Comparison of Polarimetric Radar Drop Size Distribution Retrieval Algorithms

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ABSTRACT

Recently, two physically based algorithms, the “beta” (β) method and the “constrained-gamma” method, have been proposed for retrieving the governing parameters of the gamma drop size distribution (DSD) from polarimetric radar measurements. The β method treats the drop axis ratio as a variable and computes drop shape and DSD parameters from radar reflectivity (Z), differential reflectivity (Z_{DR}), and specific differential phase (K_{DP}). The constrained-gamma method assumes that the axis ratio relation is fixed and computes DSD parameters from reflectivity, differential reflectivity, and an empirical relation between the DSD slope and shape parameters. In this paper, the two approaches are evaluated by comparing retrieved rain DSD parameters with disdrometer observations and examining derived fields for consistency. Error effects on the β method retrievals are analyzed. The β approach is found to be sensitive to errors in K_{DP} and to be inconsistent with observations. Large retrieved β values are found to associate with large retrieved DSD shape parameters and small median drop diameters. The constrained-gamma DSD method provides reasonable rain DSD retrievals that agree better with disdrometer observations.

1. Introduction

Accurate remote estimation of rainfall has long been a goal of radar meteorologists. For decades rainfall estimates were derived from a single radar measurement—radar reflectivity factor (Z). Success was limited and progress impeded by the variety of raindrop size distributions (DSDs) and sampling errors. Recent research has turned to polarimetric radar which provides additional parameters—for example, differential reflectivity (Z_{DR}), differential propagation phase (ϕ_{DP}), and its range derivative (K_{DP}). Parameters such as Z_{TH} and K_{DP} depend on raindrop shape, which is directly related to drop size. Hence, these parameters contain information about DSDs that should allow more accurate estimation of rain.

Empirical polarimetric rain rate estimators R(K_{DP}), R(Z_{TH}, Z_{DR}), and R(K_{DP}, Z_{DR}) have been developed (e.g., Ryzhkov and Zrnić 1995; Brandes et al. 2002). Such relations are obtained through power-law fitting with radar and gauge measurements, disdrometer observations, or datasets generated through numerical simulations. There are problems with this approach (see, e.g., Illingworth and Blackman 2002). Power-law fitting is usually done in the logarithmic domain with a linear fit, which does not ensure unbiased rain estimates. Even though nonlinear fitting can be used, it does not necessarily improve results because the proper functional form and error distribution are not known. Because of the large difference in sample volumes, radar–gauge comparisons usually show large scatter; and the fitted relation highly depends on how the dataset is selected and thresholds applied. Simulated DSDs may not be realistic even though the DSD parameters may cover the observed range because the probability distributions of the DSD parameters are ignored or not known. Although fixed, power-law relations that include Z_{TH} show some improvement over radar reflectivity estimators, they do not capture all the variability in DSDs. An alternative method, first proposed by Seliga and Bringi (1976), is to retrieve the DSD from the radar measurements and then compute the rain rate.

Seliga and Bringi retrieved two-parameter exponential drop distributions. However, it is widely accepted that drop distributions are better represented by a three-parameter gamma distribution (Ulbrich 1983). Two approaches of retrieving gamma DSDs with S-band radars have been reported recently (Gorgucci et al. 2002; Zhang et al. 2001). With the approach of Gorgucci et al. (see also Bringi et al. 2002) the slope of the drop axis ratio relation (β) is treated as a variable to be determined from radar measurements. Gorgucci et al. (2001) hypothesize that rain estimates with the “β method” are immune to variability in the raindrop size–shape relation. Another approach, the “constrained-gamma” method (Zhang et al. 2001; Brandes et al. 2003; Vivekanandan et al. 2004), uses radar reflectivity at horizontal polarization (Z_{TH}), Z_{TH}, and a constraining rela-

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tion between the DSD shape and slope parameters to retrieve the three parameters of the gamma DSD. Once the DSD is known, other parameters, such as rain rate, rain water content, and drop median volume diameter, are readily calculated.

In this paper, we briefly describe the two retrieval methods. Retrieved DSD parameters are then compared with in situ measurements, and the spatial association among parameters is examined. A simple analysis of error propagation with the β method is then conducted, which shows sensitivity to $K_{\text{DP}}$ estimate errors.

2. Review of the DSD retrieval methods

a. Polarimetric radar variables

Among polarization radar parameters, $Z_{\text{H}}$ (mm$^4$ m$^{-3}$), vertical reflectivity ($Z_{V}$, mm$^4$ m$^{-5}$), $Z_{\text{DR}}$ (dB), and $K_{\text{DP}}$ (° km$^{-1}$) are the most important for quantitative rain estimation. These variables depend on the DSD and the drop scattering amplitudes as follows:

$$Z_{\text{H,V}} = \frac{4\pi^3}{\pi^4 |K_{\text{e}}|^2} \int_{D_{\text{min}}}^{D_{\text{max}}} (f_{\text{H,V}}(D)|)^2 N(D) \, dD$$

(1)

$$Z_{\text{DR}} = 10 \log \left( \frac{Z_{\text{H}}}{Z_{V}} \right)$$

(2)

$$K_{\text{DP}} = \frac{180\pi}{\pi} \int_{D_{\text{min}}}^{D_{\text{max}}} \text{Re}[f_{\text{H}}(0, D) - f_{\text{V}}(0, D)] N(D) \, dD$$

(3)

where $D$ (mm) is the raindrop equivalent volume diameter, $f_{\text{H,V}}(D)$ are the backscattering amplitudes of a drop at horizontal and vertical polarization, $f_{\text{H,V}}(0, D)$ are the forward scattering amplitudes, $K_{\text{e}}$ is the dielectric factor of water, $\lambda$ (cm) is the radar wavelength, and $N(D)$ (mm$^{-1}$ m$^{-3}$) is the raindrop size distribution. $D_{\text{min}}$ and $D_{\text{max}}$ are the diameters of smallest and largest drops in the distribution. Ulbrich (1983) suggested raindrops were represented by the gamma distribution

$$N(D) = N_{\alpha} D^{\alpha} \exp(-\Lambda D).$$

(4)

With three parameters [$N_{\alpha}$ (mm$^{-1}$ m$^{-3}$), $\mu$, and $\Lambda$ (mm$^{-1}$)] the gamma distribution is capable of describing a variety of raindrop size distributions and has been widely accepted by the radar meteorology community.

