Multiwavelet Discontinuous Galerkin-Accelerated Exact Linear Part (ELP) Method for the Shallow-Water Equations on the Cubed Sphere

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ABSTRACT

In this paper a new approach is presented to increase the time-step size for an explicit discontinuous Galerkin numerical method. The attributes of this approach are demonstrated on standard tests for the shallow-water equations on the sphere. The addition of multiwavelets to the discontinuous Galerkin method, which has the benefit of being scalable, flexible, and conservative, provides a hierarchical scale structure that can be exploited to improve computational efficiency in both the spatial and temporal dimensions. This paper explains how combining a multiwavelet discontinuous Galerkin method with exact-linear-part time evolution schemes, which can remain stable for implicit-sized time steps, can help increase the time-step size for shallow-water equations on the sphere.

1. Introduction

Large-scale scientific computing is continuing to experience exponential growth of throughput, but this is occurring more through ever-expanding parallelism rather than individual processor speed, which is stagnating (Drake et al. 2008). This expansion allows reasonable computational speeds with finer spatial resolutions; however, throughput gains will eventually be limited by a time barrier. Thus, the development of new algorithms that can effectively scale in spatial resolution while maintaining adequate throughput and accuracy are needed.

The discontinuous Galerkin (DG) method has an elegant and flexible formulation that can provide high-order-accurate solutions to complicated models (Cockburn and Shu 1998, 2001). The DG method has extended the ideas of numerical fluxes and slope limiters, which were developed within high-resolution finite differences and finite volumes, into the finite-element framework to produce a method that is locally conservative and is able to capture physically relevant shocks and discontinuities. DG is a scalable method because numerical information of each element is only passed locally through numerical fluxes to the nearest neighbors. In a body of work, the DG method was successfully implemented on the sphere for advection models (Nair et al. 2005b) and the shallow-water equation (Nair et al. 2005a).

Multiwavelets are a discontinuous, orthogonal, compactly supported, multiscale set of functions with vanishing moments that yield high-order hp-adaptive approximations of $L^2$ functions (Alpert 1993). Combination of multiwavelets with the DG method results in a computationally fast and effective multiscale adaptive DG method (Archibald et al. 2010). Additionally, the hierarchical structure in location and scale of multiwavelets allows for the use of spectral methods, such as spectral filtering, locally in space and time.

The time-stepping scheme used in this work can be effectively represented by exploiting the local nature of both the DG method and multiwavelet basis. The exact linear part (ELP) method has been demonstrated to be particularly effective and efficient for multiwavelet-based schemes (Alpert et al. 2002; Beylkin et al. 1998) since the operators generated for the ELP method remain sparse in a multiwavelet representation. Additionally, the ELP is a high-order-accurate time-stepping scheme that has a single-step representation and remains stable for large time steps.

The ELP time-stepping scheme is a part of the active research in time integrators for DG methods. Specifically related to this work is the advance in space–time
expansion discontinuous Galerkin (STE-DG) methods (Klaij et al. 2006; Ambati and Bokhove 2007; Lörcher et al. 2007; Gassner et al. 2008). The main idea in STE-DG is that, for a given system, the space–time Taylor series expansion produces higher-order time and mixed space–time derivatives that can be reformulated using the Cauchy–Kovalevskaya (CK) procedure into pure spatial derivatives. As a benefit of STE-DG, the global time-step restriction of all explicit DG schemes is avoided and only local stability restrictions need to be satisfied. Related to this work is the use of operator-integrating factor splitting (Maday et al. 1990) in spectral elements methods applied to geophysical flows (St-Cyr and Thomas 2005). Unlike these semi-implicit and implicit methods, the ELP method for this study uses both space–time Taylor expansion and operator splitting to generate an explicit method where the generated operators are calculated once and remain static over the course of the simulation. The explicit ELP method requires only one evaluation per time step, in contrast to Runge–Kutta or semi and fully-implicit methods, which require multiple serial evaluations or iterative solves, respectively.

This paper extends the work done in Archibald et al. (2009) to nonlinear test cases, and is organized as follows. Section 2 presents a brief overview of the shallow-water model on the cubed sphere and section 3 introduces the multiwavelet basis and its key features. In section 4 we describe the DG method for the cubed sphere and further demonstrate how multiwavelets are incorporated. Section 5 describes ELP for the multiwavelet DG method, section 6 demonstrates the time acceleration of advection problems on the cubed sphere, and section 7 is a discussion of the results.

2. Shallow-water equations on the cubed sphere

This section begins by describing the cubed-sphere transform that is used to define the computational domain and finishes by giving the shallow-water equation in flux form on the cubed sphere, following Nair et al. (2005b).

a. Cubed-sphere geometry

The cubed sphere, first developed in Sadourny (1972), has proven to be a particularly useful gridding technique for solving partial differential equations on the sphere (Giraldo et al. 2002; Nair et al. 2005a,b; Taylor et al. 1997). Figure 1 depicts the cubed-sphere geometry, where the transformation between the inscribed cube and the sphere is determined by the gnomonic (center) projection from the sphere to each face of the cube.

