\]

(8.10)

For low zonal wavenumber \((s < 5)\) cases examined in this test, the conditions (8.10) are satisfied with an accuracy of at least \(10^{-13}\).

Figure 3 shows an example of latitudinal profiles of the field variables, \(\tilde{U}_{j,k}^s, \tilde{V}_{j,k}^s, \tilde{W}_{j,k}^s, \tilde{P}_{j,k}^s\), and \(\tilde{Q}_{j,k}^s\) calculated from (8.5) - (8.9) at specific levels indicated by \(\ell\) for the external \((k = 0)\), first symmetric \((j = 1)\), zonal wavenumber \((s = 1)\) rotational mode of the DNH model. We see that \(\tilde{U}_{j,k}^s, \tilde{V}_{j,k}^s, \text{ and } \tilde{P}_{j,k}^s\) are decreasing functions of altitude. In fact, these profiles are very similar to those of the SNH model for the RO mode with the same \(j, k, \text{ and } s\). For example, the vertical distributions of \(\tilde{U}_{j,k}^s, \tilde{V}_{j,k}^s, \text{ and } \tilde{P}_{j,k}^s\) are approximately proportional to the vertical structure function \(\xi_{\ell-\frac{1}{2}}\) as given by (7.27). One significant difference between the external rotational modes of the DNH and SNH models is that in the SNH model the variables \(\tilde{W}_{j,k}^s\) and \(\tilde{Q}_{j,k}^s\) are identically zero, while in the DNH model \(\tilde{W}_{j,k}^s\) and \(\tilde{Q}_{j,k}^s\) have non-zero values, though they are relatively small in magnitude compared to \(\tilde{U}_{j,k}^s, \tilde{V}_{j,k}^s, \text{ and } \tilde{P}_{j,k}^s\).

Figure 4 shows the same as Fig. 3, except for the second \((k = 2)\), first symmetric \((j = 0)\), zonal wavenumber \((s = 1)\) eastward-propagating acoustic mode of the DNH model. This particular mode is chosen, because Thuburn et al. (2002a) also presented the eigenfunction distributions for the same case as their Fig. 2. We can see that their latitude-height structures of the field variables resemble well with ours. Note that the
Figure 3: Latitudinal profiles of the field variables at specific levels indicated by $\ell$ for the external ($k = 0$), symmetric ($j = 1$), zonal wavenumber ($s = 1$) rotational mode of deep NH model.
Figure 4: Latitudinal profiles of the field variables at specific levels indicated by $\ell$ for the internal $(k = 2)$, symmetric $(j = 0)$, zonal wavenumber $(s = 1)$ eastward-propagating acoustic mode of deep NH model. Otherwise, this is the same as Fig. 3.
magnitudes of $\tilde{U}_{j,k}^s$ and $\tilde{V}_{j,k}^s$ are rather small compared with $\tilde{P}_{j,k}^s$, $\tilde{W}_{j,k}^s$, and $\tilde{Q}_{j,k}^s$. This is a feature of acoustic modes as we will discuss the energetics of modes later.

Figure 5 shows the same as Fig. 4, except for the field variables of the SNH model. Again, this figure should be compared with Fig. 3 of Thuburn et al. (2002a) for the same mode. Note that the latitudinal distributions of $\tilde{U}_{j,k}^s$ look like constant, but actually there are slight variations with latitude. It is interesting that, by comparing Figs. 4 and 5 here, we see rather notable differences in the global structures of the field variables for this mode.

Although the difference in structure of the eigenfunctions between the deep and shallow NH models is noticeable for the acoustic modes of zonal wavenumber $s = 1$, the difference becomes smaller as zonal wavenumber increases in agreement with the finding of Thuburn et al. (2002a). Moreover, the eigen-structures of the IG modes of the deep and shallow NH models are very similar, except that the deep Lamb modes have non-zero $\tilde{W}_{j,k}^s$ and $\tilde{Q}_{j,k}^s$ fields, while the shallow Lamb modes have vanishing $\tilde{W}_{j,k}^s$ and $\tilde{Q}_{j,k}^s$ in the same way as those of the hydrostatic primitive equation model. Thus, it is fair to say that except for planetary-scale acoustic modes, the differences in eigen-structures of various modes between the deep and shallow NH models are relatively small for the terrestrial atmosphere.

Another notable difference in the eigenfunctions between the deep and shallow NH models which is not easily visible in Figs. 3 to 5 is that the vertical structures of eigenfunctions of the DNH model vary latitudinally, while those of the SNH model do not, i.e., the vertical structures of $\tilde{U}_{j,k}^s$, $\tilde{V}_{j,k}^s$, and $\tilde{P}_{j,k}^s$ are common. Since the formulation of the basic matrix algorithm does not assume the separability of solutions even in the case of SNH model, the separability of eigenfunctions can be used to check the solutions.