The difference between the scattering amplitudes at the two polarizations depends on the raindrop shape, which is related to size. The DSD and the raindrop axis ratios are crucial in the development of polarimetric rain estimators and need to be correctly modeled in order to accurately retrieve rain parameters from radar measurements. There is agreement that “radar apparent” raindrop shapes for medium-sized drops in the free atmosphere are less oblate than the equilibrium drop shape values determined by Green (1975), assuming a balance between forces of surface tension and hydrostatic pressure, or the wind tunnel measurements of Pruppacher and Beard (1970). There are, however, different approaches as to how to implement the gamma DSD model and incorporate the more spherical drop shapes. The overarching problem is how to determine the governing parameters of the DSD from remote measurements and produce accurate estimates of rain rate and other DSD properties. The β and constrained-gamma DSD retrieval methods are reviewed in the following subsections.

b. Beta method

Considering that the gamma DSD governing parameters ($N_{\alpha}$, $\mu$, and $\Lambda$) are not physical parameters like rain rate or drop median volume diameter, normalizations are often performed to yield what are considered to be more physically meaningful parameters (Willis 1984; Chandrasekar and Bringi 1987; Testud et al. 2001). With the β approach, a normalized form of the gamma distribution is used to represent rain DSDs (Testud et al. 2001):

$$N(D) = N_{\alpha} f(\mu)(D/D_{\alpha})^\mu \exp[-(3.67 + \mu)(D/D_{\alpha})],$$

(5)

with

$$f(\mu) = \frac{6 \mu}{3.67^{\mu}} \frac{(\mu + 3.67)^{\mu+4}}{\Gamma(\mu + 4)};$$

(6)

$N_{\alpha}$ (mm$^{-1}$ m$^{-3}$) is a normalized concentration parameter equivalent to that for an exponential DSD with the same water content and drop median volume diameter $D_{\alpha}$ (mm). Although normalization may impart some physical meaning to the concentration term, there does not appear to be a fundamental advantage for this approach regarding DSD retrieval.

The central idea of the β method is to treat the raindrop shape as a variable and to retrieve β from $Z_{\text{H}}$, $K_{\text{DP}}$, and $Z_{\text{DR}}$ (Gorgucci et al. 2000, 2001). Pruppacher and Beard (1970) determined that drop axis ratios (r, the ratio of the minor and major axes) could be expressed as $r = 1.03 - 0.062D$. Gorgucci et al. (2000, 2001) rewrote this relation in a more general form with a variable slope ($\beta$, mm$^{-1}$), as

$$r = 1.03 - \beta D$$

(7)

The β parameter is believed by Gorgucci et al. to account for drop canting and oscillations. Evidence for a slope other than 0.062 comes, for example, from the experimental study of Bringi et al. (1998) who used images of drop shapes obtained by aircraft to deduce that small (large) drops were even more spherical (oblate) than indicated by the relation of Pruppacher and Beard (1970).

Gorgucci et al. (2001) generated simulated gamma DSDs by allowing the distribution parameters to vary randomly over the ranges $-1 < \mu < 5$, $10^3 \leq N_{\alpha} \leq 10^5$ mm$^{-1}$ m$^{-3}$, and $0.5 < D_{\alpha} < 3.5$ mm. Radar variables were then calculated for various $\beta$s in the range 0.02 and 0.10 mm$^{-1}$ (see also Gorgucci et al. 2000). By varying the DSD parameters and the slope term, a da-
taset was obtained. [Note that while the range of “data points” may match local precipitation climatology, the frequency of individual data points probably does not.]

A nonlinear regression with the computed radar variables was performed to derive

$$\beta = 2.08Z_H^{0.065}K_{DR}^{0.380} \beta_{DR}^{0.965},$$

(8)

where $\beta_{DR} = 10^{0.12Z_{DR}}$ is the differential reflectivity in linear units (a ratio). Incorporating the $\beta$ term, expressions for the DSD governing parameters were then derived as

$$D_0 = 0.56Z_H^{0.064}K_{DR}^{0.024} \beta_{DR}^{-1.42},$$

(9)

$$\log N_w = 3.29Z_H^{0.038} \beta_{DR}^{-1.389},$$

(10)

$$\mu = \frac{2000 \beta_{1.89} D_0^{2.39} \beta_{0.079}}{(\beta_{DR} - 1)} - 3.16\beta_{0.046} K_{DR}^{0.574} \beta_{DR}^{-0.558}.$$  

(11)

Rain rate ($R$) is given by (Bringi et al. 2002)

$$R = 0.105\beta_{0.655} Z_H^{0.85} \beta_{DR}^{0.583}.$$  

(12)

