Two Unequal Spheres in Slow Viscous Linear Shear Flow

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November 1971

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One of the fundamental problems of cloud physics is the prediction of conditions under which cloud droplets will collide. Previous studies of forces on spheres in low Reynolds number flow, such as Davis (1969), can be directly applied to this problem, since they form the basis for trajectory calculations for droplets falling in still air. The question remains, however, as to what effect motions of the air itself might have on droplet collisions: if the air mass in which the droplets fall moves as a whole, no trajectory changes will occur, but if the air is subjected to shearing motion, the droplet trajectories might very well be affected. Strong shear zones are known to occur in regions of strong convection, and modifications in droplet trajectories which influence droplet growth would be of importance in the growth of precipitation elements by the coalescence process.

The present study was undertaken as the first step in determining the influence of air shear on the trajectories of small cloud droplets. Specifically it defines the forces and torques experienced by droplet pairs because of the presence of shear in the fluid medium. The results are presented in the form of tables of coefficients which enable the forces and torques on each body to be computed for any orientation of shear. The decision to publish these results in the form of an NCAR Technical Note was dictated primarily by the extensive tabular material involved.
ABSTRACT

A derivation using bispherical coordinates is presented for the forces and torques acting on two unequal stationary spheres with arbitrary orientation relative to a slow viscous flow field that, in the absence of the spheres, would be a slow linear shear. Slow expansion or contraction of either sphere is also considered. The Stokes approximation to the hydrodynamic equations is assumed, with no slip on the boundaries. Computed force and torque coefficients are tabulated for a range of size ratios and relative separations. Several independent methods are derived for checking the consistency of the coefficients which, on the basis of these tests, are believed to be accurate to four or five significant figures. In the final section, the results are related to the canonical triadic formulation of Brenner (1964).
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NOTATION

A, B  Series coefficients, $A^m_n$, $B^m_n$

C  Distance between centers; $C^m_n$ used as series coefficients

$C_1$, $C_2$, ...  Force coefficients from Davis (1969)

c  Bispherical length parameter

$D_1$, $D_2$  Distances of centers to origin; $D^m_n$ used as series coefficients

E, F, G, H  Series coefficients, $E^m_n$, $F^m_n$, etc.

$F_{1x}$, $F_{2x}$, etc.  Force components on the spheres

g  Magnitude of shear (units: velocity/distance)

$g_{1a}$, $g_{1b}$, etc.  Angular factors; see Eq. (1.6)

h  $z$ coordinate of zero velocity plane; $h_1 \equiv h/R_1$, $h_2 \equiv h/R_2$

$K_1$ ... $K_{28}$  Force and torque coefficients (dimensionless), Eq. (5.1)

m, n  Indices, integers, $N = \max(n)$

$p^m_n$  Associated Legendre polynomial

Q  Function of $(r,z)$; see Eq. (3.3)

q  Vector fluid velocity

$R_1$, $R_2$  Radii of the spheres
S  Separation of surfaces of the spheres
   \( S = C_R - R \)

r  Radial cylindrical coordinate

\( T_{1x}, T_{2x}, \text{ etc.} \)  Torque components on the spheres

t  \( \cos \eta \)

u, v, w  Functions of \( (r,z) \); see Eq. (3.3)

\( W_1, W_2 \)  Velocities of spheres in z direction

x, y, z  Cartesian coordinates fixed in spheres

\( x', y', z' \)  Cartesian coordinates characterizing shear flow

\( \beta_1, \beta_2 \)  Radial velocities

\( \mu \)  Coefficient of viscosity

\( \chi, \alpha \)  Orientation angles of shear flow relative to spheres; see Fig. 1

\( \xi, \eta \)  Bispherical coordinates

\( \Lambda^m_n ( ) \)  Defined following Eq. (3.3)

\( \lambda_n (\xi) \)  \( \sqrt{2} \exp\left[ -(n + \frac{1}{2}) |\xi| \right] \)

\( \phi \)  Azimuthal angle around z axis

\( \psi(\xi, t) \)  \( (\cosh\xi - t)^{\frac{1}{2}} \)

\( \Psi \)  Flow field; see Sect. 2 for discussion
Superscripts on Force and Torque Coefficients

* Computed using flow field, Eq. (7.1)

e Applies to case of expanding or contracting sphere

D Derived from Davis (1969)

J Derived from Jeffery (1915)

SJ Derived from Stimson and Jeffery (1926), Pshenay-Severin (1958), and Maude (1961)
1. INTRODUCTION

The flow field around two stationary spheres in a viscous fluid whose motion in their absence would be a slow uniform linear shear can be found exactly in the time-independent Stokes approximation. The method is an extension of the bispherical solution for the Stokes equations first developed by Dean and O'Neill (1963). Previous work on the shear flow problem includes that of Wakiya (1967), who formulated a solution for equal spheres; Goldman, Cox, and Brenner (1967), who treated the problem of a sphere near a plane; and O'Neill (1968), who recently derived the force and torque for the limiting case of a sphere in contact with a plane. A recent paper (Davis, 1969) on slowly translating and rotating spheres is closely related to this one.

In the present paper the forces and torques on two unequal spheres are derived for a general orientation between the two-sphere system and a slow linear shear flow field. Interrelations between force and torque coefficients here and those of Davis (1969) are discussed in Sect. 6 and used as accuracy checks. The limiting case of a sphere near an infinite plane, for comparison with results of Goldman et al. (1967) and O'Neill (1968), is discussed in Sect. 7. Forces are also derived for the case of either sphere expanding or contracting, assuming that the fluid would otherwise be motionless. The method used is based on that developed by Dean and O'Neill (1963) and by Wakiya (1967). The present analysis is related to the general treatment of bodies in Stokesian shear flow by Brenner (1964) in Sect. 8.

Recently Lin, Lee, and Sather (1970) published an analysis of the problem considered here. Their study proceeds along similar lines to the present study although it would be very difficult to apply their results to physical problems other than the ones they specifically consider. Their study and this one complement each other and together constitute a full treatment of the problem.
2. GEOMETRY OF THE PROBLEM

Consider the hydrodynamic problem of two stationary spheres in an unbounded sheared viscous fluid with no slip on the boundaries. We will refer to this as "Problem A." By definition the \((x,y,z)\) coordinate system is fixed in the spheres, the \(z\) axis along their lines of center; by convection, the larger is centered on the positive \(z\) axis (Fig. 1a).

