Spectral Properties of One-Dimensional Diffusive Systems Subject to Stochastic Forcing

H.-L. LIU

High Altitude Observatory, National Center for Atmospheric Research,* Boulder, Colorado

(Manuscript received 2 February 2006, in final form 25 May 2006)

ABSTRACT

The vertical wavenumber and frequency spectra of horizontal wind and temperature in stochastically driven systems with diffusion, either due to uniform background eddy and molecular transport, or due to adjustment processes associated with shear or convective instability, are studied. Because of the dominating role of vertical transport in a stratified fluid, one-dimensional Langevin-type equations could be ascribed to such systems in the vertical direction. The linear equation with uniform diffusion is solved explicitly, and the spectra follow power-law distributions if the stochastic force is Gaussian. The nonlinear equations with gradient (either shear or lapse rate) dependent diffusion coefficients are shown to support scale invariance, and the power-law indices of the spectra are determined from dynamic renormalization group (DRG) analysis under rather general conditions. The exact power-law indices vary with the spectrum of the stochastic force and the nonlinearity of the systems. If the wavenumber spectrum of the force is moderately red (between $k^0$ and $k^{-2}$), the spectral indices of horizontal wind and temperature and the range of their variability are in general agreement with those inferred from wind and temperature measurements. The indices in both linear and nonlinear cases are confirmed by numerical simulations. This theory may suggest an alternative explanation to the universal vertical wavenumber and frequency spectra and their variability. By relating the universal spectra to systems characterized by stochastic forcing and background diffusion or diffusive adjustment due to shear or convective instability, which are ubiquitous in a stratified fluid, the difficulty to associate the time- and location-independent spectral features directly with the highly time- and location-dependent gravity waves or wave-breaking events is avoided. If such systems are suggestive of the real atmosphere, there is a need to be cautious in making assumptions regarding gravity waves solely based on the universal spectra when analyzing and interpreting wind and temperature observations.

1. Introduction

Power-law distributions of atmospheric horizontal wind spectra versus vertical wavenumber and frequency have been well established from many observations (Endlich et al. 1969; Vinnichenko 1970; Balsley and Carter 1982; Vincent 1984; Dewan et al. 1984; Gage and Nastrom 1985; Smith et al. 1987; Tsuda et al. 1989; Collins et al. 1994; Wu et al. 2001). It is evident from these studies that the power-law distributions and their indices are quite independent of the geographical location, altitude range, and the season. They are thus termed the universal spectra by VanZandt (1982). Spectral universality has also been observed in temperature field (Tsuda et al. 1991; Hostetler and Gardner 1994) and the power-law indices are similar to those of the horizontal wind.

Several theories have been proposed to explain the universal spectra, and most of the theories made efforts to associate the universal spectra with atmospheric gravity waves, gravity wave breaking and saturation, and/or interaction of gravity waves. VanZandt (1982) pointed out that the horizontal wind power spectral density (PSD) in frequency and vertical wavenumber could be related to the PSD in horizontal wavenumber through the dispersion relation of internal gravity waves in the relevant frequency and wavenumber ranges, suggesting a possible link between the universal spectra and gravity waves. Dewan and Good (1986), Smith et al. (1987) argued that the PSD of horizontal...
wind in vertical wavenumber would have an index of \(-3\) if the gravity waves are saturated. The argument was based on the linear gravity wave saturation theory, which related the variance of the horizontal wind perturbation to the vertical wavenumber, and the vertical wavenumber PSD was derived from the variance. Hines (1991a) questioned the linear saturation theory by arguing that the strong wave–wave interaction would be expected in the spectral tail due to Eulerian advective nonlinearity. He further proposed that in a system composed of stochastic, broadband waves the Eulerian advective nonlinearity was actually responsible for the formation of the universal spectrum (Hines 1991b, 2001). The assumption of this theory, namely that each wave component is Doppler shifted by the background wind formed by the ensemble of waves and the mean flow, however, is questioned by other studies (Broutman et al. 1997; Eckermann 1997). Eckermann (1999) demonstrated numerically that linear superposition of gravity waves was capable of producing the universal spectrum in vertical wavenumber before the onset of convective instability. The merit of treating the atmospheric gravity waves as a broadband system is also debatable (Fritts and Alexander 2003).

The stochastic wave system/turbulence idea as related to the universal spectrum is also exploited from the perspectives of turbulent cascades and nonlinear diffusivity. Lumley (1964) extended the Kolmogorov inertial subrange theory to a buoyancy subrange by rendering the dissipation rate a variable due to stratification, and was able to obtain a kinetic energy spectrum that approached a slope of \(-3\) at the low wavenumber end and \(-5/3\) at the high wavenumber end. This theory was modified to include anisotropy by Weinstock (1978), and Weinstock (1985) pointed out that the spectrum of a slope between \(-2.4\) to \(-3\) could be categorized as the strong interaction range, versus the weak interaction range with a slope of \(-2\) as found in the ocean spectrum (Holloway 1981, 1983). Weinstock (1990) also found it necessary to consider the scale-dependence of the diffusion coefficient for the theory to properly resolve the observed spectrum. Based on some of these theories, spectral models have been established to constrain gravity waves and to build gravity wave parameterization schemes for general circulation models (GCM) in order to calculate deposition of momentum flux and diffusion coefficient (Fritts and Lu 1993; Gardner 1994; Hines 1997a,b; Weinstock 1990).

In these theories the universal spectra are explained in terms of gravity waves, or interactions in broadband wave system (turbulence), or both. Observations, on the other hand, indicate that the gravity waves are highly sporadic and often quasi monochromatic. For example, all-sky camera image observations of the nightglows showed that some nights there was little gravity wave activity present at all, while other nights quasi-monochromatic waves, or superposition of limited number of quasi-monochromatic waves were observed (Taylor et al. 1991, 1995; Wu and Kileen 1996; Nakamura et al. 1998), and the wave scales and characteristics could vary significantly. Gravity wave breaking was occasionally recorded (Yamada et al. 2001). Such large and seemingly random variability of the gravity wave and gravity wave breaking over space and time appears inconsistent with the universal spectra, which are nearly independent of location, height range, and time. The fact that the gravity waves are often observed to be quasi-monochromatic or sometimes completely absent (at least in the mesosphere) also calls for closer examination of the broadband wave assumption as well as assumptions concerning the strong interactions, among these waves, either through nonlinear dissipation or advection. In a case study, Sica and Russell (1999) demonstrated that even though the vertical wavenumber PSD of the kinetic energy showed a continuous power-law distribution with a spectral slope between \(-2\) and \(-3\), only a few dominant quasi-monochromatic components were identified by applying Prony analysis to the same observation and two of these components carried most of the spectral energy.