Radar estimates of $K_{DP}$ are computed as the range derivative of differential propagation phase measurements. The standard error in $K_{DP}$ varies according to the number of radar samples, the amount of range averaging performed, and the precipitation type. For strong thunderstorms Aydin et al. (1995) determined that the error in $K_{DP}$ could be as large as $\pm 0.50^\circ$ km$^{-1}$ for measurements consisting of 64 sample pairs. Ryzhkov and Zrnić (1996) determined that $K_{DP}$ errors for measurements consisting of 64 and 128 sample pairs and 17 and 49 range samples varied from 0.04 to 0.30$^\circ$ km$^{-1}$. Gorgucci et al. (2002) estimate the error to be $0.32^\circ$ km$^{-1}$. For retrieval purposes with the $\beta$ method, Gorgucci et al. (2002) set a lower $K_{DP}$ threshold of 0.20$^\circ$ km$^{-1}$ (a rain rate of $\sim 15$ mm h$^{-1}$). Bringer et al. (2002) use 0.30$^\circ$ km$^{-1}$ (a rain rate of $\sim 20$ mm h$^{-1}$) and impose additional thresholds for radar reflectivity and differential reflectivity. The procedure of Bringer et al. is adopted here with the exception that the threshold for $K_{DP}$ was set to 0.20$^\circ$ km$^{-1}. This was done to increase the number of retrievals with the $\beta$ method. For the retrieval of $N_w$ and $D_0$ for light rainfall rates Bringer et al. use power-law relations based on disdrometer observations and drop shapes as given by Beard and Chuaung (1987) and Andersager et al. (1999). The relations for $Z_H < 35$ dBZ and $Z_{DR} \geq 0.2$ dB are

$$D_0 = 1.81Z_{DR}^{0.486},$$

(13)

$$N_w = 21Z_H^{0.383} D_0^{0.351},$$

(14)

and for $Z_H < 35$ dBZ and $Z_{DR} < 0.2$ dB are

$$D_0 = \gamma Z_H^{0.136},$$

(15)

$$N_w = (1.513/\gamma)^{7.31},$$

(16)

where

$$\gamma = 1.81\left(\frac{Z_{DR}}{Z_H^{0.38}}\right)^{0.486}.$$  

(17)

Light rain rates are calculated assuming $\mu = 3$, using the retrieved $N_w$ and $D_0$, and with the drop terminal velocity relation of Ulbrich and Atlas (1977). Finally, rainwater content $W(g m^{-3})$ can be computed from

$$W = 1.73 \times 10^{-5} N_w D_0^5.$$  

(18)

c. Constrained-gamma method

It has been found that the three parameters of the gamma DSD model [Eq. (4)] are not mutually independent (Ulbrich 1983; Chandrasekar and Bringi 1987; Kozu and Nakamura 1991; Haddad et al. 1997). Correlations among DSD parameters, if real, may be useful in reducing the number of unknowns and enable the retrieval of the DSD from a pair of independent remote measurements, such as reflectivity and differential reflectivity. A $N_w$-$\mu$ relation was found and used by Ulbrich (1983), along with reflectivity and attenuation, for retrieving the three DSD parameters. The derived relation was later attributed to statistical error (Chandrasekar and Bringi 1987). Regardless, fluctuations in $N_0$ for fixed $\mu$ range over several orders of magnitude; hence, the utility of the relation is limited.

Analysis of DSD data collected in Florida during the summer of 1998 revealed a high correlation between $\mu$ and $\Lambda$ and led to the derivation of an empirical $\mu$-$\Lambda$ relation expressed as $\mu(\Lambda)$ (Zhang et al. 2001). To better retrieve small values of $\mu$ and $\Lambda$ associated with higher rain rates, the relation was re-derived based on the truncated moment method in Brandes et al. (2003) as

$$\Lambda = 0.0365 \mu^2 + 0.735 \mu + 1.935.$$  

(19)

The disdrometer data and the relation are plotted in Fig. 1 (top). Zhang et al. (2003) show that Eq. (19) does not result purely from measurement error but contains useful information. The $\mu$-$\Lambda$ relation, differential reflectivity, and reflectivity are used to retrieve the three DSD parameters. [Differential propagation phase is used for a self-consistent calibration check (Vivekanandan et al. 2003.)] The $\mu$-$\Lambda$ relation essentially reduces the three-parameter gamma DSD to a two-parameter model.

The $\mu$-$\Lambda$ relation suggests that a drop characteristic size parameter—for example, $D_0$—and the width of the raindrop mass spectrum ($\sigma_m$), as defined by Ulbrich (1983), are related. This physical relationship is shown for disdrometer observations in Fig. 1 (bottom). The fitted relation for the observations is

$$\sigma_m = 0.051D_0^2 + 0.548D_0 - 0.347.$$  

(20)

Because both $\sigma_m$ and $D_0$ are uniquely determined by $\mu$ and $\Lambda$ for a gamma DSD (Ulbrich 1983; Ulbrich and Atlas 1998), the $\mu$-$\Lambda$ relation (19) can be converted to a $\sigma_m$-$D_0$ relation that is also shown in Fig. 1b. The $\sigma_m$–$D_0$ relations for fixed $\mu$ values of 0, 3, and 6 are plotted.
for comparison. The converted relation based on Eq. (19) agrees well with observations. Results for \( \mu = 0 \) (an exponential DSD) agree with observations having large \( D_0 \) but overestimate the spectral width for small \( D_0 \). Conversely, results for \( \mu = 6 \) agree with that for small drops but not for large drops. Although \( \mu = 3 \) may be a good average value, the fit is not as good as the \( \mu-L \) relation for the whole range of the observations when truncation is taken into account.