In the analysis to follow, the spheres will be characterized by the three bispherical parameters \((\xi_1, \xi_2, c)\) which are related to their radii \((R_1, R_2)\) and by center-to-center separation, \(C\); namely, if

\[
D_1 = \frac{(C^2 + R_1^2 - R_2^2)}{2C}, \quad D_2 = \frac{(C^2 + R_2^2 - R_1^2)}{2C}
\]

then

\[
c = \left(\frac{D_1^2 - R_1^2}{2C}\right)^{1/2} = \left(\frac{D_2^2 - R_2^2}{2C}\right)^{1/2}
\]

and

\[
\xi_1 = \ln(D_1 + c) - \ln R_1, \quad \xi_2 = \ln(D_2 + c) - \ln R_2
\]

\(D_1\) and \(-D_2\) are the \(z\) coordinates of the centers. The origin of the \((x,y,z)\) coordinate system, \(0\), is taken to be the origin of this bispherical system. In particular,

\[
z = c \sinh \xi (\cosh \xi - \cos \eta)^{-1}
\]

\[
x = c \sin \eta (\cosh \xi - \cos \eta)^{-1} \frac{\cos \phi}{\sin \phi}
\]

The undisturbed flow field, \(\Psi_0\), is assumed to be a linear shear flow with arbitrary orientation relative to the spheres. In the coordinate system \((x', y', z')\) the fluid velocity in the flow field \(\Psi_0\) is

\[
q = gz'i'
\]
Fig. 1 Geometry of the two spheres and the shear flow field.
where $i'$ is a unit vector in the $x'$ direction (Fig. 1b), and $g$ is the magnitude of the shear. The fluid velocity vanishes on the $(x'y')$ plane and is parallel to the $x'$ axis. In general the $(x',y',z')$ coordinate system will be displaced and twisted relative to the $(x,y,z)$ system. To express the fluid velocity in terms of $(x,y,z)$, first shift the origin along the $z$ axis a distance $h$ such that the new origin will lie in the $(x'y')$ plane, thus defining an $(x'',y'',z'')$ coordinate system, with $x'' = x$, $y'' = y$, $z'' = (z-h)$. (This will always be possible unless the $z$ axis is parallel to the $[x'y']$ plane, a special case that can be treated by superposing a constant flow in the $x'$ direction of a magnitude such that the zero plane of the combined flow moves over to the $z$ axis.) Since the origin of the $(x',y',z')$ coordinate system is at an arbitrary location on the $(x'y')$ plane, generality is not lost if we take it to be at $0'$, the point of intersection of the $z$ axis and the $(x'y')$ plane (or at $0$ in the special case just mentioned). Since the absolute orientation of the $(x,y)$ axes is also arbitrary, no loss of generality occurs if the $y''$ axis is defined to lie in the $(x'y')$ plane. Now define $\alpha$ to be the angle from the $y'$ axis to the $y''$ axis, and $\chi$ to be the angle from the $z'$ axis to the $z$ axis. Figure 1c shows these angular relationships. The flow field, $\Psi_0$ (Eq. 2.3), expressed in terms of unit vectors in the $(x,y,z)$ system, becomes

$$\mathbf{q} = g \left[ -x \sin \chi + (z-h) \cos \chi \right] \left[ \cos \alpha \cos \chi \mathbf{i} + \sin \alpha \mathbf{j} + \cos \alpha \sin \chi \mathbf{k} \right] \quad (2.4)$$

The underlying linearity of the hydrodynamic equations allows superposition of solutions, a principle much used in the present analysis. We will decompose Eq. (2.4) into separate flows and solve for the forces and torques due to each. Results can then be combined to give the force and torque that would arise in a given geometry. Write:

$$\frac{1}{g} \mathbf{q} = \left[ q_{1a} + q_{1b} + q_{1c} \right] + \left[ q_{2a} + q_{2b} + q_{2c} \right] + \left[ q_{3a} + q_{3b} + q_{3c} \right] \quad (2.5)$$
\( q_{1a} = g_{1a}^{x^1}, \quad q_{1b} = g_{1b}^{z^1}, \quad q_{1c} = h_{1c}^i \) \\
\( q_{2a} = g_{2a}^{x^j}, \quad q_{2b} = g_{2b}^{z^j}, \quad q_{2c} = h_{2c}^j \) \\
\( q_{3a} = g_{3a}^{x^k}, \quad q_{3b} = g_{3b}^{z^k}, \quad q_{3c} = h_{3c}^k \)  \( (2.6) \)

The angular factors \( g_{1a}, g_{1b} \), etc., are, from Eq. (2.4):

\[
\begin{align*}
  g_{1a} &= -\cos \chi \sin \chi \cos \alpha, &
  g_{1b} &= \cos^2 \chi \cos \alpha, &
  g_{1c} &= -\cos^2 \chi \cos \alpha \\
  g_{2a} &= -\sin \chi \sin \alpha, &
  g_{2b} &= \cos \chi \sin \alpha, &
  g_{2c} &= -\cos \chi \sin \alpha \\
  g_{3a} &= -\sin^2 \chi \cos \alpha, &
  g_{3b} &= \cos \chi \sin \chi \cos \alpha, &
  g_{3c} &= -\cos \chi \sin \chi \cos \alpha
\end{align*}
\( (2.7) \)

In "Problem A," flows \( q_{1a} \) and \( q_{3b} \) necessarily appear only in the combination \( (q_{1a} - q_{3b}) \), as can be seen from Eq. (2.7). This combined flow satisfies the equation of continuity, while \( q_{1a} \) and \( q_{3b} \) individually do not. To solve "Problem A," first determine the fluid velocity that would appear on the two surfaces \((\xi = \xi_1, -\xi_2)\) in the absence of the physical bodies, namely \( q(\xi_1), q(-\xi_2) \) from Eq. (2.4). Then pose the boundary value problem of finding the flow field that vanishes at infinity and assumes the prescribed values \(-q(\xi_1), -q(-\xi_2)\) on the two surfaces. We will refer to this as "Problem B" and the flow field as \( \psi^1 \), to distinguish it from the original flow field of Eq. (2.4), \( \psi_0 \). Superposition of solutions for flows \( \psi^1 \) and \( \psi_0 \) solves "Problem A." Forces and torques on the bodies, however, can only be due to flow field \( \psi^1 \), since \( \psi_0 \) has no singularities within the spheres (Brenner, 1962). Therefore we need only treat "Problem B" in the subsequent analysis. The function of flow field \( \psi_0 \) in this analysis is to prescribe boundary values for the fluid velocity in \( \psi^1 \).
The reinterpretation of our task, from "Problem A" to "Problem B," has one noteworthy consequence: there is no restriction that \( q_{1a} \) and \( q_{3b} \), now considered as prescribed boundary conditions for "Problem B," must occur together as \((q_{1a} - q_{3b})\). It is proper to treat them separately, since a solution for \( \psi^1 \) can in principle be found that satisfies any continuous prescribed boundary conditions on the two spheres and satisfies the equation of continuity. Even though these two flows necessarily occur together in "Problem A," as already noted, we will deal with them separately, since the connection can then be made with the problem of expanding or contracting spheres (discussed in Sect. 4).