As mentioned above, frequency PSD of wind and temperature also display persistent power-law behavior, and as the vertical wavenumber PSD they are also approximately independent of time and location of the measurements. In most of the theories (VanZandt 1982 as an exception), however, the frequency spectrum is either not explicitly resolved or given a priori as an assumption. Hence it is not clear if and how the frequency spectrum is related to the same mechanism that leads to the universal vertical wavenumber spectrum. Furthermore, the power-law indices of both the vertical wavenumber and frequency spectra show variability over certain ranges from the observations. Is there any physical significance underlying the variability? If there is, it was overlooked by most of the theories mentioned above. The buoyancy subrange theory (Lumley 1964; Weinstock 1978, 1985) did predict a change of the spectral index as the vertical wavenumber decreases from \(k_b\) to \(0.1k_b\), where \(k_b\) is the wavenumber dividing the turbulence inertial subrange and the buoyancy subrange (or strong wave interaction subrange). The observed variability of the vertical wavenumber spectral index, however, seems to be even larger than this prediction (as will be shown in section 5).

In this study, we examine the statistical behavior of simplified one-dimensional (in the vertical direction)
systems with the following features: 1) They are driven by a force that is random in space and time, which can be regarded as an approximation to various perturbations, including but not limited to gravity waves; 2) The systems undergo diffusive transport. Two scenarios of diffusive transport are considered here, with the diffusive coefficient being uniform, and being dependent on the gradient of the field (horizontal wind shear or temperature lapse rate). The former can be thought to describe a system with stable background stratification and weak perturbations so that the main diffusion comes from molecular diffusion or from background eddy diffusion. With large perturbations, on the other hand, the local gradient of the superposed wind or temperature can approach or exceed a certain stability threshold, which can trigger a rapid turbulent adjustment. This adjustment process can be approximated by eddy diffusion, with the coefficient dependent on the gradient or controlled by the threshold. Because the two features mentioned above are ubiquitous in a stratified fluid, we would like to examine if the universal spectra could possibly be related to such systems. It should be noted that these features are also essential features of a Lorentzian system, which has been used to approximate the incompressible Navier–Stokes velocity field (Kraichnan 1987).

This phenomenological description emphasizes the dominating vertical transport process during the adjustment involving shear or convective instability, because the main impact of the adjustment is to restore local stability by changing the vertical gradient of horizontal wind and temperature. We thus assume that the horizontal dimensions could be ignored if the main focus is on scales larger than the turbulence scale, and reduce the system to one dimension. Furthermore, with the bulk motions due to advection (thus advective nonlinearity) and pressure gradient excluded, the statistical features of the system can then be tied solely to the linear or nonlinear diffusive adjustment.

In section 2, we will build a one-dimensional stochastic equation for the horizontal wind based on the idea described above. The vertical wavenumber and frequency spectra of the wind will then be derived, first when only the linear diffusion is considered and then when the nonlinear diffusion to the leading order is included. A similar equation for the temperature will be discussed in section 3. The spectra obtained from the analysis will be compared with numerical results in section 4. In section 5, comparisons of the vertical wavenumber and frequency spectra with observations will be made. Variability of the spectral indices as related to the stochastic forcing and stability of the background state, and the interpretation and implication of the analyses and the comparisons will be discussed. Summary and conclusions will be presented in section 6.

2. Horizontal wind equation

Here we consider a simple one-dimensional stratified (1D in the vertical direction) system, wherein the horizontal wind is subject to a stochastic acceleration and a relatively small Rayleigh friction, and diffusive adjustment that could be dependent on the value of the shear. The stochastic acceleration can be regarded as the superposition of various perturbative processes. This system can be described by

\[ \frac{\partial u}{\partial t} = -\alpha(u - u_0) + \frac{\partial}{\partial x} \left[ \nu'(u_c^2) \frac{\partial u}{\partial x} \right] + f, \]  

where \( t \) and \( x \) are temporal and spatial variables, \( u_0 \) is a prespecified background wind profile, the subscript \( x \) denotes partial derivative with respect to \( x \), and \( \nu' \) is the viscosity coefficient and a function of the value of shear, or equivalently \( u_c^2 \). The characteristic time of the linear relaxation \( 1/\alpha \) is assumed to be much longer than that of the diffusive transport, and will be omitted in the analysis while kept in the numerical model. When the background shear \( du_0/dx \) and the stochastic acceleration are small, the system is dynamically stable and the eddy viscosity is mainly from molecular and/or background eddy viscosity. Here \( \nu' \) can be assumed to be a constant \( \nu \). If, on the other hand, the background shear is large and/or the stochastic acceleration becomes strong, the total shear may approach or exceed the stability threshold [i.e., \( (du/dx)^2 > N^2/Ri \), with \( N \) being the buoyancy frequency, and \( Ri \), the critical Richardson number] and eddy viscosity is enhanced therein. Because of the difficulty to represent the threshold analytically, we adopt a formal asymptotic approximation

\[ \nu = \nu + \nu u_c^2 + O(u_c^3) \]  

(2)

to the leading order of \( u_c^2 \). The validity of this approximation in studying the scaling behavior of the system will be tested by numerical experiments and discussed in section 4. Equation (1) can then be written in the form (ignoring Rayleigh friction)

\[ \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + \lambda \frac{\partial u}{\partial x} + f, \]  

(3)

where \( \lambda \) is the coefficient of the nonlinear term. We will investigate the scaling behavior of this system.

The scaling behavior of the linear equation \( (\lambda \to 0) \) will be examined first. In spectral space, it is straightforward to find the solution of the linear equation
\[ \hat{u}(k, \omega) = G \hat{f}, \]

where \( G \) is the linear propagator
\[ G^{-1} = -i\omega + \nu k^2 \]

and \( \hat{f} \) the Gaussian noise
\[ \langle \hat{f}(k, \omega) \hat{f}^{\ast}(k', \omega') \rangle = 2\pi D k^{-\gamma} \delta(k + k') \delta(\omega + \omega'), \]

where the caret denotes complex conjugate, the angle brackets represent a two-point correlation, \( D \) measures the strength of the stochastic driving, and \( r = 0 \) for white noise. From this solution the wavenumber PSD can be derived,
\[ |\hat{u}(k)|^2 = \int_{-\infty}^{\infty} d\omega |\hat{u}(k, \omega)\hat{u}^*(k, \omega)|^2 = \frac{2\pi^2 D}{\nu} k^{-2-r}. \]

If \( r \approx -2 \), the frequency PSD can also be found as
\[ |\hat{u}(\omega)|^2 = \int_{-\infty}^{\infty} dk |\hat{u}(k, \omega)\hat{u}^*(k, \omega)|^2 \approx D \nu^{r-1/2} \omega^{-(3+r)/2}. \]

Therefore, for a system with stochastic driving and uniform diffusion the wavenumber and frequency PSD of the horizontal wind display power-law distributions. The slopes of the PSD are a function of the “color” of the Gaussian random force.