The spherical wind vector \( \mathbf{u}(\lambda, \theta) = (u, v) \) can be expressed for each face of the cube in terms of contravariant vectors \((u^1, u^2)\) as

\[
\begin{pmatrix}
u \\
u
\end{pmatrix} = \mathbf{A} \begin{pmatrix} u^1 \\
u^2
\end{pmatrix}, \quad \text{with} \quad \mathbf{A} = R \begin{pmatrix}
\frac{\partial \lambda}{\partial x^1} & \frac{\partial \lambda}{\partial x^2} \\
\frac{\partial \theta}{\partial x^1} & \frac{\partial \theta}{\partial x^2}
\end{pmatrix}.
\]

Here, the longitudinal angle is given by \( \lambda \), the latitudinal angle by \( \theta \), the coordinates on each face of the cube in terms of \( x^1 \) and \( x^2 \), and the Jacobian of the transformation (the metric term) is \( \sqrt{G} = \det(\mathbf{A}^T \mathbf{A})^{1/2} \).
b. Shallow-water equations

Again following Nair et al. (2005b), the shallow-water equations on the rotating sphere in flux form are given as

$$\frac{\partial U}{\partial t} + \frac{\partial F_1(U)}{\partial x^1} + \frac{\partial F_2(U)}{\partial x^2} = S(U),$$

where the state vector $U$ and the flux vectors $F_1$ and $F_2$ are defined by

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \sqrt{Gh} \end{pmatrix}, \quad F_1(U) = \begin{pmatrix} E \\ 0 \\ \sqrt{Ghu^1} \end{pmatrix}, \quad \text{and} \quad F_2(U) = \begin{pmatrix} 0 \\ E \\ \sqrt{Ghu^2} \end{pmatrix}.$$  (3)

Here, $E = \Phi + (\varphi/2)(u_1u_1^2 + u_2u_2^2)$ is defined in terms of the covariant $(u_1, u_2)$ and contravariant $(u_1^1, u_2^2)$ wind vectors, with free-surface geopotential height (above sea level) given as $\Phi = gh(x, h)$, where $g$ is the gravitational acceleration, $h$ is the depth of the fluid, and $h(x)$ is the height of the underlying topography. The advantage of this formulation is that all differential operators are taken on the cube face. Finally, the source term is given as

$$S(U) = \begin{bmatrix} \sqrt{Ghu^1}(f + \xi) \\ -\sqrt{Ghu^1}(f + \xi) \\ 0 \end{bmatrix},$$

where $f$ is the Coriolis force, and the divergence $\delta$ and relative vorticity $\xi$ are defined as

$$\delta = \frac{1}{\sqrt{G}} \left[ \frac{\partial \sqrt{G}u_1}{\partial x^1} + \frac{\partial \sqrt{G}u_2}{\partial x^2} \right]$$
and

$$\xi = \frac{1}{\sqrt{G}} \left[ \frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2} \right].$$

This study uses the following parameters for the earth:

$$a = 6.371 \times 10^6 \text{ m},$$
$$\Omega = 7.292 \times 10^{-5} \text{ s}^{-1},$$
$$g = 9.806 \text{ m s}^{-2}.$$  (6)

3. Multiwavelet bases

In this section we briefly summarize the important properties of the multiwavelet basis derived and developed in Alpert (1993), and we introduce notation as given in Alpert et al. (2002). We begin by defining $V^k_n$ as a space of piecewise polynomial functions, for $k = 1, 2, \ldots, n = 0, 1, 2, \ldots, n$ as

$$V^k_n = \{ f : f \in \Pi_k(I^*_l) \text{ for } l = 0, \ldots, 2^n - 1, \text{ and supp}(f) = I^*_l \},$$

where $\Pi_k(I^*_l)$ is the space of all polynomials of degree less than $k$ on the interval $I^*_l = [2^{-l}n, 2^{-l}(n + 1)]$. Using this space, we can describe not only multiwavelets, but the solution space that the DG method uses for approximation. The multiwavelet subspace $W^k_n, n = 0, 1, 2, \ldots,$ is defined as the orthogonal complement of $V^k_n$ in $V^k_{n+1}$, or

$$V^k_n \oplus W^k_n = V^k_{n+1}, \quad W^k_n \perp V^k_n.$$  (8)

The immediate result of this definition of the multiwavelet subspace is that it splits $V^k_n$ into $n + 1$ orthogonal subspaces of different scales, as

$$V^k_n = W^k_0 \oplus W^k_1 \oplus W^k_2 \oplus \cdots \oplus W^k_{n-1}.$$  (9)

Given a basis $\phi_0, \ldots, \phi_{k-1}$ of $W^k_0$, the space $V^k_n$ is spanned by $2^nk$ functions, which are obtained from $\phi_0, \ldots, \phi_{k-1}$ by dilation and translation,

$$\phi_{jl}(x) = 2^{n/2} \phi_j(2^n x - l), \quad j = 0, \ldots, k - 1, \quad l = 0, \ldots, 2^n - 1.$$  (10)