The boundary conditions that arise from Eq. (2.6), establish that terms (1c), (2c), and (3c) are constant velocities; (1c) and (2c) are perpendicular to the z axis; and (3c) is parallel to the z axis. The forces and torques for (1c) and (2c) are available from the results of Davis (1971); forces for (3c) come directly from Stimson and Jeffery (1926).

By symmetry the force and torque coefficients for (2b) and (1b) will be similar, so only (1b) will be explicitly treated here. Flow (2a) does not produce forces on the spheres; the torques that arise if Jeffery's (1915) \( \omega_1, \omega_2 \) are related to our \( g_{2a} \) are the same as those derived by Jeffery for two spheres rotating about their lines of center:

\[
\omega_1 = \omega_2 = g_{2a}
\]

Specifically, from Jeffery (1915),

\[
T_{1z} = 4\pi c^3 g_{2a} \sum_{n=0}^{\infty} \left\{ \text{csch}^2 \left[ \left( n+1 \right) \xi_1 + n \xi_2 \right] - \text{csch}^2 \left[ \left( n+1 \right) \left( \xi_1 + \xi_2 \right) \right] \right\}
\]

(2.8)

with a similar expression for \( T_{2z} \) with the subscripts 1 and 2 interchanged. The remainder of the present analysis will be aimed at deriving the forces and torques that arise from (1a), (1b), (3a), and (3b).
3. FORMULATION FOR LINEAR SHEAR

The two-sphere problem in linear shear flow is formulated in the bispherical coordinate system in much the same manner as the problem of two spheres translating or rotating perpendicular to their lines of center. The present treatment follows those of Wakiya (1967) and Davis (1969).

With no slip on the boundaries assumed, the boundary conditions on sphere 1, $\xi = \xi_1$, for those parts of Eq. (2.6) under consideration, are:

\[ q_r = -g_{1a} r \cos^2 \phi, \quad q_\phi = g_{1a} r \cos \phi \sin \phi, \quad q_z = 0 \]  
(1a)

\[ q_r = -g_{1b} z \cos \phi, \quad q_\phi = g_{1b} z \sin \phi, \quad q_z = 0 \]  
(1b)

\[ q_r = q_\phi = 0, \quad q_z = -g_{3a} r \cos \phi \]  
(3a)

\[ q_r = q_\phi = 0, \quad q_z = -g_{3b} z \]  
(3b)

with similar boundary conditions on sphere 2.

The velocity components are decomposed in terms of functions of $(r,z)$: $Q_m, u_m, v_m, w_m$, as in Wakiya (1967). Thus,

\[ q_r = \frac{1}{2} \sum_{m=0}^{2} \left( \frac{r}{c} Q_m + u_m + v_m \right) \cos^m \phi \]

\[ q_\phi = \frac{1}{2} \sum_{m=0}^{2} \left( u_m - v_m \right) \sin^m \phi \]  
(3.2)

\[ q_z = \frac{1}{2} \sum_{m=0}^{2} \left( \frac{z}{c} Q_m + 2v_m \right) \cos^m \phi \]
This decomposition assures symmetry about the (xz) plane (where $\phi=0$), a condition met by flows (1a) and (1b) which are in the x direction, and also by (3a) and (3b) which are in the z direction and are independent of $\phi$. In terms of bispherical coordinates,

$$
\begin{align*}
 w_m &= \psi(\xi, t) \sum_{n=m}^{\infty} \Lambda_n^m (A, B, \xi) P_n^m(t) \\
 v_m &= \psi(\xi, t) \sum_{n=m-1}^{\infty} \Lambda_n^m (C, D, \xi) P_n^{m-1}(t) \\
 u_m &= \psi(\xi, t) \sum_{n=m+1}^{\infty} \Lambda_n^m (G, H, \xi) P_n^{m+1}(t)
\end{align*}
$$

(3.3)

where

$$
\begin{align*}
 t &= \cos \eta \\
 \psi(\xi, t) &= (\cosh \xi - t)^{1/2} \\
 \Lambda_n^m (A, B, \xi) &= A_n^m \exp \left[ (n+\frac{1}{2}) \xi \right] + B_n^m \exp \left[ -(n+\frac{1}{2}) \xi \right]
\end{align*}
$$
For the problem at hand, \( m = 0, 1, 2 \). The \((m=2)\) terms that arise in (1a), although required for a complete description of the flow field, will not be evaluated since they produce neither forces nor torques. When the \( m = 2 \) terms in (1a) are omitted, comparison of Eqs. (3.1) and (3.2) gives:

\[
\begin{align*}
\frac{r}{c} Q_0 + u_0 + v_0 &= -g_{1a} r \\
\frac{z}{c} Q_0 + 2w_0 &= 0
\end{align*}
\]

(1a)

\[
\begin{align*}
\frac{r}{c} Q_1 + u_1 + v_1 &= -2g_{1b} z \\
u_1 - v_1 &= 2g_{1b} z \\
\frac{z}{c} Q_1 + 2w_1 &= 0
\end{align*}
\]

(1b)

\[
\begin{align*}
\frac{r}{c} Q_1 + u_1 + v_1 &= 0 \\
u_1 - v_1 &= 0 \\
\frac{z}{c} Q_1 + 2w_1 &= -2g_{3a} r
\end{align*}
\]

(3a)

\[
\begin{align*}
\frac{r}{c} Q_0 + u_0 + v_0 &= 0 \\
\frac{z}{c} Q_0 + 2w_0 &= -2g_{3b} z
\end{align*}
\]

(3b)

Each set is to hold on each sphere, i.e., for \( \xi = \xi_1 \) and \( \xi = -\xi_2 \).
When these equation sets are solved for the four cases,

\[ Q_0 = -2c(w_0/z) \]

\[
\begin{align*}
(u_o + v_0) &= -g_{1a}r + 2w_0(r/z) \\

(1a) \end{align*}
\]

\[ Q_1 = -2c(w_1/z) \]

\[
\begin{align*}
u_1 &= r(w_1/z) \\
u_1 &= r(w_1/z) \\
(1b) \end{align*}
\]

\[ Q_1 = -2g_{3a}c(r/z) - 2c(w_1/z) \]

\[
\begin{align*}
u_1 &= v_1 = g_{3a}(r^2/z) + w_1(r/z) \\

(3a) \end{align*}
\]

\[ Q_0 = -2g_{3b}c - 2c(w_0/z) \]

\[
\begin{align*}
(u_0 + v_0) &= 2g_{3b}r + 2w_0(r/z) \\

(3b) \end{align*}
\]

for \( \xi = \xi_1, -\xi_2 \).