For the nonlinear equation, a perturbative method is used to analyze the scaling behavior of the system. To simplify the perturbative analysis, let \( A = u_x \), and the horizontal wind equation is then transformed into an equation of the shear,
\[ \frac{\partial A}{\partial t} = \nu \frac{\partial^2 A}{\partial x^2} + \lambda \frac{\partial^2 A^3}{\partial x^2} + f_x. \]

It is noted that Eq. (9) is similar to the simplest form of the transport equation near marginal stability in tokamak confinement physics (Diamond and Hahm 1995), which was shown to be self-organized critical (Bak et al. 1987) and display scale invariance. The similarity and differences will be further discussed in section 5.

To determine if this nonlinear system supports scale invariance and if so to quantify the invariance, the dynamic renormalization group (DRG) analysis is used (Forster et al. 1977; Hwa and Kadar 1992; Diamond and Hahm 1995). The renormalization group (RG) method has been applied to the study of scaling laws in critical phenomena (Wilson 1970), and extended to the study of dynamic equations (Ma and Mazenko 1975).

DRG is a coarse-graining procedure that can be used to study the large-scale behavior of a system characterized by scale invariance. For instance, Forster et al. (1977) applied this method to study the large-distance and long-time properties of a randomly stirred fluid. A recent review focusing on its application to turbulence analysis can be found in Smith and Woodruff (1998). Here we briefly outline the idea and the procedures of the DRG analysis. More details of the method can be found in the above references.

Fluid motions often involve a wide range of scales and scale couplings through nonlinearity. Detailed analysis of the scale couplings of such complex systems is difficult. In spite of such complexity and vastly varying details of different fluid systems, universal scaling laws have been observed over various scales, including the Kolmogorov \(-5/3\) law in the turbulence inertial range and the numerous scaling laws from the atmospheric and oceanic observations. The scaling laws suggest self-similarity over the applicable ranges and the universality indicates that details of the various flow motions are likely irrelevant for understanding such self-similarity. These are the characteristics that DRG can exploit: Because of the irrelevance of details, the small scales (above a nominal wavenumber) of a flow can be eliminated through coarse-graining and only the effects of the small scales on large scales (i.e., scale coupling across the nominal wavenumber) should be considered and represented as a correction to the large scales. Because of the self-similarity, this elimination-correction process can be iteratively applied from small scales to large scales in the applicable range. In this process (RG transformation), the small scales are systematically eliminated and what is left is only the large-scale features with small-scale correction (renormalization).

The implementation of DRG consists of two principal steps: The first one is elimination of small scales through coarse-graining, and the calculation of the modification of equation coefficients at large scales by the small scales using perturbative method. The second step involves rescaling of the modified coefficients. In physical space, the second step is to interpolate the coarse-grained quantities obtained in the first step back onto the finer scales before coarse-graining, in order to facilitate comparisons between quantities before and after the transformation. In a self-similar system, these quantities should be almost identical under the transformation. These procedures are most conveniently implemented in spectral space (Fig. 1): The problem is considered in a spectral sphere \(|\hat{k}| < \Lambda\), with \( \hat{k} \) being the wavenumber and \( \Lambda \) its nominal cutoff value. The spectral sphere is then divided into two parts, the inner...
is the linear propagator as defined in Eq. (5), and again \( \lambda \) is considered as a formal perturbation parameter and assumed to be independent of \( x \) and \( t \). This nonlinear equation has to be solved using a perturbative method (appendix A), and the modification of the propagator, the nonlinear coefficient (or vertex, because of its geometrical location in diagram method), and the field strength in the iterative process needs to be determined. In this analysis our main focus is to determine if this system is scale invariant and the power-law indices if it is, but not to explicitly calculate the coefficients. We thus further simplify the analysis by adopting the first Kraichnan–Wyld approximation in the perturbative expansion (appendix A; Morton and Corrsin 1970), where only the modification to the propagator \( G \) (through the modification of viscosity) is considered, while the vertex and the field strength are unchanged to this order. This procedure is given in appendix A, and the propagator, field strength and vertex to this order are

\[
(G^I)^{-1} = -i\omega + \nu^I k^2, \quad (11a)
\]

\[
D^I = D, \quad (11b)
\]

\[
\lambda^I = \lambda, \quad (11c)
\]

where the modified viscosity coefficient is found to be (appendix A):

\[
\nu^I = \begin{cases} 
\nu \left[ 1 + \frac{3}{2} \rho^2 \frac{\exp(r-1)l-1}{r-1} \right] & \text{if } r \neq 1 \\
\nu \left( 1 + \frac{3}{2} \rho^2 l \right) & \text{if } r = 1
\end{cases}
\]

(12)

The superscript I indicates that these are variables in the intermediate step (the first step as mentioned above) of the DRG transformation. Here the nominal cutoff wavenumber \( \Lambda \) has been set to 1, without loss of generality. The dimensionless parameter \( \rho \) can be regarded as the normalized nonlinear coupling coefficient,

\[
\rho = \frac{\sqrt{\Lambda D}}{\nu}. \quad (13)
\]

It can also be viewed as the Reynolds number for this system. From Eq. (12), it is seen that the convergence of the perturbative expansion is critically dependent on the magnitude of \( \rho \). It is thus important to study the behavior of \( \rho \) in the process of DRG transformation.