By construction, similar properties hold for multiwavelets. If the piecewise polynomial functions $\psi_0, \ldots, \psi_{k-1}$ form an orthonormal basis for $W^k_0$, then by dilation and translation the space $W^k_n$ is spanned by $2^nk$ functions:

$$\psi_{jl}(x) = 2^{n/2} \psi_j(2^n x - l), \quad j = 0, \ldots, k - 1, \quad l = 0, \ldots, 2^n - 1.$$  (11)

A function $f \in V^k_n$ can be represented by the following expansion of scaling functions:

$$f(x) = \sum_{l=0}^{2^n-1} \sum_{j=0}^{k-1} s_{jl} \phi_{jl}(x),$$

where the coefficients $s_{jl}$ are computed as

$$s_{jl} = \int_{2^{-l}n}^{2^{-l}(n+1)} f(x) \phi_{jl}(x) \, dx.$$  (13)

The decomposition of $f(x)$ has an equivalent multiwavelet expansion given by
This research sets the scaling functions, \( \tilde{f}(x) = \sum_{j=0}^{k-1} s_{j,0} \phi_j(x) + \sum_{m=0}^{n-1} \sum_{i=0}^{2^m-1} d_{m,i}^j \psi_{m,i}^j(x) \), \( (14) \)

with the coefficients

\[
d_{m,i}^j = \int_{x = l}^{2^{-n(j+1)} f(x) \psi_{m,i}^j(x) \, dx. \tag{15}
\]

It is demonstrated in Alpert (1993) how fast transforms between (12) and (14) can be developed using multiwavelets with \( k \) vanishing moments can be constructed on consecutive levels \( m \) and \( m + 1 \) through repeated application of

\[
s_{m,j,2l}^j = \sum_{j=0}^{k-1} [h_{j,0}^{(0)} s_{j,2l}^{m+1} + h_{j,1}^{(1)} s_{j,2l}^{m+1}],
\]

\[
d_{m,j,2l+1}^j = \sum_{j=0}^{k-1} [g_{j,0}^{(0)} s_{j,2l+1}^{m+1} + g_{j,1}^{(1)} s_{j,2l+1}^{m+1}], \tag{16}
\]

using the scaling coefficients \( h_{j,0}^{(0)} \) and \( g_{j,0}^{(0)} \) for \( i, j = 0, \ldots, k - 1 \). The vanishing-moment property is used in Alpert (1993) as an additional restriction in the development of multiwavelets that aids in the sparsity of integral and exponential operators that are generated from multiwavelets (Alpert et al. 2002). The inverse operation that takes expansion coefficients of (14) to (12) is given by

\[
s_{m,j,2l}^{m+1} = \sum_{j=0}^{k-1} [h_{j,0}^{(0)} s_{j,2l}^m + g_{j,0}^{(0)} d_{m,j}^j],
\]

\[
d_{m,j,2l+1}^{m+1} = \sum_{j=0}^{k-1} [h_{j,1}^{(1)} s_{j,2l+1}^m + g_{j,1}^{(1)} d_{m,j}^j], \tag{17}
\]

for the scaling coefficients \( h_{j,0}^{(0)} \) and \( g_{j,0}^{(0)} \), for \( i, j = 0, \ldots, k - 1 \). Multiwavelets are generated from the scaling function and coefficients as

\[
\psi_i(x) = \sqrt{2} \sum_{j=0}^{k-1} [h_{j,0}^{(0)} \phi_j(2x) + h_{j,1}^{(1)} \phi_j(2x-1)],
\]

\[
i = 0, \ldots, k - 1. \tag{18}
\]

This research sets the scaling functions, \( \phi_0, \ldots, \phi_{k-1} \), to be the Legendre polynomials that are scaled and normalized to the interval \([0, 1]\) as

\[
\phi_j(x) = \begin{cases} \sqrt{2j+1} P_j(2x-1) & \text{if } x \in (0, 1), \\ 0 & \text{otherwise}, \end{cases} \tag{19}
\]

for \( j = 0, \ldots, k - 1 \). Figure 2 depicts the multiwavelets \( \psi_0, \ldots, \psi_{k-1} \) for order \( k = 3 \), using (18) and (19).

The total numbers of expansion coefficients in (12) and (14) are the same, but the number of significant expansion coefficients for a given error tolerance level \( \epsilon \) will be different. A benefit of using the multiwavelet expansion (14) is that much fewer significant expansion coefficients are generally needed. A result of this property when multiwavelets are used in DG methods is an increase in computational speed and efficiency (Alpert 1993). Current limitations of multiwavelets used in this study are the restriction of multiwavelets to quadrilateral spectral element grids with dimension \( 2^n \times 2^m \) and the added complexity required to support the multiscale hierarchical structure.

In this paper we use the thresholding technique explained in Beylkin et al. (2008), where for a given tolerance \( \epsilon \), the multiwavelet expansion (14) is truncated such that

\[
\| f - \tilde{f} \|_\infty < \epsilon. \tag{20}
\]

Details of the algorithm are given in the appendix. The same tolerance level is used for both multiwavelet expansions of solutions and ELP operators.

4. Multiwavelet discontinuous Galerkin method

In this section we describe the multiwavelet DG method (Alpert 1993) in two dimensions using scalar components. This derivation can be directly applied to the system of equations [(2)] for the shallow-water equations on the sphere.