In addition to the equations that result from the boundary conditions, the equation of continuity imposes the relations

\[
\begin{align*}
3 + r \frac{\partial}{\partial r} + z \frac{\partial}{\partial z} Q_m + c \left[ \left( \frac{\partial}{\partial r} + \frac{m+1}{r} \right) u_m \right] \\
+ \left( \frac{\partial}{\partial r} - \frac{m-1}{r} \right) v_m + 2 \frac{\partial w_m}{\partial z} &= 0
\end{align*}
\]

\[
(3.6)
\]
Because \( q_0 \) vanishes identically for \( m = 0 \), (1a) and (3b) involve the sum \( (u_0 + v_0) \), not \( u_0 \) and \( v_0 \) individually. It would be convenient to express \( (u_0 + v_0) \) in a series resembling the \( u_0 \) series if the procedure can be shown to be valid. By Eq. (3.3), \( u_0 \) is expressible as a series in \( P_1(t) \), while \( v_0 \) is expressible as a series in \( P_{-1}(t) \) which, for \( n \geq 1 \), is proportional to \( P_1(t) \). Hence for \( n \geq 1 \), both \( v_0 \) and \( u_0 \) can be expressed as series in \( P_1(t) \).

The formal possibility of terms in the \( v_0 \) series in \( P_{-1}(t) \) and \( P_0(t) \) is ruled out because these functions are unbounded for \( t = -1 \), a value not excluded in the present problem (MacRobert, 1967; Chapter XVIII, Eqs. [11] and [53]), and so coefficients of \( (m=-1; n=-1, 0) \) terms must vanish. The \( v_0 \) series can therefore be rewritten to resemble the \( u_0 \) series and no loss of generality will result if for convenience we set

\[
(v_0 + u_0) = 2u_0
\]  

(3.7)

and so eliminate \( v_0 \) from further consideration.
The boundary condition equations (3.5), rewritten in terms of bi-spherical coordinates, and combined, become:

\[
\Lambda_n^0(C,D,\xi) = -2g_{sb} c \lambda_n(\xi) + 2 \csc h \xi \left[ \frac{(n+1)}{2n+3} \lambda_{n+1}^0(A,B,\xi) \right. \\
\left. - \cosh \xi \lambda_n^0(A,B,\xi) + \left( \frac{n}{2n-1} \right) \lambda_{n-1}^0(A,B,\xi) \right], \quad (n \geq 0)
\]

\[
\Lambda_n^0(G,H,\xi) = -\frac{1}{2} (g_{1a} - 2g_{3b}) c \csc h \xi \left[ \lambda_{n-1}(\xi) - \lambda_{n+1}(\xi) \right] \\
+ \csc h \xi \left[ \lambda_{n-1}^0(A,B,\xi)/(2n-1) - \lambda_{n+1}^0(A,B,\xi)/(2n+3) \right], \quad (n \geq 1)
\]

\[
\Lambda_n^1(C,D,\xi) = -2g_{3a} c \csc h \xi \left[ \lambda_{n-1}(\xi)/(2n-1) - \lambda_{n+1}(\xi)/(2n+3) \right] \\
+ 2 \csc h \xi \left[ \frac{(n+2)}{2n+3} \lambda_{n+1}^1(A,B,\xi) - \cosh \xi \lambda_n^1(A,B,\xi) \\
+ \left( \frac{n-1}{2n-1} \right) \lambda_{n-1}^1(A,B,\xi) \right], \quad (n \geq 1)
\]

\[
\Lambda_n^1(G,H,\xi) = 2g_{3a} c \csc h \xi \left[ \lambda_{n-1}(\xi)/(2n-1) - \lambda_{n+1}(\xi)/(2n+3) \right] \\
+ \csc h \xi \left[ \lambda_{n-1}^1(A,B,\xi)/(2n-1) - \lambda_{n+1}^1(A,B,\xi)/(2n+3) \right], \quad (n \geq 2)
\]

\[
\Lambda_n^1(E,F,\xi) = -2g_{1b} (2n+1) \lambda_n(\xi) \text{sgn}(\xi) \\
+ 2g_{3a} c \csc h \xi \left[ (n+1)(n+2) \lambda_{n+1}(\xi)/(2n+3) \right. \\
\left. - n(n-1) \lambda_{n-1}(\xi)/(2n-1) \right] + \csc h \xi \left[ \lambda_{n+1}^1(A,B,\xi) \left( \frac{(n+1)(n+2)}{2n+3} \right) \\
- \lambda_{n-1}^1(A,B,\xi) \left( \frac{n(n-1)}{2n-1} \right) \right], \quad (n \geq 0)
\]

evaluated for \( \xi = \xi_1, -\xi_2 \), where

\[
\lambda_n(\xi) \equiv \sqrt{2} \exp \left[ -(n+1) |\xi| \right], \quad \text{sgn}(\xi) \equiv \xi/|\xi|
\]
The continuity equation (3.6) provides two additional equations for both \(m = 0\) and \(m = 1\) (Wakiya, 1967). For \(m = 0\), one equation is

\[
5C_n^0 - nC_{n-1}^0 + (n+1)C_{n+1}^0 - 4n(n+1)G_n^0 \\
+ 2n(n-1)G_{n-1}^0 + 2(n+1)(n+2)G_{n+1}^0 + 2(2n+1)A_{n+1}^0 \\
- 2nA_n^0 - 2(n+1)A_{n+1}^0 = 0
\]

and, for \(m = 1\),

\[
5C_n^1 - (n-1)C_{n-1}^1 + (n+2)C_{n+1}^1 + 2E_n^1 - E_{n-1}^1 - E_{n+1}^1 \\
- 2(n-1)(n+2)G_n^1 + (n-2)(n-1)G_{n-1}^1 + (n+2)(n+3)G_{n+1}^1 \\
+ 2(2n+1)A_{n+1}^1 - 2(n-1)A_{n-1}^1 - 2(n+2)A_{n+1}^1 = 0
\]

The other two equations (in \(D_n^0, H_n^0, B_n^0; D_n^1, F_n^1, H_n^1, B_n^1\)) are analogous, with opposite signs for the \((B_0^0, B_1^1)\) terms. \(E_0^0, F_0^0\) terms do not appear because of the elimination of \(v\) by Eq. (3.7).