In the second step, the inner spectral sphere (with radius \( e^{-l} \)) needs to be rescaled back to the original
cutoff wavenumber (at 1, see Fig. 1), thus a scaling factor $e^l$ is multiplied to $k$,

$$k_s = e^l k,$$  \hspace{1cm} (14)

where the subscript $s$ denotes the variable as rescaled. The corresponding scalings for frequency and shear are assumed to be

$$\omega = e^{-c_1 l} \omega_s,$$  \hspace{1cm} (15a)

$$\hat{A} = e^{c_1 + c_2 + 1} \hat{A}_s,$$  \hspace{1cm} (15b)

where $c_1$ and $c_2$ are to be determined. Note that Eq. (15b) is obtained by Fourier transforming the scaling $A = e^{c_2} A_s$ in physical space. Then, by replacing $k$, $\omega$, and $\hat{A}$ in Eq. (10) with the rescaled variables $k_s$, $\omega_s$, and $\hat{A}_s$, respectively, using Eqs. (14) and (15), we can determine the scalings of the viscosity coefficient, the vortex, and the forcing strength

$$\nu = \nu e^{c_1 - 2l},$$  \hspace{1cm} (16a)

$$\lambda = \lambda e^{c_1 + 2c_2 - 2l},$$  \hspace{1cm} (16b)

$$D = D e^{c_1 - 2c_2 + r - 3l},$$  \hspace{1cm} (16c)

We can combine these results from the rescaling step with the results from the intermediate step to obtain the renormalized coefficients (denoted by the R superscript)

$$\nu^R = \nu e^{c_1 - 2l},$$  \hspace{1cm} (17a)

$$\lambda^R = \lambda e^{c_1 + 2c_2 - 2l},$$  \hspace{1cm} (17b)

$$D^R = D e^{c_1 - 2c_2 + r - 3l}.$$  \hspace{1cm} (17c)

By taking derivatives of both sides of Eqs. (17) with respect to $l$, and using the results from the intermediate step, we find the differential recursion relations of the these renormalized coefficients as

$$\frac{1}{\nu^R} \frac{d \nu^R}{dl} = c_1 - 2 + \frac{3}{2} (\rho^R)^2,$$  \hspace{1cm} (18a)

$$\frac{1}{\lambda^R} \frac{d \lambda^R}{dl} = c_1 + 2c_2 - 2,$$  \hspace{1cm} (18b)

$$\frac{1}{D^R} \frac{d D^R}{dl} = c_1 - 2c_2 + r - 3.$$  \hspace{1cm} (18c)

Equations (11), (12), and (13) have been used to obtain Eq. (18a), and $\rho^R$ is assumed to be small in the transformation. It should be noted that Eq. (18a) is valid for both $r \neq 1$ and $r = 1$. The differential recursion relation for $\rho^R = \sqrt{\lambda^R D^R} \nu^R$, which determines the convergence of the perturbative expansion as mentioned above, is then derived from Eqs. (18),

$$\frac{d \rho^R}{dl} = \frac{\rho^R}{2} \left[ \epsilon - 3(\rho^R)^2 \right]$$  \hspace{1cm} (19)

where $\epsilon = r - 1$. From Eq. (19) it is seen that $\rho^R$ has a stable fixed point at 0 when $\epsilon \leq 0$, and a stable fixed point at $\sqrt{3}$ when $\epsilon > 0$. This assures the smallness of $\rho$ in the DRG transformation, given the spectral slope is not too steep when $r > 1$. The system can therefore be renormalized and display self-similarity under DRG transformation. It is also recognized that the differential recursion relation of the nonlinear coupling coefficient is similar to that obtained by Forster et al. (1977) for their model A, even though the dynamic equation [Eq. (9)] and the nonlinearity involved are quite different from those in Forster et al. (1977).

To determine the scaling indices $c_1$ and $c_2$, we need to further examine the recursion relations reflected in Eq. (18). It is plausible to assume that the work done to the system by the stochastic driving does not change under the transformation. Then from Eq. (18c) we have

$$c_1 - 2c_2 + r - 3 = 0,$$  \hspace{1cm} (20)

when $l \to \infty$. It is not immediately evident if we could assume that either the viscosity or the nonlinear coefficient is invariant under DRG transformation. Under the latter assumption and for $\epsilon < 0$, however, the renormalized viscosity is scale-dependent and diverges because $\rho^R \to 0$ as $l \to \infty$. For $\epsilon \equiv 0$, on the other hand, the viscosity and the nonlinear coefficient have the same scaling because $\rho^R$ converges to a nontrivial constant, and either assumption yields the same scaling indices in conjunction with Eq. (20). To avoid the divergence of the viscosity, we thus assume that the viscosity is invariant under DRG transformation [the same assumption made by Forster et al. (1977)]. Then two sets of scaling indices are obtained for $\epsilon < 0$ and $\epsilon \equiv 0$ with the crossover at $r = 1$:

$$c_1 = \begin{cases} 2 & \text{if } \epsilon < 0 \\ 2 - \epsilon/2 & \text{if } \epsilon \equiv 0 \end{cases}$$  \hspace{1cm} (21)

$$c_2 = \begin{cases} \epsilon/2 & \text{if } \epsilon < 0 \\ \epsilon/4 & \text{if } \epsilon \equiv 0. \end{cases}$$  \hspace{1cm} (22)

When $\epsilon < 0$, the renormalized nonlinear coefficient (and thus the nonlinear term) approaches zero as $l \to \infty$. From $c_1$ and $c_2$, the scalings of $A$ and $u$ can be determined. From Eqs. (14) and (15) we can rewrite $\hat{A}$ as

$$\hat{A}(k, \omega) = k^{-(c_1 + 2c_2 + 1)} f_A \left( \frac{\omega}{k^{c_1}} \right),$$  \hspace{1cm} (23)

where $f_A$ is a scaling function. The scaling of the correlation function of $u$ can then be found to be
\[
C_u(k, \omega) = \frac{1}{2\pi} \frac{\langle \hat{u}(k, \omega) \hat{u}^*(k', \omega') \rangle}{\delta(k + k') \delta(\omega + \omega')}
\approx k^{-c_1-2c_2-3} \mathcal{J}_u \left( \frac{\omega}{k^{c_1}} \right),
\]
where \(\mathcal{J}_u\) is again a scaling function. The correlation function, Eq. (24), can be integrated over frequency (wavenumber) to obtain the wavenumber (frequency) PSD of \(u\) for the nonlinear equation

\[
S_u(k) \propto \begin{cases} 
  k^{-2-r} & \text{if } r < 1 \\
  k^{-(5+r)/2} & \text{if } r \geq 1 
\end{cases}
\]
\[
S_u(\omega) \propto \begin{cases} 
  \omega^{-(3+r)/2} & \text{if } r < 1 \\
  \omega^{-8(5-r)} & \text{if } r \geq 1.
\end{cases}
\]  

Both wavenumber and frequency PSD show the same scaling as in the linear case [Eqs. (7) and (8)] for \(r < 1\), consistent with the diminishing nonlinear coefficient and nonlinear term in the DRG transformation. For \(r > 1\), the scalings of the nonlinear equation is different than those in the linear case. At the crossover value of \(r = 1\) (corresponding to a stochastic driving with a pink spectrum), the spectral slopes of the wavenumber PSD in both linear and nonlinear cases are \(-3\) and those of the frequency PSD are \(-2\).