Consider the two-dimensional scalar nonlinear conservation law:

\[
u_t + \nabla \cdot f(\nu) = 0, \quad (0, 1)^2 \times (0, T). \tag{21}
\]
We restrict our attention to uniform Cartesian meshes since they provide the most natural representation for multiwavelets; other mesh choices are possible, but the implementation becomes more challenging (Coulth 2000).

\[
\int_{\Omega} \int_{\Pi} \frac{\partial u}{\partial t} \psi^n_{\mu,p}(x) \psi^n_{\mu,p}(y) \, dx \, dy = \int_{\Omega} \int_{\Pi} f(u) \frac{\partial \psi^n_{\mu,p}(x)}{\partial x} \psi^n_{\mu,p}(y) \, dx \, dy + \int_{\Omega} \int_{\Pi} f(u) \psi^n_{\mu,p}(x) \frac{\partial \psi^n_{\mu,p}(y)}{\partial y} \, dx \, dy
\]

\[
- \int_{\partial[\Pi \times \Pi]} f(u) \cdot \mathbf{n} \psi^n_{\mu,p}(x) \psi^n_{\mu,p}(y) \, ds,
\]

for \( j, j = 0, \ldots, k - 1 \) and for \( l, \ell = 0, 1, \ldots, 2^n - 1 \), where \( \mathbf{n} \) is the outward-facing unit-normal vector on the element boundary \( \partial[\Pi \times \Pi] \). We assume that the nonlinear conservation law (21) has a solution in the form of the following two-dimensional multiwavelet expansion,

\[
u_h(x, y, t) = \sum d^{m,\mu}_{\mu,\mu,j} \psi^n_{\mu,p}(x) \psi^n_{\mu,p}(y),
\]

with summation taken over \( j, j = 0, \ldots, k - 1 \) and \( m, \mu = -1, 0, \ldots, n - 1 \); and \( l = 0, 1, \ldots, \min(0, 2^m - 1) \). Given a fixed-order \( k \geq 0 \) and resolution \( n \geq 0 \), the variational formulation of the DG method is derived by multiplying (21) by the test functions \( \phi^n_{\mu,p} \mathbf{V}^k_h \) and integrating to obtain

\[
\int_{\Pi} \int_{\Pi} \frac{\partial u_h}{\partial t} \phi^n_{\mu,p}(x) \phi^n_{\mu,p}(y) \, dx \, dy = \int_{\Pi} \int_{\Pi} f(u) \frac{\partial \phi^n_{\mu,p}(x)}{\partial x} \phi^n_{\mu,p}(y) \, dx \, dy + \int_{\Pi} \int_{\Pi} f(u) \phi^n_{\mu,p}(x) \frac{\partial \phi^n_{\mu,p}(y)}{\partial y} \, dx \, dy
\]

\[
- \int_{\partial[\Pi \times \Pi]} f(u) \cdot \mathbf{n} \phi^n_{\mu,p}(x) \phi^n_{\mu,p}(y) \, ds,
\]

where \( f(u_h) \) is a monotone numerical flux, the focal point for the only communication between elements. We note that the boundary integral in (24) is taken on all boundaries where the corresponding wavelet is discontinuous. The evaluation of the nonlinear components of (24) is performed in physical space, which can be accomplished in \( O(k2^n) \) operations (Beylkin et al. 1991).

Throughout this paper we use the well-known simple Lax–Friedrichs flux (LeVeque 1990) and Gauss–Lobatto quadrature for integration. An explicit solution of (24) results directly from the orthogonality of multiwavelets, where \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) are defined in (3) and \( \mathbf{F}(u_h) \) is the numerical flux. The source term \( \mathbf{S}(\mathbf{U}) \), is given in (4), and

\[
\frac{\partial d^{m,\mu}_{\mu,\mu,j}}{\partial t} = \int_{\Pi} \int_{\Pi} \mathbf{F}_1(U_h) \frac{\partial \Psi^n_{\mu,p}(x)}{\partial x} \Psi^n_{\mu,p}(x) \, dx \, dx + \int_{\Pi} \int_{\Pi} \mathbf{F}_2(U_h) \Psi^n_{\mu,p}(x) \frac{\partial \Psi^n_{\mu,p}(y)}{\partial y} \, dx \, dx
\]

\[
+ \int_{\Pi} \int_{\Pi} \mathbf{S}(U_h) \Psi^n_{\mu,p}(x) \Psi^n_{\mu,p}(y) \, dx \, dx - \int_{\partial[\Pi \times \Pi]} \mathbf{F}(U_h) \cdot \mathbf{n} \Psi^n_{\mu,p}(x) \Psi^n_{\mu,p}(y) \, ds,
\]

for all index values given previously.