So far we have been dealing with the eigenfunctions of the symmetric modes. The eigenfunctions of antisymmetric modes are constructed from the eigenvector $Y$ of (6.14). Other than the fact that $\tilde{U}_{j,k}^s$, $\tilde{P}_{j,k}^s$, $\tilde{W}_{j,k}^s$, and $\tilde{Q}_{j,k}^s$ are antisymmetric and $\tilde{V}_{j,k}^s$ is symmetric with respect to the equator, the observations made on the symmetric modes here are
(a) zonal velocity ($\tilde{U}$);
(b) meridional velocity ($\tilde{V}$);
(c) pressure ($\tilde{P}$);
(d) vertical velocity ($\tilde{W}$);
(e) log-potential temperature ($\tilde{Q}$).

Figure 5: Same as Fig. 4, except for the mode from the shallow NH model.
equally applicable to the antisymmetric modes and we will not dwell on our discussion further.

8.3 Energetics of normal modes

Eigenfunctions of normal modes exhibit unique differences among various kinds of normal modes. The normalization condition (8.2) of the eigenvectors that ensures the integrated total energy is invariant for each mode provides a means to characterize different modes by the following energy products (Platzman 1992):

\[ U \ast \ast 2 + V \ast \ast 2 = \frac{1}{L} \sum_{\ell=1}^{L} \int_{-1}^{1} \left[ \left\{ \tilde{U}_{j,k}^{s}(\mu, \ell - \frac{1}{2}) \right\}^2 + \left\{ \tilde{V}_{j,k}^{s}(\mu, \ell - \frac{1}{2}) \right\}^2 \right] d\mu, \]  

(8.11)
as a measure of the horizontal kinetic energy,

\[ W \ast \ast 2 = \frac{1}{L} \sum_{\ell=1}^{L} \int_{-1}^{1} \left[ \tilde{W}_{j,k}^{s}(\mu, \ell) \right]^2 d\mu, \]  

(8.12)
as a measure of the vertical kinetic energy,

\[ P \ast \ast 2 = \frac{1}{L} \sum_{\ell=1}^{L} \int_{-1}^{1} \left[ \tilde{P}_{j,k}^{s}(\mu, \ell - \frac{1}{2}) \right]^2 d\mu, \]  

(8.13)
as a measure of the elastic energy, referred to by Eckart (1960) following the convention of acoustics, and

\[ Q \ast \ast 2 = \frac{1}{L} \sum_{\ell=1}^{L} \int_{-1}^{1} \left[ \tilde{Q}_{j,k}^{s}(\mu, \ell) \right]^2 d\mu. \]  

(8.14)

Eckart (1960) called the above quantity the thermobaric energy. In the hydrostatic primitive equation model, the elastic energy and thermobaric energy can be combined as a single energy product that is called available potential energy (Lorenz 1955). As an afterthought, we should have taken the sum of the elastic and thermobaric energies and presented them as the available potential energy. Here, we will show the two forms of energy separately.

Table 7 shows an example of normalized energy products defined above for each mode of zonal wavenumber \( s = 1 \) of symmetric DNH model. The vertical mode index \( (k) \) and
Table 7. Normalized energy products defined by (8.11) - (8.14) for each mode of s = 1 of symmetric DNH model for vertical mode (k) and meridional mode (j).

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60
meridional mode index \((j = n - s)\) are listed in the last two columns. As we can anticipate from Fig. 4, the horizontal kinetic energy of acoustic modes is very small and the majority of kinetic energy resides in the vertical motion. The rest of energy of the acoustic modes is split in the elastic and thermobaric energy categories and the partition between the two categories of energy depends on the vertical mode.

On the contrary, the contribution of the vertical kinetic energy is rather small for the IG and RO modes. Particularly, for the Lamb modes and the external RO modes, the vertical kinetic energy and the thermobaric energy are very small. (In the SNH model both the vertical kinetic energy and the thermobaric energy are found to be zero within \(10^{-26}\).

For the IG modes, the horizontal kinetic energy occupies roughly one half of the total energy and the rest is shared by the elastic and thermobaric energies and the partition between the two categories of energy depends on the vertical mode.

For the RO modes, the share of the horizontal kinetic energy in the total energy becomes larger than the IG modes. In fact, as seen from Table 8 that shows the same as Table 7, except for zonal wavenumber 10, the share of the horizontal kinetic energy in the total energy greatly increases by the expense of the elastic and thermobaric energies as wavenumber increases.