### 3. Temperature equation

A 1D temperature equation can be constructed similarly, using convective instability as a threshold to replace the threshold due to shear instability in the horizontal wind equation. Specifically, eddy diffusion will be turned on whenever \(dT/dx < \Gamma\), the adiabatic lapse rate. This system, with Newtonian cooling and a stochastic driving, can be described by

\[
\frac{\partial T}{\partial t} = -\alpha(T - T_0) + \frac{\partial}{\partial x} \left[ \mathcal{K}(T) \frac{\partial T}{\partial x} \right] + f. \tag{27}
\]

It should be noted that the eddy heat flux here may not only be a parameterization of turbulent heat flux, but also of the sensible heat flux of breaking gravity waves (e.g., Walterscheid 1981; Liu 2000). In the linear case, this equation is the same as the linear horizontal wind equation and the scaling behavior is thus exactly the same [Eqs. (7) and (8)]. When nonlinearity and gradient dependency is considered, it is recognized that the eddy diffusion coefficient \(\mathcal{K}\) is dependent on the gradient of the field rather than on the value of the gradient, and this dependency is valid only when \(T_s < 0\). Therefore, the eddy diffusion coefficient may be approximated to the leading order by \(\mathcal{K} = \kappa + \lambda T_s + O(T_s^2)\) (\(\lambda < 0\)) if \(T_s < 0\) everywhere in the domain. This will be the case when the background lapse rate is large and the stochastic forcing is very weak. Because of the weak forcing, it is most likely that the problem can be treated as a linear one. In reality, however, it is quite unlikely that the temperature gradient is negative everywhere in the domain.

If the temperature gradients change signs in the domain, higher-order terms have to be considered, \(\mathcal{K} = \kappa + \lambda_1 T_s + \lambda_2 T_s^2 + O(T_s^3)\). To exclude negative diffusion, the second-order term (and in general the sum of terms of even orders) should dominate near \(T_s = 0\). The scaling of temperature will then be similar to that of the horizontal wind, if we only take the second-order term into consideration. We also note that if the second- and higher-order terms are truncated in the expansion of \(\mathcal{K}\) the resulting temperature equation will be unphysical when \(T_s > 0\), and this is reflected in the DRG analysis. For \(r \leq 1\), the nonlinear correction of the diffusion coefficient diminishes in the DRG transformation and the scaling is the same with the linear equation, as in the case of the horizontal wind. For \(r > 1\), however, the DRG transformation does not have any stable fixed point which indicates that the nonlinearity cannot be renormalized. This is because for \(r > 1\) the nonlinear term become comparable to the linear diffusive term so that the total diffusion coefficient can become negative (see appendix B).

The phenomenological description of the temperature by Eq. (27) only considers the diffusive process in the vertical direction and ignores the advective transport by the wind. It is thus not in any way coupled to the horizontal wind Eq. (1), and is a different approach from that presented in Forster et al. (1977).

### 4. Comparison with numerical experiments

Numerical experiments have been conducted to test the analyses above. In these numerical experiments, three Gaussian spectra with different colors are used as the stochastic forcing: \(r = 0\) (white), \(r = 1\) (pink), and \(r = 2\) (red) for cases with \(r\) below at, and above the crossover value of 1, respectively. Each of these spectra are applied to the horizontal wind equation with linear viscosity (linear case), nonlinear viscosity [Eq. (3)] (nonlinear case), and viscosity controlled by a threshold shear value (threshold case). To numerically integrate these equations, Euler–Maruyama method (e.g., Kloeden and Platen 1999) is used for the time integration for a given time discretization \(\Delta t\). The diffusion term, as the deterministic term in the Euler–Maruyama method, is treated explicitly at the current time step, although the time stepping may be split into appropriate smaller steps to assure numerical stability. The
δ-correlation of the stochastic forcing is thus valid up to the given time discretization ∆t. For each step of the simulations, the Gaussian noise is produced using the Box–Muller method (Press et al. 1996). The constant viscosity coefficient for the linear case is set to 50 m² s⁻¹. In the nonlinear case, the diffusion coefficient \( \nu' = 1 + 333(u/c_i) \) m² s⁻¹ where \( c_i \) is the threshold value \( C_i^2 = N^2/Ri_c \). In this case \( \nu' \) is capped at 300 m² s⁻¹ so that the time step will not be too small. In the threshold case, the diffusive coefficient \( \nu' \) is set to a background value of 1 m² s⁻¹ if \( u_c^2 \leq C_i^2 \) and 300 m² s⁻¹ otherwise. The buoyancy frequency \( N \) is assumed to be a constant 0.015 s⁻¹, and the critical Richardson number \( Ri_c \) is 0.25. The threshold value of the wind shear is thus 0.03 s⁻¹ in this numerical experiment. The initial wind velocity is 10 m s⁻¹, and the relaxation coefficient is 1 × 10⁻⁶ s⁻¹. These numerical values are chosen to reflect the terrestrial atmosphere and otherwise quite arbitrary.

The integration for each experiment is carried for 40 000 s, with the results from the last 20 000 s being used to calculate the wavenumber and frequency PSD. The computational domain includes 2048 grid points with grid size of 20 m. The wavenumber (frequency) spectra at multiple times (locations) are averaged to reduce the variance of the PSD. The PSD from these numerical experiments, each normalized by the spectral value at wavenumber (frequency) 1, are shown in Fig. 2. In the linear case, the wavenumber and frequency PSD display power-law distributions and the slopes corresponding to the three prescribed drivings all agree well with the values obtained from Eqs. (7) and (8) (except at the lowest wavenumbers and frequencies probably due to distortion by FFT and finite size cutoff).

The PSD in the nonlinear case also show good agreement with the scaling indices given by Eqs. (25) and (26). What is significant is that the numerical experiments confirm the change of scaling of wavenumber PSD at the crossover value of \( r = 1 \) for the nonlinear case predicated by the DRG analysis: the scaling is the same as the linear case below this value while displays shallower slopes than the linear case above this value. This change of scaling is also expected for the frequency PSD according to Eq. (26), though it is difficult to discern from the numerical experiments. The agreement between the numerical results and the DRG analysis for the nonlinear case implies that the first Kraichnan–Wyld approximation and the distant interaction assumption (see appendix B for details), used in this DRG analysis as well as in previous studies of turbulence (Kraichnan 1987, and references therein), are valid at least for this one-dimensional system. The wavenumber PSD in the threshold case are similar to those in the nonlinear case, with a similar change of scaling at the crossover value. The similarity between of the spectra from the threshold and nonlinear cases indicates that approximating the former by the latter in our analysis [Eq. (2)] is a reasonable approach for studying the general scaling behavior of the system. It is also recognized that the wavenumber PSD in the nonlinear case and the threshold case show larger variance with pink and red driving.