The method assumes that the raindrop axis ratio relation when averaged over a radar sampling volume is constant. A fixed axis ratio relation was derived from measurements of Pruppacher and Pitter (1971), Chandrasekar et al. (1988), Beard and Kubesh (1991), and Andsager et al. (1999). The measurements were fitted with a polynomial function (Brandes et al. 2002) to obtain

\[
\begin{align*}
    r &= 0.9951 + 0.02510D - 0.03644D^2 \\
    &+ 0.005030D^3 - 0.0002492D^4. \quad (21)
\end{align*}
\]

This relation includes oscillations due to vortex shedding; but oscillations due to collisions are generally absent, except perhaps for the study of Chandrasekar et al. The measurements used in the derivation of (21) are presented in Fig. 2. The axes ratio relation, \( r = 1.03 - 0.062D \), of Pruppacher and Beard (1970) is also shown. Even with a variable slope \( \beta \) the linear relation (7) does not capture the nonlinear nature of the axis ratio relation for raindrops, especially for sizes smaller than 2 mm, which contribute significantly to rain rate and the specific differential phase.

With the constraining relation (19) and the fixed shape–size relation (21), \( Z_{\text{DR}} \) uniquely determines the DSD shape parameter \( \mu \). Hence, \( \mu \) can be retrieved from the measured \( Z_{\text{DR}} \), and \( \Lambda \) and \( N_0 \) are subsequently calculated from (19) and radar reflectivity. For computational convenience simple relations are derived based on the constrained-gamma model. We calculated radar parameters (\( Z_H \) and \( Z_{\text{DR}} \)) for \( \Lambda \) in the range 1–13 mm\(^{-1}\). The ratio of \( R \) and \( Z_H \) is independent of \( N_0 \) and can be expressed solely in terms of \( \Lambda \) (or \( \mu \)). Thus, for a constrained-gamma DSD the ratio is a function only of \( Z_{\text{DR}} \).

After taking the logarithm of the ratios and applying a polynomial fit, we obtain

\[
\begin{align*}
    \log(R/Z_H) &= 0.165Z_{\text{DR}}^0 - 0.897Z_{\text{DR}}^1 - 2.12 \quad (22) \\
    R &= 7.60 \times 10^{-3} \times Z_H \times 10^{0.165Z_{\text{DR}}^-0.897Z_{\text{DR}}}. \quad (23)
\end{align*}
\]
Expressions for median drop diameter and rain water content are (Brandes et al. 2004)

\[ D_0 = 0.171Z_{\text{DR}}^2 - 0.725Z_{\text{DR}} + 1.479Z_{\text{DR}} + 0.717 \]  

(24)

\[ W = 5.589 \times 10^{-4}Z_{\text{HH}} \times 10^{0.233Z_{\text{HH}} - 1.124Z_{\text{DR}}} \]  

(25)

The units are linear for \( Z_{\text{HH}} \) and dB for \( Z_{\text{DR}} \). The corresponding relation for \( \mu \) is

\[ \mu = 6.084D_0^5 - 29.85D_0 + 34.64 \]  

(26)

The above expressions work best for \( 0.3 \leq Z_{\text{DR}} \leq 3.0 \) dB. For simplicity \( N_w \) is computed from \( W \) by inverting Eq. (18).\(^1\) The modeled data points and derived relations (22) and (24) are presented in Fig. 3. Note that the polynomial relations with the constrained-gamma method are fundamentally different from those generated with the \( \beta \) method where the simulations involve randomly distributed values over specified ranges. The polynomial relations derived with the constrained-gamma method are unique for the axis ratio relation (21); and issues regarding data selection (thresholds) and fitting errors, other than that used in the derivation of Eq. (19), are not involved.

3. Retrieval comparison

Data used to compare the two approaches were collected in east-central Florida during the summer of 1998 in a special experiment (PRECIP98) to evaluate the potential of polarimetric radar for estimating rain in a subtropical environment. Measurements were made with the National Center for Atmospheric Research’s S-band, dual-polarization radar (S-Pol) over a video disdrometer located at a distance of 38 km. Radar measurements examined here were obtained at an elevation angle of 0.5°.

Figure 4 shows time series of \( Z_{\text{HH}}, Z_{\text{DR}}, \) and \( K_{\text{DP}} \) obtained with radar. The heavy line shows measurements filtered over five range gates (a linear average) as used in implementation of the constrained-gamma method. Range gates were 0.15 km in length. The thin line shows values filtered over 20 gates as used with the \( \beta \) method. Even with heavy filtering the smoothed \( K_{\text{DP}} \) trace is noisy.
Retrieved parameters using the two approaches (as described in section 2) are shown in Figs. 5 and 6. Results from the disdrometer observations are included for comparison. It can be seen that relatively few retrievals are made with the \( \beta \) method (indicated by small circles); that is, only those measurements meeting the thresholds \( Z_H \geq 35 \text{ dBZ}, Z_{DR} \geq 0.2 \text{ dB}, \) and \( K_{DP} \geq 0.2^{°} \text{ km}^{-1}. \) Nevertheless, for measurements to which the method is applied the slope of the raindrop axis ratio relation or \( \beta \) term fluctuates considerably from minute to minute. Much of the variation is believed to relate to errors in the estimate of \( K_{DP} \) rather than rain physics (section 4). Inspection reveals that the estimated \( \beta \) are all \( >0.04 \text{ mm}^{-1}. \) Imposing a threshold for \( K_{DP} (0.2^{°} \text{ km}^{-1}) \) removes the statistical information contained in measurements that are smaller than the threshold and results in an overestimate of \( K_{DP} \) in regions of low \( K_{DP} \) (Ryzhkov and Zrnić 1996). Comparisons for \( D_0 \) and \( R \) are generally good. Comparisons for \( \mu \) and \( N_w \) reveal that the retrievals with the constrained-gamma method agree better with the disdrometer measurements than those with the \( \beta \) approach even for heavy rain rates. Gorgucci et al. (2002) and Brandes et al. (2003) both remark on the difficulty of estimating \( \mu \) in the presence of measurement error.