Thus, there are four infinite equation sets for the \(m = 0\) case, and six equation sets for \(m = 1\). Since for sufficiently large \(n\) the coefficients \(A_n^0, B_n^0, A_n^1, B_n^1\) decrease in magnitude as \(n\) increases, we assume that it is possible to choose an index \(N\) such that \(A_{N+1}^0, B_{N+1}^0, A_{N+1}^1, B_{N+1}^1\) and all succeeding coefficients are small enough to be ignored. If the equations are truncated by setting the coefficients with subscript \((N+1)\) and larger equal to zero, a finite complete set remains that can be solved recursively for the unknown coefficients \(A_0^0, \ldots, A_N^0\), etc. For a particular configuration, this process is most conveniently carried out in a digital computer by use of subroutines without deriving an explicit algebraic solution for the equations.
The forces and torques for sphere 1 are found, given the series coefficients, from the following expressions adapted from Wakiya (1967):

\[
F_{1z} = -6\pi \mu R_1 (4 \sqrt{2}/3) \sinh \xi_1 \sum_{n=0}^{N} A_n^0 \\
F_{1x} = -6\pi \mu R_1 (2 \sqrt{2}/3) \sinh \xi_1 \sum_{n=0}^{N} E_n^1 \\
T_{1y} = -4\pi \mu R_1^2 (\sqrt{2}) \sinh^2 \xi_1 \sum_{n=0}^{N} (2n+1-\coth \xi_1) E_n^1
\]

(3.10)

Forces and torques on sphere 2 are found by means of similar formulas:

\[
F_{2z} = -6\pi \mu R_2 (4 \sqrt{2}/3) \sinh \xi_2 \sum_{n=0}^{N} B_n^0 \\
F_{2x} = -6\pi \mu R_2 (2 \sqrt{2}/3) \sinh \xi_2 \sum_{n=0}^{N} F_n^1 \\
T_{2y} = 4\pi \mu R_2^2 (\sqrt{2}) \sinh^2 \xi_2 \sum_{n=0}^{N} (2n+1+\coth \xi_2) F_n^1
\]

(3.11)
4. EXPANDING AND CONTRACTING SPHERES

A different but related problem is to find the forces on two unequal stationary spheres that are slowly expanding or contracting in a viscous fluid that would otherwise be motionless. As before, time-independent Stokes hydrodynamics is assumed. If the radial velocities of the two spherical surfaces are $\beta_1$, $\beta_2$, and $\theta$ is the polar angle on either sphere, then the boundary conditions are:

\[
\begin{align*}
q_r &= \beta \sin \theta \\
q_z &= \beta \cos \theta \\
q &= 0
\end{align*}
\]

which lead to equations:

\[
\begin{align*}
u_0 &= \beta \sin \theta - (r/z) \beta \cos \theta + (r/z) w_0 \\
Q_0 &= -2c(w_0/z) + 2c\beta \cos \theta/z
\end{align*}
\]

which are analogous to Eq. (3.5).

Expressed in terms of the bispherical coordinate system:

\[
\begin{align*}
cos \theta &= (t \cosh \xi_1 - 1) (\cosh \xi_1 - t)^{-1} \text{sgn}(\xi_1) \\
\sin \theta &= \sinh \xi_1 \sqrt{1 - t^2} (\cosh \xi_1 - t)^{-1} \text{sgn}(\xi_1)
\end{align*}
\]
whence, after reduction,

\[
\Lambda_n^0(C,D,\xi) = 2\beta \text{sgn}(\xi) \coth \xi \left[ \left( \frac{n+1}{2n+3} \right) \lambda_{n+1}(\xi) - \text{sech} \xi \lambda_n(\xi) \right] \\
+ \left( \frac{n}{2n-1} \right) \lambda_{n-1}(\xi) + 2\text{csch} \xi \left[ \right], \quad (n \geq 0)
\]

\[
\Lambda_n^0(G,H,\xi) = -2\beta \text{sgn}(\xi) \coth \xi \left[ \left( \frac{n+2}{2n+3} \right) \lambda_{n+1}(\xi) - \cosh \xi \lambda_n(\xi) \right] \\
+ \left( \frac{n-1}{2n-1} \right) \lambda_{n-1}(\xi) + \text{csch} \xi \left[ \right], \quad (n \geq 1)
\]

(4.4)

evaluated for \( \xi = \xi_1, -\xi_2 \). The omitted terms in brackets are the same as analogous terms in Eq. (3.8). Forces are calculated as in Eq. (3.10).

The case of two spheres moving individually along their lines of center could be treated similarly. However, a better approach which leads to a more rapidly converging computational procedure is that of Stimson and Jeffery (1926) together with the extension made by Pshenay-Severin (1958) and Maude (1961). The relevant formulas can be found in their publications; Table 2 presents the numerical results for reference.
5. COMPUTED FORCE AND TORQUE COEFFICIENTS

The force and torque components on either sphere, for a given orientation of the two spheres with respect to the shear flow, can be found from the dimensionless coefficients $K_1 \ldots K_{20}$, identified according to Eq. (2.6) in Table 1, and tabulated in Table 2. They enable forces and torques to be calculated using the following formulas:

\[
\begin{align*}
F_{1x} &= -6\pi\mu R^2 \cos\alpha \sin^2 \chi K_2 \cos^2 \chi (K_1 + h K^D) \\
F_{1y} &= -6\pi\mu R^2 \sin \alpha \cos \chi \left[ h K^D - K_1 \right] \\
F_{1z} &= -6\pi\mu R^2 \cos \alpha \sin \chi \cos \chi \left[ (K_4 - K_5) + h K^D \right] \\
F_{2x} &= -6\pi\mu R^2 \cos \alpha \sin^2 \chi (K_7 - h K^D) \\
F_{2y} &= -6\pi\mu R^2 \sin \alpha \cos \chi \left[ h K^D - K_7 \right] \\
F_{2z} &= -6\pi\mu R^2 \cos \alpha \sin \chi \cos \chi \left[ (K_{10} - K_{11}) + h K^D \right] \\
T_{1x} &= -4\pi\mu R^3 \sin \alpha \cos \chi \left[ h K^D - K_{13} \right] \\
T_{1y} &= -4\pi\mu R^3 \cos \alpha \sin^2 \chi (K_{15} - h K^D) \\
T_{1z} &= -4\pi\mu R^3 \sin \alpha \sin \chi \left[ K^J \right] \\
T_{2x} &= -4\pi\mu R^3 \sin \alpha \cos \chi \left[ h K^D - K_{17} \right] \\
T_{2y} &= -4\pi\mu R^3 \cos \alpha \sin^2 \chi (K_{19} - h K^D) \\
T_{2z} &= -4\pi\mu R^3 \sin \alpha \sin \chi \left[ K^J \right]
\end{align*}
\]
where \( h = h/R \), \( h = h/R \). The coefficients \( K \ldots K \) are functions only of the two-sphere geometry and therefore depend only on the two dimensionless ratios \( (R_1/R_2) \) and \( (S/R_2) \).

For expanding stationary spheres, with radial velocities \( \beta_1, \beta_2 \),

\[
F(e)_{1z} = 6\pi\mu R \left[ \beta_1 K^e_{11} + \beta_2 K^e_{22} \right]
\]
\[
F(e)_{2z} = 6\pi\mu R \left[ \beta_1 K^e_{12} + \beta_2 K^e_{22} \right]
\]

while for two spheres translating along their lines of center with velocities \( W_1, W_2 \),

\[
F(t)_{1z} = 6\pi\mu R \left[ W_1 K^{SJ}_{11} + W_2 K^{SJ}_{22} \right]
\]
\[
F(t)_{2z} = 6\pi\mu R \left[ W_1 K^{SJ}_{12} + W_2 K^{SJ}_{22} \right]
\]

Coefficients \( K_{11}, \ldots, K_{20} \) are given in Table 3.