Similar numerical experiments are also carried out for temperature Eq. (27). The eddy diffusion coefficient is set to be proportional to the temperature gradient when it is negative, and to a background value otherwise. Figure 3 shows the wavenumber and frequency PSD from a numerical experiment where the background atmosphere is isothermal, so that the total temperature gradient fluctuates around 0 in the domain. The scalings of the spectra are similar to the nonlinear case of the horizontal wind equation. On the other hand, wavenumber PSD of temperature perturbation has a steeper slope near \(-4\) when \( r = 2 \) (similar to the linear case) if the background lapse rate is set to a very large value so that total temperature gradient is negative at most places (not shown). These results are consistent with the discussion in section 3.

In the numerical experiments, we also tested various stochastic driving that is not strictly Gaussian. For example, at each time step a random location is selected and acceleration (or heating) with a random strength is applied to a certain area around this location. The wind and temperature also display robust power-law scaling in these numerical experiments, though the indices show more variability.

5. Discussion

From the analyses and the numerical experiments in sections 2, 3, and 4, we see that the power-law slopes of the vertical wavenumber and frequency PSD are dependent on the spectrum of the stochastic driving and nonlinearity of the equation. When the spectrum of the stochastic driving changes from white (\( r = 0 \)) to red (\( r = 2 \)), the slope of the wavenumber PSD varies between \(-2\) and \(-4\) and that of the frequency PSD between \(-1.5\) and \(-2.5\) for the linear case. In the nonlinear case and the threshold case, the ranges of variation of the spectral slopes are between \(-2\) and \(-3.5\) (wavenumber) and \(-1.5\) and \(-2.6\) (frequency). At the crossover value of \( r = 1 \), corresponding to the pink spectrum of the stochastic driving, the wavenumber PSD has a slope of \(-3\) and frequency PSD has a slope of \(-2\). The
spectra are also affected by the background atmosphere, because the background shear and background lapse rate influence the atmospheric stability and thus the eddy adjustment.

When examining the vertical wavenumber PSD of the horizontal wind from previous measurements, it is found that the reported spectral slopes are around −3 but vary in the range between −2 and −4: −3 (Endlich et al. 1969; VanZandt 1982), −2.9 to −3.2 (Collins et al. 1994), −2.04 to −3.92 (Wu et al. 2001). In the study by Wu et al. (2001), the PSD slopes were calculated from 33 foil chaff rocket measurements, and the slope averaged over all 33 measurements was −3.07 ± 0.44. Among the 33 measurements, 29 are in the range be-
between -2 and -3.5. For the frequency PSD, most of the reported values are near -2 and vary between -1.5 and -2.5: -5/3 (-1.67) (VanZandt 1982), -1.8 to -2.4 (Collins et al. 1994), -2 (Hertzog and Vial 2001). The vertical wavenumber and frequency PSD of temperature are similar to those of the horizontal wind. The vertical wavenumber spectra from radiosonde measurements by Tsuda et al. (1991) showed spectral slopes between -2.91 and -3.24 in the troposphere and shallower slopes between -2.16 and -3.06 in the stratosphere. Stratospheric temperature from balloon measurements by F. Dalaudier (Staquet and Sommeria 2002) had a slope of -3 down to the range of ~10 m. Airborne lidar temperature measurements by Hostetler and Gardner (1994) showed vertical wavenumber spectra with slope between -2.3 and -2.7 in the upper stratosphere and between -2.9 and -3.3 in the upper mesosphere. The vertical wavenumber and frequency spectra from simultaneous ground observations displayed slopes of -2.6 (upper stratosphere)/-2.8 (upper mesosphere) and -1.8, respectively.

From these comparisons, it is seen that the observed slopes and the range of their variability of both wavenumber and frequency PSD are in general agreement with the results from our analyses and numerical experiments, assuming that wavenumber spectrum of the stochastic driving is moderately red. This assumption indicates that the forcing is more prominent at large scales, which is physically plausible. This general agreement thus points to a possible cause of the power-law PSD structure and the cause of the variability of the power-law indices in terms of the variability of the forcing and nonlinearity, with the latter being determined by the gradient dependent diffusive adjustment, which is in turn closely associated with the strength of the forcing and the background atmosphere. Furthermore, because viscosity/diffusion either caused by molecular transport, background eddy adjustment, and/or eddy adjustment associated with instability is a ubiquitous feature in a stratified atmosphere irrespective of specifics of fluid motion, it is plausible to tie them to the universality of the spectra.

It is of course open to further discussion whether this theory represents the physics behind the universal spectra, and these highly simplified models certainly cannot reproduce all the physics. On the other hand, in these reduced systems with only limited physics, namely diffusive adjustment in a stochastic system, and by excluding the bulk motion due to advection and pressure gradient, we hope to isolate the possible causes of the universal spectra. Even though gravity wave is not a resolvable feature in the systems defined by Eqs. (1) and (27), the PSD obtained from these systems are in reasonable agreement with the observations. The implication, therefore, is that the gravity waves, wave breaking, and wave interaction may not be the direct cause leading to the universal power-law distributions, and the PSD with such distributions obtained from observations do not necessarily bear information of gravity wave spectra.

We will now discuss some assumptions involved in this analysis and how they may affect the analytical results. First, it is worth noting that the scaling indices are closely related to the form of the diffusion. In this study, the adjustment threshold considered is tied to the gradient of the variables (signed or absolute value). The transport flux here is thus proportional to the gradient [Fick’s law; Eqs. (1) and (27)], and the form of the diffusion is of second order [Eqs. (3)]. If, on the other hand, curvature is used as the threshold, then the form of the diffusion is of fourth order (hyperdiffusion or biharmonic diffusion). Such a system is discussed in Liu...
et al. (2002), and the values of its scaling indices are different than those shown here.

In section 2, it is assumed that the viscosity is invariant under DRG transformation both below and above the crossover value of $r = 1$. From this assumption, the scaling indices $c_1$ and $c_2$ are determined and the change of scaling at $r = 1$ in the nonlinear case is revealed. The numerical experiments presented in section 4 confirm these analytical results, and thus confirm the scale independence of the transport coefficient. This scale independence stems from the renormalizability and the dominance of the viscous (diffusive) term when $r < 1$, or being comparable to the nonlinear term when $r \geq 1$ in the process of the renormalization. The scale independence of the diffusivity is also implied in Forster et al. (1977).