The multiwavelet discontinuous Galerkin method for the shallow-water equations can be stated, using the cubed sphere of section 2a, for the six local Cartesian coordinate systems for each face of the cube. Considering the subdomain of one cube face the weak Galerkin formulation results in finding the solution:

\[
\mathbf{U}_h(x^1, x^2, t) = \sum d^{m,\mu}_{\mu,\mu,j} \Psi^n_{\mu,p}(x^1) \Psi^n_{\mu,p}(x^2),
\]

with
\[
\Psi^m_n(x) = \begin{bmatrix}
\psi^m_n \left( \frac{2x}{\pi} + \frac{1}{2} \right) \\
\psi^m_n \left( \frac{2x}{\pi} + \frac{1}{2} \right) \\
\psi^m_n \left( \frac{2x}{\pi} + \frac{1}{2} \right)
\end{bmatrix},
\]

(27)

and the dyadic interval is rescaled as \( \tilde{I}_n = [2^{-n-1}\pi l, 2^{-n-1}\pi(l + 1)] - (\pi/4) \).

5. Time discretization

We use a method of time stepping that has been demonstrated to be particularly effective and efficient for multiwavelet schemes (Alpert et al. 2002; Beylkin et al. 1998). The idea behind the development of these schemes is to convert differential equations of the form

\[ u_t = \mathcal{L}u + \mathcal{N}(u), \]

(28)

where the system is split into a linear operator \( \mathcal{L} \) and nonlinear operator \( \mathcal{N}, \) into the equivalent integral equation,

\[ u(t) = e^{\mathcal{L}t}u_0 + \int_0^t e^{(t-\tau)\mathcal{L}}\mathcal{N}(u)\,d\tau. \]

(29)

The multiwavelet basis allows fast scaling and squaring methods that produce sparse and highly accurate approximations to the exponential linear operator. These time-stepping schemes are therefore called exact linear part (ELP) schemes. The explicit discrete scheme is determined by approximating (29) as

\[ u_{n+1} = e^{\Delta t\mathcal{L}}u_n + \Delta t \sum_{m=0}^{M-1} \beta_m N_{n-m}, \]

(30)

where \( M + 1 \) is the number of time levels involved, \( l \leq M, \) \( N_n = \mathcal{N}(u_n), \) and \( u_n = u(x, n\Delta t). \)

The first step in determining the coefficients in the discrete scheme (30) is to write a Taylor series expansion:

\[ u_{n+1} = \sum_{k=0}^{\infty} u_n^{(k)} \Delta t^k \left( \frac{\Delta t}{k!} \right)^k, \]

(31)

where

\[ u_n^{(k)} = \left. \frac{\delta^k}{\delta t^k} u(t) \right|_{t = t_n}. \]

(32)

Next, the Taylor series expansion coefficients are expressed using the differential equation (28) as

\[ \begin{array}{c|c|c|c|c}
M & \beta_0 & \beta_1 & \beta_2 & \text{Order} \\
\hline
1 & Q_1 & 0 & 0 & 1 \\
2 & Q_1 + Q_2 & -Q_2 & 0 & 2 \\
3 & Q_1 + (3Q_2/2) + Q_3 & -2(Q_2 + Q_3) & (Q_2/2) + Q_3 & 3 \\
\end{array} \]

Table 1. Coefficients of explicit ELP schemes for \( l = 1. \)

\[ u_n^{(k)} = \mathcal{L}^ku_n + \mathcal{N}(u_n), \]

and

\[ u_n^{(k)} = \mathcal{L}^k u_n + \sum_{j=0}^{k-1} \mathcal{L}^{k-j-1}N_n^{(j)}, \]

(33)

where

\[ N_n^{(k)} = \left. \frac{\delta^k}{\delta t^k} N(u(t)) \right|_{t = t_n}. \]

(34)

Rearranging the substitution of (31) into the discrete scheme (30) with \( l = 1 \) produces

\[ u_{n+1} = e^{\Delta t\mathcal{L}}u_n + \Delta t \sum_{j=0}^{M-1} \beta_j Q_j(\Delta t)N_n^{(j)}, \]

(35)

where

\[ Q_j(x) = \frac{e^x - \sum_{k=0}^{j-1} x^k}{x^j}. \]

(36)

It is demonstrated in Beylkin et al. (2008) that (36) can be computed using a scaling and squaring method based on the following two-scale relationships:

\[ Q_1(2x) = \frac{1}{2} [Q_0(x)Q_1(x) + Q_1(x)], \]
\[ Q_2(2x) = \frac{1}{4} [Q_1(x)Q_1(x) + 2Q_2(x)], \]

(37)

Using a Taylor series expansion on the nonlinear terms in (30) and equating to (35) will produce Table 1 for explicit time stepping, or when \( \gamma = 0, \) and Table 2 for implicit time stepping.

We finish this section by showing how the exponential operator \( e^\mathcal{L} \) can be approximated by using a scaling and squaring method. Suppose we are given the matrix \( \mathcal{L} \) and an error tolerance \( \epsilon; \) the scaling and squaring method that approximates the exponential linear operator is as follows:
(a) compute the exponent \( j \) such that \( \| \mathcal{L} \|_2 < \epsilon \).
(b) compute the approximation \( e^{\mathcal{L}t} = I + (\mathcal{L}t/2) + (\mathcal{L}^2t/6) + \cdots \), and
(c) \( e^{\mathcal{L}t} \) is squared \( j \) times to obtain \( e^{\mathcal{L}t} \).