A comparison can also be made between the constrained-gamma method and the retrievals of Bringi et al. (2002) for light precipitation (the squares in Figs. 5 and 6). For the latter, \( N_w \) and \( D_0 \) are retrieved with power-law relations [Eqs. (13)–(17)]. Similarities are to be expected with the constrained-gamma method because both approaches rely on \( Z_H \) and \( Z_{DR} \) measurements. The primary differences being that the \( \mu-\Lambda \) relation is a constraint in the approach of Zhang et al. (2001) and that the disdrometer observations used for model development are not the same. The power-law estimators of Bringi et al. [Eqs. (13)–(17)] are derived from observations obtained in Brazil and are not specifically tuned for Florida storms. Examination shows that the range in \( N_w \) computed with the relations of Bringi et al. (2002) tends to be large with significant over and underestimates. This result extends to \( D_0 \) as well, especially for small \( D_0.\) The biases may be tied to \( \mu \) which is not retrieved by Bringi et al. (2002). When the observed (disdrometer) estimate of \( \mu \) is large (\( >3 \)), \( N_w \) is overestimated and \( D_0 \) tends to be underestimated. Such biases are symptomatic of retrievals with fixed \( \mu \) (Brandes et al. 2003).

An example of the spatial distribution of retrieved DSD parameters with the \( \beta \) method for a strong thunderstorm is shown in Fig. 7. Maximum reflectivity is \( \sim 50 \text{ dBZ}. \) Radar estimates of \( K_{DP} \) vary from small negative values to about \( 2^{°} \text{ km}^{-1}. \) In regions where \( \beta \) is not retrieved, the power-law relations of Bringi et al. (2003) are used for retrieval. In the central storm core (the core is defined as the region with \( K_{DP} > 0.2^{°} \text{ km}^{-1} \) and roughly with \( Z_H > 40 \text{ dBZ} \) retrieved \( \beta \) are noisy but relatively small, suggestive of drops with more spherical shapes in the mean; \( D_0.\) are large implying that forced oscillations may be responsible. At the periphery of the \( \beta \) domain, \( \beta \) becomes large, an indication of flattened drops or drops with reduced oscillations. However, this collocation of large \( \beta \) and smaller \( D_0.\) seems spurious. Betas of \( 0.06-0.07 \text{ mm}^{-1} \) are characteristic of large drops rather than the associated \( D_0.\) of \( 1-1.5 \text{ mm} \) (Fig. 2). Decreased rain rates and drop sizes...
should result in fewer collisions and smaller $\beta$s. The large $\beta$s may relate to an overestimate of $K_{\text{DP}}$ in this region of the storm. In addition to the bias associated with imposition of a $K_{\text{DP}}$ threshold, bias may be introduced by the filtering of $\varphi_{\text{DP}}$ performed to compute $K_{\text{DP}}$. Filtering reduces the magnitude of estimated $K_{\text{DP}}$ in the storm core and spreads the signal into regions of smaller $K_{\text{DP}}$. The resulting distribution of radar-derived parameters may not match that used in the simulations from which Eqs. (8)–(12) were derived.

Retrieved shape parameters within central portions of the storm core range from 2 to 3. An interpretation is that accretion and drop breakup in the storm core create DSDs that are relatively broad. This region of relatively small $\mu$s is bounded, especially in the west (on the left), by a band of near exponential DSDs ($0 \leq \mu \leq 2$). At the edge of the $\beta$ domain, $\mu$ tends to be large, an indication of narrow (more monodispersed) DSDs. While this basic pattern seems plausible, the retrieved $\mu$s are highly dependent on $\beta$ [Eq. (11)]. The drop concentration parameter exhibits large values in the storm core ($\log N_w > 4.5$). There is considerable small-scale variation in the field, regardless of whether $N_w$ is retrieved from the $\beta$ method or the power-law relations. Rainwater content and rain-rate retrievals show the expected large water contents and heavy rain rates in the storm core and smaller values at the edge of the storm.

The high noise level in the $\beta$, $\mu$, $N_w$, and $D_0$ fields is attributed to errors in $K_{\text{DP}}$. This is difficult to see from the $K_{\text{DP}}$ field presented because it is dominated by a strong signal for heavy rain rates. A retrieval for the stratiform component of this rain event is shown in Fig. 8. Note that the noise in the $K_{\text{DP}}$ field is transferred to the retrieved DSD parameters. Also, an incompatibility exists between the retrievals of $N_w$ and $D_0$ with the $\beta$ method and those derived with power-law estimators.

The constrained-gamma model retrieval for the strong thunderstorm complex is presented in Fig. 9. The basic patterns are similar to those with the $\beta$ method, but they differ in detail. Retrieved DSD shape parameters are relatively small in the storm core where $\mu$ approaches zero. Small negative $\mu$s at the leading edge of the storm are associated with new storm cells having large $D_0$s—for example, near $x = -23$, $y = 33$ km and at $x = -27$, $y = 23$ km. The implication is that DSDs in the developing cells are dominated by small numbers of large drops, causing the distribution, when plotted in $\log N(D)$–$D$ space, to be concave upward. $D_0$s decrease toward the rear of the storm (left-hand side of the figure) where precipitation is light and more stratiform. Large $\mu$s, suggestive of narrow DSDs, characterize this region of the storm. Retrievals here are noisy due to increased influence of ground clutter and the relative growth of

![Fig. 7. DSD retrieval with the $\beta$ method for a thunderstorm observed at 1926 UTC 17 Sep.](image-url)
measurement error as the precipitation signal weakens—a problem with all retrieval methods. Drop concentrations (log $N_w$) approach 4.5 in the storm core and decrease toward fringe areas. Maximum rainwater content and rain rate are a little higher with the constrained-gamma method than those in Fig. 7 possibly because there is less smoothing of the $Z_H$ and $Z_{DR}$ fields. Agreement between the constrained-gamma method and the $\beta$ method for the retrieved physical parameters; for example, $W$, $R$, and $D_0$ is fair. There is less correspondence for the retrieved DSD parameters $N_w$ and $\mu$. Retrievals with the constrained-gamma method show greater spatial continuity and appear to have a lower noise level even though the radar measurements have been subjected to less filtering.