Weinstock (1985) pointed out that the slope of the vertical spectrum could be a manifestation of the strength of interactions of either gravity wave or turbulence, and that the slope between $-2.4$ and $-3$ fall in the subrange of strong interactions. This is somewhat analogous to the nonlinear case in this theory. The wavenumber spectrum in the linear case, on the other hand, is similar to the wave spectrum in the weak wave subrange (Holloway 1981, 1983). However, in this study, the interaction in the nonlinear case should still be considered a weak interaction because the spatial and temporal scales under consideration are much larger than those relevant to turbulent structures. For the same reason, the turbulent adjustment should be regarded as sporadic and intermittent in the spatial and temporal scale we are interested in. This is also a necessary assumption for the one-dimensional simplification: the horizontal transport will become important if the horizontal distribution of turbulent regions is contiguous or nearly contiguous.

As mentioned in section 2, the mathematical form of the systems studied here bear apparent similarity to the equation proposed by Diamond and Hahm (1995), which was shown to display self-organized criticality (SOC). Both systems are characterized by a stochastic driving and sporadic diffusive adjustment due to threshold instability. However, an important feature in SOC systems is their overall marginal stability and the bursty discharge events on large scales: exponential growth of perturbations on large scales triggered by local adjustment. The atmosphere, on the other hand, is not in a marginal stable state under general condition. It could be locally in the troposphere and mesosphere, considering their temperature lapse rates and the latter being a surf zone for gravity wave and wave breaking. It is not clear if diffusive adjustment due to convective/dynamical instability in the atmosphere can trigger such discharge event and on what scales.

6. Summary and conclusions

In this study, we investigate the spectral structures of a diffusive system driven by stochastic forcing. The purpose is to examine if such systems support scale invariance and what are the implications for the study of spectra of atmospheric wind and temperature. Two simple systems with such characteristics are postulated and analyzed. These systems phenomenologically describe atmospheric horizontal wind and temperature, respectively, in the vertical direction, and the viscosity/diffusion is either a constant or dependent on threshold instability (shear or convective instability). It is found that such systems can indeed support scale invariance, and the actual power-law indices are closely related to the nonlinearity of the system and the spectrum of the stochastic forcing. The scaling indices are either solved explicitly (linear case), or inferred by the dynamical renormalization group (DRG) method, and tested by numerical models. In the DRG analysis, the systems with threshold instability are approximated to the leading order by nonlinear diffusive equations. For the linear case, the scaling index is $-2 - r$ for the wavenumber PSD and $-(3 + r)/2$ for the frequency PSD, with $k^{-r}$ being the wavenumber scaling of the Gaussian noise. The scaling for the nonlinear case, as determined from the DRG analysis, changes at a crossover value of $r = 1$. For $r < 1$, the viscous/diffusive term dominates in DRG transformation and the scaling is the same as the linear case. For $r > 1$, the nonlinear term is comparable to the viscous/diffusive term, and the scaling for wavenumber and frequency PSD are $-(5 + r)/2$ and $-8/(5 - r)$, respectively. For all valid values of $r$ the viscosity/diffusivity is scale invariant under DRG transformation. These values, especially the change of scaling in the nonlinear case, are confirmed by numerical experiments for stochastic driving with white and moderately red spectra ($r = 0$, $r = 1$, and $r = 2$). The numerical experiments also demonstrate that the scaling indices in the case with threshold diffusive adjustment are similar to the nonlinear case, indicating the validity to approximate the former by the latter when studying general spectral properties.

By examining reported vertical wavenumber and frequency PSD scaling from horizontal wind and temperature measurements, it is seen that the average values are about $-3$ and $-2$ with the ranges of variation between $-2$ and $-4$, and $-1.5$ and $-2.5$, respectively. The scaling indices of both vertical wavenumber and frequency PSD derived from this analysis are in reason-
able agreement with these observed values if the wave-number spectrum of the stochastic driving is moderately red (0 < r < 2). At the crossover value of r = 1, the analysis yields the average values of the observed spectral indices (−3 for wavenumber PSD and −2 for frequency PSD).

The implication of this agreement for the study of atmospheric universal spectra is twofold: It may suggest an alternative explanation to the universal spectra and its variability, and it circumvents the difficulty in previous studies to associate the persistent universal spectra directly to the highly intermittent and sporadic gravity waves or wave-breaking events, as well as the evident discrepancy between the broadband wave assumption and the often discrete and quasi-monochromatic feature of observed waves. This analysis also demonstrates that the universal spectrum can be produced in these simplified systems that do not support gravity wave propagation or breaking, and that the universal spectrum is possibly a statistical feature of the system with certain characteristics (diffusive adjustment in a stochastic system), rather than a statistical feature of the forcing. It thus cautions against the use of the universal spectra alone to gauge gravity wave activities without careful examination of other parameters. Indeed, the pitfall of interpreting universal spectra in terms of gravity waves was demonstrated by Sica and Russell (1999). The same caution was also raised by Fritts and Alexander (2003) by summarizing the numerous observations demonstrating continuous spectra and discrete waves.

**Acknowledgments.** The author thanks three anonymous reviewers for insightful and stimulating comments. The author’s effort is in part supported by the NASA Sun Earth Connection Theory Program (S-13753-G).

**APPENDIX A**

**Calculation of the Modified Propagator: Eq. (10)**

Equation (10) could be formally solved by expanding $A$ in a power series in $\lambda$, $\sum_{i=0}^{\infty} A_i$, with $i$ being the order of $\lambda$; $A_i(i > 0)$ can be expressed by the lower-order terms and ultimately by the linear solution ($i = 0$) in terms of the linear propagator $G$ and random force $f$. It turns out that the expansion can be collected into subgroups of related terms, each containing terms of all orders in $\lambda$, a process referred to as consolidated expansion. Consolidated equations could then be formulated for the generalized propagator, the generalized strength of the force, and the generalized vertex (e.g., Morton and Corrsin 1970). Conceivably the expansion becomes increasingly complex with increasing orders, and it is difficult to recognize and consolidate structurally related terms. As an alternative, diagrammatic representation has often been used for the perturbative analysis. The diagrams are simply compact notations for representing the terms in the expansion series, and, as noted by Morton and Corrsin (1970), the diagrammatic representations are used for three reasons: 1) It is much easier to write down the diagrams for the higher-order terms than to write out the terms themselves. 2) These diagrams prove to be a viable language for expressing equations for the statistical properties; and 3) they are heuristically valuable for combining terms according to their structural properties for consolidated expansion series. Morton and Corrsin (1970) also gave a detailed treatment of the diagrammatic method and construction of consolidated equations using this method. For weakly nonlinear systems ($\lambda$ is small), the expansion series are likely to converge, in which case asymptotic solutions may be found from properly truncated consolidated equations. A low-order truncation, the first Kraichnan–Wyld approximation discussed in Morton and Corrsin (1970), is used in the current analysis.