The ELP method is an independent time-integration method that can be applied to any spatial method; however, the sparsity and cost are dependent upon the particular combination. Using the multiwavelet bases, all operators described in this section are sparse and the cost of evaluating these operators is \( O(k^2n) \) (Alpert et al. 2002).

6. Numerical results

In this section we consider advection and shallow-water equations on the sphere, which have specific importance to the development of climate models. Specifically, we will consider test cases one, two, and five of Williamson et al. (1992), the moving and stationary vortices on the sphere test of Nair and Jablonowski (2008), the cross-polar flow test proposed in McDonald and Bates (1989), and the barotropic instability test proposed in Galewsky et al. (2004). Throughout, we employ the ELP time-stepping scheme outlined in section 5 with multiwavelet DG. We refer to this method as multiwavelet DG (MWDG) ELP. For comparison we use fourth-order-in-time Runge–Kutta time stepping (RK4; Abramowitz and Stegun 1972, chapter 9), using DG with a Legendre basis; we refer to this method as DG RK4. The number of elements on each face is \( 2^n \), which is determined by \( n \), the level of multiwavelet refinement. The Courant–Friedrichs–Lewy (CFL) condition is given as

\[
CFL = \frac{u_0 \Delta t}{\Delta x},
\]

where \( \Delta t \) is the time step and \( \Delta x = (1/4^n) \). All simulations are performed in the MATLAB scientific computing environment [using a quad-core AMD Opteron 1354 (Budapest) processor running at 2.1 GHz, with 8 GB of DDR2-800 memory].

a. Cosine-bell advection

Given the advecting field \( h \), the equation for advection in flux form is

\[
\frac{\partial h}{\partial t} + \mathbf{v} \cdot (h \mathbf{v}) = 0.
\]

The first test in Williamson et al. (1992) is to solve (39) on the surface of a sphere, with initial conditions given in spherical coordinates as

\[
h[r(\lambda, \theta) = \begin{cases} h_0 \left[ 1 + \cos \left( \frac{\pi r}{R} \right) \right] & \text{if } r < R, \\ 0 & \text{otherwise,} \end{cases}
\]

for \( r(\lambda, \theta) = a \arccos[\sin \theta \sin \theta + \cos \theta \cos \theta \cos(\lambda - \lambda_c)] \) and advecting wind

\[
\mathbf{v} = u_0 \begin{pmatrix} \cos \theta \cos \alpha + \sin \theta \cos \lambda \sin \alpha \\ -\sin \lambda \sin \alpha \end{pmatrix}.
\]

Here, the parameters are set to \( h_0 = 1000 \) m, \((\lambda_c, \theta_c) = (3\pi/2, 0), R = a/3, u_0 = 2\pi a/12 \) days and \( \alpha = \pi/4 \). We note that this choice of \( \alpha \) represents a particularly difficult problem, since the advecting cosine bell passes through four corners and along two edges of the cubed-sphere grid during each full revolution.

Figure 3 depicts the MWDG ELP solution and relative error for cosine-bell initial conditions, with \( n = 4, k = 3, \) and CFL = 18.2. It can be seen that using ELP time stepping provides a stable solution for time steps that significantly exceed the CFL requirement for explicit methods. Since pure advection is a linear problem, it is not surprising that ELP allows very large time steps. The difference between the MWDG ELP solution and the exact solution is no more than a fraction of a percent.

Table 3 gives more detailed insights into the properties of MWDG ELP and a comparison to DG RK4. Our first observation from Table 3 is that the \( L_2 \) error and order of convergence are comparable for each time-stepping method and CFL number. We note that the convergence rates and errors are similar to the results published in Nair et al. (2005b) for the same problem, with cosine-bell initial conditions, using a DG method with CFL = 0.1 and a third-order Runge–Kutta method. Finally, it can be seen that the MWDG ELP method can significantly increase the time step while preserving accuracy. This increase in time step resulted in a decrease in wall-clock time, which showed approximately a 13-fold

<table>
<thead>
<tr>
<th>( M )</th>
<th>( \gamma )</th>
<th>( \beta_0 )</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( Q_2 )</td>
<td>( Q_1 - Q_2 )</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>( (Q_2/2) + Q_4 )</td>
<td>( Q_1 - 2Q_2 )</td>
<td>( Q_1 - (Q_2/2) )</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>( (Q_4/32) + Q_4 )</td>
<td>( Q_1 + (Q_2/2) - 2Q_3 - 3Q_4 )</td>
<td>( -Q_2 + Q_3 + Q_4 )</td>
<td>( (Q_2/6) - Q_4 )</td>
<td>4</td>
</tr>
</tbody>
</table>
improvement in wall-clock time for a 16-fold increase in time step, and approximately a 25-fold improvement in wall-clock time for a 60-fold increase in time step.