9. Conclusions

This report describes a method of calculating normal modes of a linear system of nonseparable differential equations, based on a nonhydrostatic, compressible, stratified, rotating deep global atmospheric model. Earlier, Thuburn et al. (2002a) investigated the identical problem by using a finite-difference approach in meridional and radial directions, while harmonic solutions are assumed in longitude. Here, a traditional spectral approach is used in the horizontal direction using a spherical harmonics expansion. A finite-difference scheme is used to discretize the variables in the radial direction. One advantage in adopting the spectral approach in the horizontal direction is that the present undertaking can be
Table 8. Same as Table 7, except for zonal wavenumber $s = 10$.

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<th>$P^2$</th>
<th>$Q^2$</th>
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<td>(k) (j)</td>
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Eastward Inertio-Gravity Symmetric Modes

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Westward Inertio-Gravity Symmetric Modes

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Westward Acoustic Symmetric Modes

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62
regarded as a generalization of the Laplace-Taylor problem dealing with the normal modes of hydrostatic primitive equation model.

A test of the numerical schemes is conducted for the case of isothermal basic state at rest with a constant gravitational acceleration. In order to verify the working algorithms for solution of the deep nonhydrostatic (NH) problem, the normal mode problem of the shallow NH model is considered in parallel. The solutions of the shallow NH problem can be obtained from two eigenvalue problems, formulated separately in the meridional and radial directions, though the two eigenvalue problems are coupled. However, the shallow NH problem can also be solved without separating the variables by using the same block matrix algorithm as adopted for the deep NH problem. Thus, the verification of solutions from two different approaches helps to check the block matrix formulation of the deep NH problem.

The test is conducted with the terrestrial conditions similar to those used by Thuburn et al. (2002a). Therefore, despite differences in the numerical methods, our results are in agreement with theirs in many respects. For example, one of the most significant results is that the species of normal modes of the deep NH model are found to be identical to those of the shallow NH model. Namely, there are three kinds of internal modes; inertia-gravity (IG, 1st kind), rotational (RO, 2nd kind), and acoustic (AC, 3rd kind) modes and two kinds of external modes. In the shallow NH model, one kind of the external modes is referred to as the Lamb waves that are horizontal motions and their frequencies are close to the IG modes for large-scale motions and asymptote to the AC modes for small-scale motions. The other kind of the external modes belongs to the RO modes.

In the deep NH model, there are the same number of internal modes as in the shallow NH model, though their corresponding frequencies and eigen-structures are slightly different. Two kinds of external modes are no longer purely horizontal, though their vertical motions are very weak compared to the horizontal motions and the vertical structures of horizontal velocity and pressure are very similar to those of the shallow NH model. Again,
one kind of external modes is rotational and the other can be regarded as deep Lamb modes.

Some differences are found in the eigen-structures of normal modes between the deep and shallow NH models. For example, the eigen-structures of large-scale AC modes of the deep NH model are markedly different from those of the shallow NH model. This presumably occurs due to the incorporation of the \( \cos \phi \) Coriolis term in the zonal equation of motion due to a relatively large contribution of vertical velocity, as noted also by Thuburn et al. (2002a).

Earlier, Eckart (1960, p. 135) made a conjecture concerning the role of the \( \cos \phi \) Coriolis parameter on free oscillations of an isothermal atmosphere on plane level surfaces. He said that, “There are effects that depend on \([2\Omega \cos \phi]\), and these can be very marked for frequencies in the neighbourhood of \([\text{frequency} = 2\Omega \sin \phi]\).” This conjecture is recently corroborated by Thuburn et al. (2002b) and Kasahara (2003a,b) who showed the existence of a new kind of oscillations whose frequencies are very close to the inertial frequency, \(2\Omega \sin \phi\).

Eckart then went on to make another conjecture that

“This has been established only for plane level surfaces. If the level surfaces are spheres, \([2\Omega \sin \phi]\) will depend on latitude. It may be argued that the effects in question will then occur only in a relatively narrow band of latitudes and that this band will be different for each value of \([\text{frequency}]\). At any given latitude, those values of \([\text{frequency}]\) for which these effects are large may be relatively unimportant compared to the much larger range of frequencies for which the effects are small.”

Judging from the present results of the spherical geometry and those of the tangent-plane case mentioned earlier, it seems that the Eckart’s conjecture summarizes our findings succinctly. It appears that the \( \cos \phi \) Coriolis terms have negligible effects on low-frequency rotational modes. Likewise, the \( \cos \phi \) Coriolis terms influence very little on high-frequency
AC modes, except perhaps for very large-scale motions. However, the influence of the $\cos \phi$ Coriolis terms on IG modes are more subtle. On the one hand, as long as the atmosphere is very stable, the effects of rotation on the IG modes are categorically small, except for very large-scale motions. On the other hand, if one asks which component of the Coriolis vector influences more on the IG modes for all scales, then the $\cos \phi$ component seems to be the likely winner.
References

Gordon and Breach, 200 pp.