\( K^{SJ}_{25-28} \) were calculated using the equations of Stimson and Jeffery (1926), Pshenay-Severin (1958), and Maude (1961). The superscript SJ was attached to these coefficients to emphasize that they were computed using the Stimson-Jeffery theory. The superscript J on coefficients \( K^J_{16}, K^J_{20} \) indicated that they were computed using the Jeffery formula restated here as Eq. (2.8). Coefficients \( K^D_3, K^D_6, K^D_9, K^D_{12}, K^D_{14}, \) and \( K^D_{18} \) were obtained from the Cs given by Davis (1969):

\[
K^D_3 = -(C_1 + C_5), \text{ etc.}
\]

Several of the coefficients behave in an unexpected way. For example, the change in sign of \( K_{19} \) for \( (R_2/R_1) = 5, 10 \) is puzzling and it may be noted that \( K \) is not a monotonic function of \( S/R_2 \). However, as discussed below (Sect. 6), check sums (6.2) and (6.3) should give a
good indication of the accuracy of the calculations, and any error in individual coefficients should clearly appear. The check sums do not, however, indicate that any of the Ks in question are in error and so the apparently anomalous behavior is a real consequence of the nature of the flow field.
6. ACCURACY CHECKS

Although Wakiya (1967) does not carry his analysis of the shear flow problem to the point of obtaining forces and torques, he does tabulate values for individual terms in the $A_n$, $B_n$ series for several separations of equal spheres. Terms computed using the computer program developed for the work reported here check with those reported by Wakiya to four decimal places.

As shown in Davis (1969), one way that the accuracy of calculations for unequal spheres can be estimated is by using check sums from Wakiya (1967):

\[
\begin{align*}
\sum_{n=0}^{N} \left[ 2A_n^0 - (2n+1)C_n^0 \right] &= \sigma_a \\
\sum_{n=0}^{N} \left[ 2B_n^0 + (2n+1)D_n^0 \right] &= \sigma_b \\
\sum_{n=1}^{N} \left[ 2n(n+1)A_n^1 + (2n+1)B_n^1 \right] &= \sigma_c \\
\sum_{n=1}^{N} \left[ 2n(n+1)B_n^1 - (2n+1)F_n^1 \right] &= \sigma_d
\end{align*}
\]

(6.1)

all of which should vanish if carried to large enough $N$. For the results reported, these check sums turned out to be less than $10^{-4}$ in magnitude except for those coefficients that apply to the larger sphere for cases where $(S/R_2) \leq 5$. Failure of this check-sum test does not, however, necessarily invalidate the results, since they are not directly related to the quantities which we are attempting to calculate. A better test on the consistency of the results which involves the
coefficients directly is described in the next paragraphs. It should be possible to achieve any required degree of accuracy for any specified size ratio and separation provided that \( N \) could be made large enough. The way the problem was programmed, \( N \) was limited to values less than 1,000. This limitation could have been removed, but it is unlikely that the improvement in the results would justify the reprogramming effort involved.

By superposition it is possible to deduce interconnections among the force coefficients here derived and those reported by Davis (1969). If we superimpose (3a) with \( g = -1 \), (1b) with \( g = 1 \), and constant translations in the \( x \) direction with velocity \((D_1 \ g)\) for sphere 1 and \((-D_2 \ g)\) for sphere 2, the result should be the same as for two spheres rotating in a still fluid with unit angular velocities, both in the \(-y\) direction. Thus the following relationships between \( K_{ij} \) of the present paper and \( C_{ij} \) of Davis should hold:

\[
\begin{align*}
R_1 \left( K_{12} - K_{12} \right) & = \left[ \frac{D \ C - D \ C}{2} - R_2 \left( C + C \right) \right] = \delta_a \\
R_2 \left( K_{78} - K_{78} \right) & = \left[ \frac{D \ C - D \ C}{2} - R_2 \left( C + C \right) \right] = \delta_b \\
R_1 \left( K_{13} - K_{15} \right) & = 2 \left[ \frac{D \ C - D \ C}{2} - R_2 \left( C + C \right) \right] = \delta_c \\
R_2 \left( K_{17} - K_{19} \right) & = 2 \left[ \frac{D \ C - D \ C}{2} - R_2 \left( C + C \right) \right] = \delta_d
\end{align*}
\]

\( \delta_a = \delta_b = \delta_c = \delta_d = 0 \). These relationships can also be derived by treating separately the antisymmetric part of the fluid velocity gradient tensor (see Sect. 8).

Similarly the coefficients derived flow coefficients. If (1a), (3b), and coefficients for a flow \((y|)\), which will be the same as for

\[\text{Eq. 6.2}\]

\(\text{Eq. 5.3}\)

\(\text{Eq. 6.1}\)

\(\text{Eq. 5.1}\)

\footnote{The factor 2 in Eqs. (5.3) and (6.1) (the last two equations) arises from a difference in the definition of the torque coefficients between this and the author's previous paper (Davis, 1969): compare the Eqs. (5.1) of both papers.}
case (1a), are combined with constant translations of the spheres in the z direction of \((D_g)_1\) for sphere 1 and \((-D_g)_2\) for sphere 2, which are computed via the Stimson-Jeffery theory, the result should be the same as for sphere 1 contracting with \(\beta_1 = -R_1 g\), and sphere 2 contracting with \(\beta_2 = -R_2 g\). Thus the following relationship should also hold:

\[ R_1 \left[ 2K + K_e \right] + R K_e^{12} + R K_e^{22} - \left( S J D K_SJ - D K_{\text{SJ}}^{12} \right) = \delta_e \]

\[ R_2 \left[ 2K + K_e \right] + R K_e^{13} + R K_e^{24} - \left( S J D K_SJ - D K_{\text{SJ}}^{28} \right) = \delta_f \]

(6.3)

\[ \delta_e = \delta_f = 0. \]

Table 4 gives results for checks of Eqs. (6.2) and (6.3) by tabulating \(\delta_a, \delta_b, \ldots, \delta_f\) where greater in magnitude than \(10^{-7}\) using the \(C_j\) values reported in Davis (1969) and \(K_j\) values from Tables 2 and 3. The results show very good consistency except for the coefficients related by Eq. (6.3) with \((R_1 / R_2) = 5, 10\) and \((S/R_2) = 0.001\). These coefficients have therefore been enclosed in parentheses in Tables 2 and 3 to indicate that one or more of them have lower accuracy.
Unfortunately the present results cannot be directly compared with those of Goldman et al. (1967) and O'Neill (1968). It might appear that the case of a sphere near or touching an infinite plane in a shear flow parallel to the plane would be the limiting case of (1b) in the present treatment with \( R \to \infty \). However, for the undisturbed flow field defined by Eq. (2.3), if sphere 1 is significantly larger than sphere 2, fluid originally on the \( z > 0 \) side of the \((xy)\) plane will be forced over into the negative \( z \) half-space. Thus, results completely different from those of Goldman et al. and of O'Neill are obtained as \( R \to \infty \) since the flow field is not at all the same. A comparison becomes possible, however, if we assume for the undisturbed flow field that a linear shear exists from \( z = -\infty \) through increasing \( z \) to the surface of sphere 1, and that the fluid is motionless elsewhere. That is:

\[
\begin{align*}
q &= 0, \text{ for } (z-s_1) \geq 0 \\
q &= g(z-s_1)i, \text{ for } (z-s_1) < 0
\end{align*}
\]