Figure 10 presents the constrained-gamma method retrieval for the stratiform component of the storm. The retrieved fields are relatively uniform compared to the strong convection in Fig. 9. Higher reflectivity regions generally correspond with large median volume diameters and DSDs characterized by small distribution shape factors. There is a slight tendency for these regions to have reduced drop concentrations. Perhaps the accretion of small drops is taking place. At the edges of the storm system $D_0$s decrease to $\sim 1$ mm and $\mu$s become large, suggesting that the drop distributions are narrow.

4. Discussion

a. Error propagation

The radar retrievals and disdrometer comparisons are subject to measurement error, DSD retrieval model error, and sampling volume differences. If possible, these error effects should be quantified. Here, we examine the impact of radar measurement errors on estimated rain parameters for the $\beta$ retrieval method. A simple analysis of error propagation for the $\beta$ estimator [Eq. (8)], based on the assumption that the errors in the radar estimates $Z_H$, $Z_{DR}$, and $K_{DR}$ are uncorrelated, is presented in the appendix. The analysis yields a relative standard error given by

$$\frac{\text{std}(\beta)}{\beta} = \left[0.365^2 \frac{\text{var}(Z_H)}{Z_H^2} + 0.965^2 \frac{\text{var}(K_{DR})}{K_{DR}} + 0.380 \frac{\text{var}(K_{DR})}{K_{DR}} \right]^{1/2},$$

where std( . . . ) is the standard deviation and var( . . . )
represents the variances of the parameter estimates ($\sigma$). For a well-calibrated radar and with range filtering over five gates, $Z_H$ and $Z_{DR}$ can be measured accurately such that $\text{std}(Z_H) < 0.5$ dB and $\text{std}(Z_{DR}) < 0.1$ dB. Based on the discussion in section 2b and Fig. 4 we conservatively estimate the standard error in $K_{DP}$ to be $0.20^\circ$ km$^{-1}$. The contribution to the error in $\hat{\beta}$ for the three parameters is shown in Fig. 11 (top). Note that the contributions of the expected measurement error in reflectivity and differential reflectivity to the error in $\hat{\beta}$ are constant. This is because the measurements and errors, in dB, are relative terms. Error associated with reflectivity is larger than that with differential reflectivity because of the relative measurement error level and the weighting in (8). In contrast, the error contribution to $\hat{\beta}$ from $K_{DP}$ depends on the magnitude of $K_{DP}$. The error contribution from $K_{DP}$ dominates at low values and becomes smaller than that from reflectivity only for $K_{DP} > -1.5^\circ$km$^{-1}$. This suggests that the small-scale variation in the retrieved $\hat{\beta}$ (Figs. 5, 7, and 8) originates with measurement error rather than physical processes.

The question remains as to the influence of the error in $\hat{\beta}$ on retrieved DSD parameters. An error analysis similar to that for $\hat{\beta}$ is presented in the appendix for the rain-rate estimator Eq. (12). Results are shown in Fig. 11 (bottom) for an assumed $Z_H$ of 40 dBZ, a $Z_{DR}$ of 1.0 dB, and the range in $K_{DP}$ shown in the upper panel. In this case the error contributions from $Z_H$ and $Z_{DR}$ vary slightly due to the dependence of $R$ on $\beta$. Errors in $\hat{\beta}$ associate with a large corresponding error in the estimated rain rate. Even at relatively small error levels in $\hat{\beta}$ (large $K_{DP}$), the error contribution from $K_{DP}$ is significant.

b. $\mu$–$\beta$ correlation

Retrieved values of $\mu$ with the $\beta$ method are positively correlated with $\beta$ (Figs. 5, 7, and 8). Correlation arises from the strong dependence of $\mu$ on $\beta$ in Eq. (11). The relationship between the error in the estimate of $\beta$ and the correlation between variables, also derived in the appendix, is shown in Fig. 12. Correlation between $\mu$ and $\beta$ is high even at relatively low error levels. Moreover, a strong physical correlation between $\mu$ and $\beta$ does not seem reasonable. Disdrometer observations generally show that large $\mu$s (narrow drop distributions) associate with small $D_s$ (e.g., Figs. 5 and 6). Observations also show that the axis ratio relation slope for small drops ($D < 2$ mm) is smaller than that for large drops (Fig. 2). Hence, the observational evidence in-
dicates that small drops (typically characterized by narrow DSDs with large $\mu$) would normally associate with small values of $\beta$. Inspection of the figures also reveals several retrievals with large retrieved values of $\beta$, large $\mu$, and large $N_w$ (e.g., near 2100 UTC in Fig. 5). Large $\mu$ would normally associate with small drop concentrations. Perhaps the spurious $\beta-\mu$ and $\mu-N_w$ points arise from the simulations upon which the method is based. Gorgucci et al. (2000) allowed $\beta$ to vary from 0.02–0.10 mm$^{-1}$. This range (see Fig. 2) may be too large and cause compensating adjustments in other parameters of the retrieved DSD.