Figure A1 defines the diagrammatic elements and displays Eq. (10) in diagram form. The superscript I denotes the intermediate step of the renormalization procedure, as explained in section 2, which the consolidated expansions intend to approximate. The diagrammatic representation of the first Kraichnan–Wyld expansion is shown in Fig. A2. As seen from the diagrams, the first Kraichnan–Wyld approximation includes the first two terms of the consolidated expansion of the
generalized (modified) propagator, but only the first terms of the expansions of vertex and forcing strength,

\[ G^4(k, \omega) = G(k, \omega) - 3G(k, \omega)\lambda k^2 M^2 G^2(k, \omega), \]  
(A1)

where

\[ \mathcal{M} = \frac{1}{(2\pi)^2} \int_{k_1, \omega_1} dk_1 d\omega_1 G^4(k_1, \omega_1)G^4(-k_1, -\omega_1) \]

\[
\langle -i k_1 \hat{f} \hat{k}_1 \hat{f}^* \rangle \tag{A2}
\]

and \( G \) is the linear propagator [Eq. (5)]. From Eq. (A1) we have

\[ (G^4)^{-1} = G^{-1} + 3\lambda k^2 \mathcal{M} \]

\[ = -i \omega + (\nu + 3\lambda \mathcal{M}) k^2, \]
(A3)

which indicates

\[ \nu = \nu + 3\lambda \mathcal{M}. \]
(A4)

Reinsert Eq. (A3) in Eq. (A2), and using Eq. (6) we have

\[ \mathcal{M} = \frac{D}{2\pi} \int_{k_1, \omega_1} dk_1 d\omega_1 \frac{k_1^{-r} - r}{\omega_1 + (\nu + 3\lambda \mathcal{M}) k_1^2}. \]  
(A5)

By integrating with respect to \( \omega_1 \) over \( -\infty \) to \( +\infty \) we have

\[ \mathcal{M} = \frac{D}{2} \int_{e^{-l}}^{1} dk_1 \frac{k_1^{-r} - r}{\nu + 3\lambda \mathcal{M}}. \]  
(A6)

The integration range of the wavenumber \( k_1 \) is over the spectral shell (small scales), \( e^{-l} < k_1 < 1 (l > 0, \text{and the nominal cutoff wavenumber has been set to 1 without loss of generality}) \) to account for the small-scale modification to the large scales,

\[ \mathcal{M} = \begin{cases} 
\frac{D}{2} \left( \frac{1}{\nu + 3\lambda \mathcal{M}} \right)^{-l} - 1 & \text{if } r \neq 1 \\
\frac{D}{2} \frac{1}{\nu + 3\lambda \mathcal{M}} & \text{if } r = 1
\end{cases} \tag{A7}
\]

\( \mathcal{M} \) is then solved from the resulting quadratic equation. By plugging \( \mathcal{M} \) back into Eq. (A3), it is found that among the two solutions of \( \mathcal{M} \) the positive solution yields a modified viscosity \( \nu + 3\lambda \mathcal{M} \) that is a small perturbation to the original \( \nu \), while the negative solution yields a near zero viscosity. The positive solution is thus selected,

\[ \mathcal{M} = \frac{D}{2} \left( \frac{1}{\nu + 3\lambda \mathcal{M}} \right)^{-l} - 1 \quad \text{if } r \neq 1 \\
\frac{D}{2} \frac{1}{\nu + 3\lambda \mathcal{M}} \quad \text{if } r = 1 \tag{A8}
\]

where \( \rho = \sqrt{\lambda D/\nu} \) as defined in Eq. (13) and assumed to be small in the approximation. The modified viscosity [Eq. (12)] is then determined from Eqs. (A4) and (A8).

APPENDIX B

Recursion Relation of the Temperature Equation

If we approximate \( \mathcal{R} \) by \( \kappa + \lambda T_x \), the Eq. (27) becomes

\[ \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} + \lambda \frac{\partial T_x}{\partial x} + f. \]  
(B1)

It is necessary to assume that

\[ \kappa > \left| \frac{\partial T_x}{\partial x} \right|, \]  
(B2)

so that the total diffusion coefficient is positive. By taking partial derivative of Eq. (B1) with respect to \( x \), we get the equation of temperature gradient \( B = T_x \). Its diagrammatic representation and the first Kraichnan–Wyld expansion is shown in Fig. B1. The propagator modification translates into

\[ G^4(k, \omega) = G(k, \omega) + 4G(k, \omega) \frac{(-\lambda k^2)}{(2\pi)^2} \mathcal{G}^2(k, \omega), \]  
(B3)
and $G^i$ can be solved from Eq. (B5).

$$ (G^i)^{-1} = -i\omega + \left( \frac{2\lambda^2 D}{\pi} \right) k^2. \tag{B9} $$

Combining Eqs. (B8) and (B9) the modified diffusion coefficient can be obtained

$$ \kappa' = \begin{cases} 
\kappa \left( \rho \frac{1 - e^{-r^{-1/3}}}{1 - r} \right) & \text{if } r \neq 1, \\
\kappa(1 - \rho^r) & \text{if } r = 1
\end{cases} \tag{B10} $$

where $\rho = \lambda^{2/3} D^{1/3}/\kappa$. Considering $\lambda^i = \lambda$ and $D^i = D$ from Fig. B1, and using Eq. (B10), the differential recursion relation of the renormalized $\rho$, $\rho^R = (\lambda^R)^{2/3}$ $(D^R)^{1/3}/\kappa^R$ is obtained:

$$ \frac{d\rho^R}{dl} = \frac{\rho^R}{3} [ \epsilon + 3(\rho^R)^2], \tag{B11} $$

with $\epsilon = r - 1$. At $r \leq 1$, $\rho^R$ has a stable fixed point at 0, indicating the diminishing significance of the nonlinear term in DRG transformation to larger scales. This is consistent with the assumption that the linear term is dominating [Eq. (B2)]. For $r > 1$, $\rho^R$ does not have any fixed stable point, and $\rho^R$ actually diverges in the DRG transformation. This implies the increasing significance of the nonlinear term, which violates the assumption expressed in Eq. (B2) and can lead to negative diffusion.

REFERENCES