(7.1)

where \( s_1 \) is the \( z \) coordinate of the point on sphere 1 nearest the origin \((s_1 = D - R)\). This flow field approaches that assumed by Goldman et al. and O'Neill as \( R \to \infty \). As a check on the present theory, forces and torques on sphere 2 for the flow field of Eq. (7.1) were calculated using Eq. (3.8) for case (1b) with the added stipulation that the fluid velocity vanish for \( \xi = \xi_1 \). This modification accounted for all but the shift of the zero velocity plane over to the surface of sphere 1. But such a shift is readily accomplished using the force coefficients \( K^D_{9}, K^D_{18} \). To conform with the notation of Goldman et al.

\[
\begin{align*}
F^S &= -\left[ K^*_{7} -s K^D_{19} \right] / [S+R] \\
T^S &= \left[ K^*_{17} -s K^D_{18} \right]
\end{align*}
\]

(7.2)

The superscript * indicates use of the flow field of Eq. (7.1).
The forces and torques here computed (Table 5) using the flow field of Eq. (7.1) do indeed approach those of Goldman et al. (1967) and O'Neill (1968).
The shear flow field prescribed by Eq. (2.4) can be decomposed into a pure shear and a pure (rigid) rotation of the fluid. Write the velocity gradient diad of Eq. (2.4)

\[ G = g k' l' \]  \hspace{2cm} (8.1)

as the sum of a symmetric diadic \( S \) and an antisymmetric diadic \( R \),

\[ G = S + R \]  \hspace{2cm} (8.2)

where

\[ S = \frac{1}{2} g (i' k' + k' i'), \quad R = \frac{1}{2} g (-i' k' + k' i') \]  \hspace{2cm} (8.3)

\( S \) represents a pure fluid shear, \( R \) represents a pure rotation. Because of the linearity of the problem, force and torque coefficients for these two flow fields can be obtained by linear combinations of the \( K_4 \) computed for Eq. (2.4). Since the \( C_4 \) computed by Davis (1969) apply to the case of spheres translating or rotating, or to the case of the fluid streaming uniformly or rotating near stationary spheres, the force and torque coefficients that apply to that part of the velocity field characterized by diadic \( R \) are derivable as linear combinations of appropriate \( C_4 \). Equations (6.2) display explicitly the relationships between those combinations of the \( K_4 \) that apply to the \( R \) part of the shear field and linear combinations of the \( C_4 \) that were derived by Davis for translating and rotating spheres. Had we been interested only in the \( R \) part of the original shear flow field, no new calculations would have been necessary. But those linear combinations of the \( K_4 \) that apply to the symmetric part of Eq. (2.4), characterized by diadic \( S \), are "new" in the sense that they cannot be derived from linear combinations of the \( C_4 \) of Davis.
For a body in a pure shear flow field, Brenner (1964) writes

\[ F = -\mu \phi_0 : S \]  
\[ T = -\mu \tau_0 : S \]  

where \( \phi_0 \) and \( \tau_0 \) are the pure shear force and torque triads. (The subscript, \( \text{zero} \), indicates that they refer to a particular point on the body.)

If the body has rotational symmetry, Brenner (1964) shows that \( \phi_0 \) has two independent nonvanishing components, while \( \tau_0 \) has only a single independent nonvanishing component. To conform with his convention, let subscripts \((1,2,3)\) refer to \((z,y,x)\) in our notation. Then by splitting Eq. (2.4) into \( S \) and \( R \) and comparing components with Eqs. (2.6) and (5.1), it is not difficult to show that

\begin{align*}
\phi^{(1)}_{33} &= \phi^{(1)}_{31} = \phi^{(1)}_{22} = \phi^{(1)}_{21} = -3\pi R^2 \left[ (K + K - h)_{1 \, 1 \, 3} \right] \\
\phi^{(1)}_{11} &= 6\pi R^2 \left[ (K_{1 \, 5} - h)_{1 \, 6} \right] \\
\phi^{(2)}_{33} &= \phi^{(2)}_{31} = \phi^{(2)}_{22} = \phi^{(2)}_{21} = -3\pi R^2 \left[ (K + K - h)_{2 \, 9} \right] \\
\phi^{(2)}_{11} &= 6\pi R^2 \left[ (K_{10} - K_{11})_{2 \, 12} \right] \\
\tau^{(1)}_{213} &= \tau^{(1)}_{231} = -\tau^{(1)}_{312} = -\tau^{(1)}_{321} = -2\pi R^3 \left[ (K + K - h)_{1 \, 14} \right] \\
\tau^{(2)}_{213} &= \tau^{(2)}_{231} = -\tau^{(2)}_{312} = -\tau^{(2)}_{321} = -2\pi R^3 \left[ (K + K - h)_{2 \, 18} \right]
\end{align*}
For clarity, the subscript zero has been omitted from the components of $\Phi_0$ and $\tau_0$. Instead they are identified with superscripts (1) and (2) to indicate whether they refer to sphere 1 or sphere 2.
Table 1

IDENTIFICATION OF $K_1, K_2, \ldots$ WITH EQ. (2.6)

<table>
<thead>
<tr>
<th>Case</th>
<th>Force Coefficients</th>
<th>Torque Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sphere 1</td>
<td>Sphere 2</td>
</tr>
<tr>
<td>(1b), (2b)</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>(3a)</td>
<td>2</td>
<td>8</td>
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<tr>
<td>(1c), (2c)</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>(1a)</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>(3b)</td>
<td>5</td>
<td>11</td>
</tr>
<tr>
<td>(3c)</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>(2a)</td>
<td>--</td>
<td>--</td>
</tr>
</tbody>
</table>

(a) From Stimson and Jeffrey (1926).
Table 2

FORCE AND TORQUE COEFFICIENTS FOR SHEAR FLOW

\[ \frac{R_1}{R_2} = 1.0 \]