5. Summary and conclusions

The “$\beta$” and “constrained-gamma” methods of retrieving gamma DSDs from polarimetric radar measurements were compared. With the $\beta$ method the axis ratios of oscillating drops are computed from the radar measurements of $Z_H$, $K_{DP}$, and $Z_{DR}$. The procedure is founded on simulations with variable $\beta$ and random distributions of the governing parameters of the gamma DSD that are used to establish relationships with the radar variables $Z_H$, $Z_{DR}$, and $K_{DP}$. The constrained-gamma method incorporates the $Z_H$ and $Z_{DR}$ measurements and an empirical relation between the slope and shape parameters of the DSD as determined from disdrometer measurements. The $\mu-\lambda$ relation reduces the gamma DSD to a two-parameter distribution in that only two radar measurements are required. Axis ratios are assumed constant for the radar measurement volume and based on a relation believed to account largely for the oscillations attributable to vortex shedding.

The attractiveness of the $\beta$ method stems from ostensibly retrieving the slope factor $\beta$ and the three parameters of the gamma distribution from radar measurements alone. The principal disadvantages are that the three radar measurements used for retrieval may not be independent (Illingworth and Blackman 2002) and that errors in estimates of $K_{DP}$ and errors in the modeled DSD restrict its utility to heavy rainfall rates. As Illingworth and Blackman point out, distributions of randomly generated gamma DSD parameters and use of a linear axis ratio relation may not be representative of natural rain. Further, the power-law fitting employed does not guarantee an unbiased estimator with minimum error because the relations may not capture the true functional form and account for the error distribution. With the present $\beta$-method formulation the errors in estimates of $K_{DP}$ cause correlation between $\beta$ and the
DSD shape parameter. The retrieval yields large $\beta$s at the edges of storms where drops are normally small. This result does not appear to be physical.

The constrained-gamma method avoids the use of simulated DSDs and the error propagation associated with $K_{DP}$. The procedure yields the expected associations between radar observables and derived DSD parameters, works reasonably well at low and high rain rates, and provides relatively accurate retrieval of DSD parameters for a broad range of DSDs. Additional studies are needed to verify the stability of the $\mu-D$ relation [Eq. (19)]. Concerns regarding the contribution of DSD moment estimate errors in the determination of such relations are being addressed (Zhang et al. 2003). Regardless, this study demonstrates that the constrained-gamma method does not create large errors in the retrieved physical parameters. That the method has a physical basis is manifest by the $\sigma_m-D_0$ relation shown in Fig. 1.

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APPENDIX

Error Propagation with the β Method

With the β method the drop axis ratio relation slope (β) is estimated from the radar variables $Z_H$, $\xi_{DR}$, and $K_{DP}$ as

$$\beta = 2.08 Z_H^{-0.365} K_{DP}^{0.380} \xi_{DR}^{0.065}.$$  \hspace{1cm} (A1)

The DSD governing parameters ($D_0$ and $\mu$) and rain rate are related to $\beta$ and the radar measurements by

$$D_0 = 0.56 Z_H^{0.064} \xi_{DR}^{0.024 \beta^{-1.42}}$$  \hspace{1cm} (A2)

$$\mu = \frac{200 \beta^{0.90} D_0^{2.23} \xi_{DR}^{0.09}}{(\xi_{DR} - 1)} - 3.16 \beta^{-0.046} \xi_{DR}^{0.374 \beta^{-0.555}}$$  \hspace{1cm} (A3)

$$R = 0.105 \beta^{0.365} Z_H^{0.93} \xi_{DR}^{0.585 \beta^{-0.701}}.$$  \hspace{1cm} (A4)

Because radar estimates $Z_H$, $\xi_{DR}$, and $K_{DP}$ contain measurement error, the estimated parameters $\hat{\beta}$, $\hat{D}_0$, $\hat{\mu}$, and $\hat{R}$ also have error. To understand how the error propagates from the radar measurements to the estimated rain
parameters, the radar and DSD parameters are expressed as sums of their expected values and fluctuation errors \([\delta(\ldots)]\), for example, as

\[
\hat{\beta} = P + \delta P. \tag{A5}
\]

Based on the first-order approximation of Papoulis (1965, section 7–3), errors in the estimated parameters can be expressed in terms of the radar measurement errors as

\[
\delta \beta = \frac{\partial \bar{\beta}}{\partial Z_H} \delta Z_H + \frac{\partial \bar{\beta}}{\partial \xi_{DR}} \delta \xi_{DR} + \frac{\partial \bar{\beta}}{\partial K_{DP}} \delta K_{DP} \tag{A6}
\]

\[
\delta D_0 = \left( \frac{\partial D_0}{\partial Z_H} + \frac{\partial D_0}{\partial \beta} \frac{\partial \beta}{\partial Z_H} \delta Z_H + \frac{\partial D_0}{\partial \xi_{DR}} + \frac{\partial D_0}{\partial \beta} \frac{\partial \beta}{\partial \xi_{DR}} \delta \xi_{DR} \right) \delta \bar{\xi}_{DR} \]

\[
+ \left( \frac{\partial D_0}{\partial K_{DP}} + \frac{\partial D_0}{\partial \beta} \frac{\partial \beta}{\partial K_{DP}} \right) \delta K_{DP}. \tag{A7}
\]

The mean of the fluctuations is zero; hence, the variance of an estimated parameter is simply an ensemble average of the square of the fluctuation errors. Assuming that the radar measurements errors are statistically uncorrelated, we obtain the variance of \(\hat{\beta}\) due to measurement error as

\[
\text{var}(\hat{\beta}) = \left( \left( \frac{\partial \bar{\beta}}{\partial Z_H} \right)^2 \text{var}(Z_H) + \left( \frac{\partial \bar{\beta}}{\partial \xi_{DR}} \right)^2 \text{var}(\xi_{DR}) + \left( \frac{\partial \bar{\beta}}{\partial K_{DP}} \right)^2 \text{var}(K_{DP}) \right)^2. \tag{A10}
\]