b. Stationary vortex

The work done by Nair and Jablonowski (2008) provides analytic solutions for both moving and stationary deformational flows on the sphere. Given the rotational deformational flow

\[ v_r(\theta') = a \omega(\theta') \begin{bmatrix} \sin \theta \cos \theta - \cos \theta \cos(\lambda - \lambda_p) \sin \theta \\ \cos \theta \sin(\lambda - \lambda_p) \end{bmatrix}, \tag{42} \]

where \((\theta_p, \lambda_p)\) defines the center of the vortex and the angular velocity is

\[ \omega(\theta') = \begin{cases} \frac{\sqrt{3} \pi \text{sech}^2 \rho \tanh \rho}{4 \rho} & \text{if } \rho \neq 0, \\ 0 & \text{if } \rho = 0. \end{cases} \tag{43} \]

TABLE 3. Convergence rates for section 6a using RK4 and ELP time stepping for the MWDG method with order \(k = 3\) and drop tolerance \(\epsilon = 10^{-8}\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>RK4 (CFL = 0.2)</th>
<th>ELP (CFL = 3.2)</th>
<th>ELP (CFL = 12.0)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(L_2) error</td>
<td>Order</td>
<td>(T) (s)</td>
</tr>
<tr>
<td>2</td>
<td>(1.98 \times 10^{-1})</td>
<td>—</td>
<td>1.81</td>
</tr>
<tr>
<td>3</td>
<td>(4.04 \times 10^{-2})</td>
<td>2.30</td>
<td>13.72</td>
</tr>
<tr>
<td>4</td>
<td>(7.53 \times 10^{-3})</td>
<td>2.42</td>
<td>1.18 (\times 10^{2})</td>
</tr>
</tbody>
</table>

Fig. 3. (a) The final MWDG ELP solution of section 6a with \(n = 4, k = 3\), and CFL = 18.2. (b) Error after 1 complete revolution. The equivalent resolution at the equator for this set of multiwavelet parameters on the cube sphere is \(360^\circ/\sqrt{2} = 1.875^\circ\).
with \( \rho = \rho_0 \cos \theta' \) all written in terms of the rotated coordinates,

\[
\lambda'(\lambda, \theta; \lambda_p, \theta_p) = \arctan\left[ \frac{\cos \theta \sin(\lambda - \lambda_p)}{\cos \theta \sin \theta_p \cos(\lambda - \lambda_p) - \cos \theta_p \sin \theta} \right]
\]

(44)

and

\[
\theta'(\lambda, \theta; \lambda_p, \theta_p) = \arcsin[\sin \theta \sin \theta_p + \cos \theta \cos \theta_p \cos(\lambda - \lambda_p)],
\]

(45)

then the solution to the conservation law (39) for the rotational deformational flow (42) is given analytically as

\[
h(\lambda', \theta', t) = 1 - \tanh\left[ \frac{\rho}{\gamma} \sin(\lambda - \theta - \omega_r t) \right],
\]

(46)

where time is measured in days, the initial field is given by \( h(\lambda', \theta', 0) \), and \( \gamma \) is a field parameter. In this study we follow the parameter choice given in Nair and Jablonowski (2008) by setting \( \rho_0 = 3, \gamma = 5 \), and \( (\theta_p, \lambda_p) = (\pi, \pi/2 - \alpha) \).

Table 4 gives more details on the performance of MWDG ELP for this stationary vortex problem. Again, we compare MWDG ELP to DG RK4. We use the same scaling in space and time as for the cosine-bell advection test, and we observe similar improvements for both time step and wall-clock time. Specifically, it can be seen that approximately a 13-fold improvement in wall-clock time for a 16-fold increase in time step, and approximately a 25-fold improvement in wall-clock time for a 60-fold increase in time step. The vortex problem is a challenging problem to resolve spatially, and it can be seen in Table 4 that the convergence rates are reduced as compared to the cosine-bell advection test.

c. Moving vortex

The stationary vortex problem was extended in Nair and Jablonowski (2008) to provide an analytic solution to the moving vortex problem. Suppose that \([\theta_p(t), \lambda_p(t)]\) is the vortex center that is moving along the axis of the solid-body rotation for the velocity field given in (41). Then, the solution to the conservation law (39) with the time-dependent velocity field,

\[
v(t) = u_0 \left[ \frac{(\cos \theta \cos \alpha + \sin \theta \cos \lambda \sin \alpha)}{\sin \lambda \sin \alpha} \right] + a_0 \theta'(t) \left[ \begin{array}{c} \sin \theta_p(t) \cos \theta - \cos \theta_p(t) \cos(\lambda - \lambda_p) \sin \theta \\ \cos \theta_p(t) \sin(\lambda - \lambda_p) \end{array} \right],
\]

(47)

is given analytically by (46) for \((\lambda', \theta')\), which are functions of \([\theta_p(t), \lambda_p(t)]\).

The stationary flow fields in the previous examples are idyllic for the MWDG ELP method. Since there is no time dependence, the cost of generating the linear exponential operator occurs only once during the setup of the simulation, and this operator remains unchanged during the simulation. The moving vortex test case, introduced in Nair and Jablonowski (2008), provides an analytic solution for a nonstationary flow field. In this case, for the MWDG ELP method, either the linear exponential operator must be updated during the simulation or treated as a nonlinear operator. Updating the linear exponential operator would allow longer time steps, but would incur computational cost greater than the cost of covering this time with a smaller-time-step Runge–Kutta method. In this study, we treat the time dependence of the flow fields in the nonlinear operator of the MWDG ELP method.