<table>
<thead>
<tr>
<th>( \frac{S}{R_2} )</th>
<th>( K_1 )</th>
<th>( K_2 )</th>
<th>( K_3 )</th>
<th>( K_4 )</th>
<th>( K_5 )</th>
<th>( K_6 )</th>
<th>( K_7 )</th>
<th>( K_8 )</th>
<th>( K_9 )</th>
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<tr>
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<td>6.3984E+00</td>
<td>3.6321E-03</td>
<td>9.4095E-01</td>
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<tr>
<td>( = 1.0 )</td>
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<td>5.0576E-02</td>
<td>7.9493E-01</td>
<td>9.6535E-02</td>
<td>2.5837E+00</td>
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<td>-5.0576E-02</td>
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</tr>
<tr>
<td>( = 0.10 )</td>
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<td>7.8377E-02</td>
<td>7.3313E-01</td>
<td>4.5695E-01</td>
<td>2.5397E+00</td>
<td>6.5090E-01</td>
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<td>-7.8377E-02</td>
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<td>7.2475E-01</td>
</tr>
</tbody>
</table>

\[ \frac{K_1}{\theta} \]

<table>
<thead>
<tr>
<th>( \frac{S}{R_2} )</th>
<th>( K_1 )</th>
<th>( K_2 )</th>
<th>( K_3 )</th>
<th>( K_4 )</th>
<th>( K_5 )</th>
<th>( K_6 )</th>
<th>( K_7 )</th>
<th>( K_8 )</th>
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<td>2.2264E-01</td>
<td>-8.9119E-01</td>
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<td>-7.1611E+01</td>
<td>-2.2264E+01</td>
<td>-8.9119E+01</td>
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<td>2.3580E-01</td>
<td>-8.8319E-01</td>
<td>9.0258E-01</td>
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<td>9.0258E-01</td>
<td>-7.0990E+01</td>
<td>-2.3580E+01</td>
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Table 2, Continued

\[ \frac{R_1}{R_2} = 2.0 \]

<table>
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<tr>
<th></th>
<th>( S/R_2 = 10.0 )</th>
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<th>( S/R_2 = 0.10 )</th>
<th>( S/R_2 = 0.01 )</th>
<th>( S/R_2 = 0.001 )</th>
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</thead>
<tbody>
<tr>
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<td>3.5155E+00</td>
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Table 2, Continued

$R_1/R_2 = 5.0$

| $S/R_2$     | $K_1$     | $K_2$     | $K_3$     | $K_4$     | $K_5$     | $K_6$     | $K_7$     | $K_8$     | $K_9$     | $K_10$    | $K_11$    | $K_12$    | $K_13$    | $K_14$    | $K_15$    | $K_16$    | $K_17$    | $K_18$    | $K_19$    | $K_20$    |
|------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 1.0        | $1.1325E+00$ | $1.7742E-02$ | $9.4607E-01$ | $3.9105E-02$ | $1.1935E+00$ | $6.8598E-01$ | $-3.7019E+00$ | $-6.9749E-01$ | $4.4253E-01$ | $-1.0833E+00$ | $-4.3844E+00$ | $1.8608E-01$ | $8.9373E-01$ | $6.4900E-02$ | $-1.0199E+00$ | $9.9820E-01$ | $1.5958E+00$ | $-1.3766E-01$ | $6.4093E-01$ |
| 0.10       | $1.0851E+00$ | $2.1135E-02$ | $9.5007E-01$ | $8.4926E-02$ | $1.1672E+00$ | $9.7847E-01$ | $-2.8895E+00$ | $-7.1580E-01$ | $3.5167E-01$ | $-2.2030E+00$ | $-4.3919E+00$ | $1.1974E-01$ | $8.9436E-01$ | $6.5264E-02$ | $-1.0270E+00$ | $9.9810E-01$ | $1.3759E+00$ | $-1.7144E-01$ | $4.7249E-01$ |
| 0.01       | $1.0807E+00$ | $2.1378E-02$ | $9.5081E-01$ | $1.4391E-01$ | $1.2194E+00$ | $9.7957E-01$ | $-2.7993E+00$ | $-7.1311E-01$ | $3.4106E-01$ | $-3.6752E+00$ | $-5.7483E+00$ | $1.1298E-01$ | $8.9510E-01$ | $6.4818E-02$ | $-1.0236E+00$ | $9.9812E-01$ | $1.3994E+00$ | $-1.7473E-01$ | $4.5157E-01$ |
| 0.001      | $1.0807E+00$ | $2.1706E-02$ | $9.5092E-01$ | $2.0671E-01$ | $1.2825E+00$ | $9.7968E-01$ | $-2.7902E+00$ | $-7.1279E-01$ | $3.3998E-01$ | $-5.2527E+00$ | $-7.3142E+00$ | $1.1230E-01$ | $8.9984E-01$ | $6.5236E-02$ | $-1.0236E+00$ | $9.9812E-01$ | $1.4017E+00$ | $-1.7505E-01$ | $4.4944E-01$ |
Table 2, Continued

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### Table 3

FORCE COEFFICIENTS FOR EXPANSION OR CONTRACTION, AXIAL TRANSLATION

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<th>S/R$_2$ = 0.10</th>
<th>S/R$_2$ = 0.01</th>
<th>S/R$_2$ = 0.001</th>
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<td>2.4896E+02</td>
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Table 3, Continued

R₁/R₂ = 5.0

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<td>R₁/R₂ = 0.01</td>
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R₁/R₂ = 10.0

| K₂₁ 1.6123E-02 | 1.3580E-01 | 9.0856E-01 | 8.3704E+00 | (8.2783E+01) |
| K₂₂ 2.5030E-03 | 3.6776E-02 | 6.7384E-01 | 7.9725E+00 | (8.2204E+01) |
| K₂₃ -2.4288E-01 | -1.4013E+00 | -9.1133E+00 | -8.3739E+01 | (-8.2777E+02) |
| K₂₄ -3.5503E-03 | -3.0622E-01 | -6.6680E+00 | -7.9654E+01 | (-8.2199E+02) |
| K₂₅ -1.0467E+00 | -1.8126E+00 | -1.9726E+00 | -9.4669E+00 | -8.3906E+01 |
| K₂₆ -2.3760E-02 | 1.8771E-01 | 9.7641E-01 | 8.4704E+00 | 8.2993E+01 |
| K₂₇ -2.3760E-01 | 1.8771E+00 | 9.7641E+00 | 8.4704E+01 | 8.2993E+02 |
| K₂₈ -1.0704E+00 | -1.9467E+00 | -9.8035E+00 | -8.4741E+01 | -8.2913E+02 |
Table 4

CHECK SUMS, Eqs. (6.2) AND (6.3) TABULATED IF $|\delta| \geq 10^{-7}$

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Table 5

FORCE AND TORQUE COEFFICIENTS TO COMPARE WITH
GOLDMAN, COX, AND BRENNER (1967), AND O'NEILL (1968)

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(a) From Goldman et al., 1967.
(b) From O'Neill, 1968.
REFERENCES