Similarly, we obtain the variances of \(\hat{D}_0\), \(\hat{\mu}\), and \(\hat{R}\) as

\[
\text{var}(\hat{D}_0) = \left( \left( \frac{\partial D_0}{\partial Z_H} \right)^2 \text{var}(Z_H) + \left( \frac{\partial D_0}{\partial \xi_{DR}} \right)^2 \text{var}(\xi_{DR}) + \left( \frac{\partial D_0}{\partial K_{DP}} \right)^2 \text{var}(K_{DP}) \right)^2.
\]

\[
\text{var}(\hat{\mu}) = \left( \left( \frac{\partial \mu}{\partial Z_H} \right)^2 \text{var}(Z_H) + \left( \frac{\partial \mu}{\partial \xi_{DR}} \right)^2 \text{var}(\xi_{DR}) + \left( \frac{\partial \mu}{\partial K_{DP}} \right)^2 \text{var}(K_{DP}) \right)^2.
\]

\[
\text{var}(\hat{R}) = \left( \left( \frac{\partial R}{\partial Z_H} \right)^2 \text{var}(Z_H) + \left( \frac{\partial R}{\partial \xi_{DR}} \right)^2 \text{var}(\xi_{DR}) + \left( \frac{\partial R}{\partial K_{DP}} \right)^2 \text{var}(K_{DP}) \right)^2.
\]
var($\hat{D}_0$) = $\left(\frac{\partial D_0}{\partial Z_H} + \frac{\partial D_0}{\partial \beta} \frac{\partial \beta}{\partial Z_H}\right)^2 \text{var}(\hat{Z}_H) + \left(\frac{\partial D_0}{\partial \xi_{DR}} + \frac{\partial D_0}{\partial \beta} \frac{\partial \beta}{\partial \xi_{DR}}\right)^2 \text{var}(\hat{\xi}_{DR}) + \left(\frac{\partial D_0}{\partial K_{DP}} + \frac{\partial D_0}{\partial \beta} \frac{\partial \beta}{\partial K_{DP}}\right)^2 \text{var}(\hat{K}_{DP})$

(A11)

var($\hat{\mu}$) = $\left(\frac{\partial \mu}{\partial Z_H} + \frac{\partial \mu}{\partial \beta} \frac{\partial \beta}{\partial Z_H}\right)^2 \text{var}(\hat{Z}_H) + \left[\frac{\partial \mu}{\partial \xi_{DR}} + \frac{\partial \mu}{\partial \beta} \frac{\partial \beta}{\partial \xi_{DR}}\right]^2 \text{var}(\hat{\xi}_{DR})$

(A12)

Also, the covariance between $\hat{\beta}$ and $\hat{\mu}$ is obtained from (A6) and (A8) as

$$\text{cov}(\hat{\mu}, \hat{\beta}) = \left\langle \frac{\partial \mu}{\partial \beta} \frac{\partial \beta}{\partial \xi_{DR}} + \frac{\partial \mu}{\partial \xi_{DR}} \frac{\partial \xi_{DR}}{\partial \beta} \right\rangle \text{var}(\hat{\xi}_{DR}) \text{var}(\hat{\beta})$$

(A13)

The partial derivatives in the above expressions, obtained from the functional relations (A1)–(A4), are

$\frac{\partial \beta}{\partial Z_H} = -0.365 \frac{\beta}{Z_H}$

(16a)

$\frac{\partial \beta}{\partial \xi_{DR}} = 0.965 \frac{\beta}{\xi_{DR}}$

(16b)

$\frac{\partial \beta}{\partial K_{DP}} = 0.380 \frac{\beta}{K_{DP}}$

(16c)

$\frac{\partial D_0}{\partial Z_H} = 0.064 \frac{D_0}{Z_H}$

(17a)

$\frac{\partial D_0}{\partial \xi_{DR}} = 0.024 \beta^{-1.42} \frac{D_0}{\xi_{DR}}$

(17b)

$\frac{\partial D_0}{\partial \beta} = -0.034 \beta^{-1.42} \ln(\xi_{DR}) \frac{D_0}{\beta}$

(17c)

$\frac{\partial \mu}{\partial \beta} = \frac{200 \beta^{0.89} D_0^{1.23} \beta^{0.039}}{(\xi_{DR} - 1)^2} \ln(\xi_{DR}) \frac{1}{\beta}$

(A18b)

$\frac{\partial \mu}{\partial \xi_{DR}} = -200 \beta^{0.89} D_0^{1.23} \beta^{0.039} \frac{1}{\xi_{DR}} - 1.18 \beta^{-0.535} \ln(\xi_{DR}) \frac{1}{\xi_{DR}}$

(18c)

$\frac{\partial R}{\partial \beta} = 0.93 \frac{R}{Z_H}$

(19a)

$\frac{\partial R}{\partial \xi_{DR}} = -0.585 \beta^{-0.703} \frac{R}{\xi_{DR}}$

(19b)

$\frac{\partial R}{\partial K_{DP}} = [0.865 + 0.411 \beta^{-0.703} \ln(\xi_{DR})] \frac{R}{\beta}$

(19c)

The relationship between the relative standard error in $K_{DP}$ and the error in the estimate of $\beta$ (A10) for assumed errors of 0.5 dB in $Z_H$, 0.1 dB in $Z_{DR}$, and 2° km$^{-1}$ in $K_{DP}$ is presented in Fig. 11 (upper panel). These errors are added as a surrogate for measurement errors. The lower panel shows the relative standard error in the estimated rainfall rate and the error in $\hat{\beta}$ (A13), assuming $Z_H = 40$ dBZ and $Z_{DR} = 1$ dB. Figure 12 shows the correlation between $\hat{\mu}$ and $\hat{\beta}$ (A15) as a function of the standard error in $\hat{\beta}$.

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