Figure 4 displays the exact solution of the moving vortex after one complete revolution on day 12 and the numerical errors for both MWDG ELP and DG RK4. For both methods we use \( n = 2 \), and \( k = 9 \). The same number of degrees of freedom and \( \alpha = \pi/4 \) were used in Nair and Jablonowski (2008) for a DG method and had a time step of 360 s. We were able to simulate the DG RK4 up to a stable time step of 405 s. The MWDG ELP was stable up to a time step of 664 s. The relative \( \ell_2 \) norm (Williamson et al. 1992) was \( 2.06 \times 10^{-3} \) for DG RK4 and \( 1.36 \times 10^{-3} \) for MWDG ELP. Examination of Fig. 4
FIG. 4. (a) Exact solution of the moving-vortex test of section 6c after 1 complete revolution on day 12. (b) Numerical error for DG RK4 with a time step of 405 s. (c) Numerical error for MWDG ELP with a time step of 664 s. For both methods, $\alpha = \pi/4$, $n = 2$, and $k = 0$. The equivalent resolution at the equator for this set of multiwavelet parameters on the cube sphere is 2.5°.
shows that the MWDG ELP does not have as many phase errors as DG RK4. The wall-clock time of the DG RK4 method was 2.31 times that of MWDG ELP.

The DG RK4 takes four function evaluations for each step, where the MWDG ELP only takes one function evaluation. Thus, in this situation the time step per function evaluation for MWDG ELP is 664 s, whereas for DG RK4 it is 101 s. We do not experience a sixfold decrease in wall-clock time because the operators for each function evaluation of the MWDG ELP method are more complex than those for DG RK4.

d. Steady-state geostrophic flow

The second test in Williamson et al. (1992) is to solve (2) on the surface of a sphere, with initial conditions given by the analytic $h$ field,

$$
gh = gh_0 - \left( a\Omega u_0 + \frac{u_0^2}{2} \right) (-\cos \lambda \cos \theta \sin \alpha + \sin \theta \cos \alpha)^2,
$$

$$
h_s = 0,
$$

with the advecting wind given in (41) with $u_0 = 2\pi a/12$ days, $gh_0 = 2940$ m$^2$ s$^{-2}$, and Coriolis parameter given as

$$
f = 2\Omega (-\cos \lambda \cos \theta \sin \alpha + \sin \theta \cos \alpha). \quad (49)
$$

Figure 5 depicts the numerical solution of “example d” with $a = \pi/4$, $n = 2$, and $k = 9$ at day 5, and the corresponding error at day 5. The same number of degrees of freedom and $a$ were used in Nair et al. (2005b) for a DG method that had a time step of 36 s. We were able to simulate the DG RK4 up to a stable time step of 45 s.

Fig. 5. (a) Numerical solution and (b) error at day 5 of section 6d for the MWDG ELP with $a = \pi/4$, $n = 2$, and $k = 9$. Here, the time step is 90 s. The equivalent resolution at the equator for this set of multiwavelet parameters on the cube sphere is 2.5°.
The MWDG ELP scheme remained stable up to a time step of 90 s. Figure 6 provides additional confirmation that the accuracy of the DG RK4 method is comparable to the MWDG ELP method, since the $2^\text{nd}$ norm was $2.81 \times 10^{-8}$ for DG RK4 and $2.73 \times 10^{-8}$ for MWDG ELP. The wall-clock time of the DG RK4 method was 1.67 times that of MWDG ELP. The time step per function evaluation for MWDG ELP is 90 s, whereas for DG RK4 it is 11 s. Again, the gains in time step per function evaluation do not fully transfer to benefits in wall-clock time since the operators for each function evaluation of the MWDG ELP method are more complex than those for DG RK4.

**e. Zonal flow over an isolated mountain**

The fifth test in Williamson et al. (1992) is to solve (2) on the surface of a sphere, with initial conditions given by the analytic $h$ field,

$$gh = gh_0 - \left(a\Omega u_0 + \frac{u_0^2}{2}\right)\sin^2\theta,$$

$$h_s = 2000 \left(1 - \frac{r}{R}\right),$$  \hspace{1cm} (50)

with the advecting wind given in (41) for $\alpha = 0, R = \pi/9$,

$$r^2 = \min\left[R^2, \left(\lambda - \frac{3\pi}{2}\right)^2 + \left(\theta - \frac{\pi}{6}\right)^2\right],$$

$$u_0 = 2\pi a/24 \text{ days}, \quad gh_0 = 5960 \text{ m}^2 \text{ s}^{-2}, \quad \text{and Coriolis parameter given as}$$

$$f = 2\Omega \sin\theta.$$  \hspace{1cm} (51)

This example simulates zonal flow impinging on a mountain. Figure 7 depicts the numerical solution at day 15 for the MWDG ELP method. For consistency, this example has the same time and spatial discretization as is used in example d.

The normalized integrals defined in Williamson et al. (1992) are computed for mass conservation,

$$\psi = h;$$

total energy,

$$\psi = h\mathbf{v} \cdot \mathbf{v} + gh\left(h + h_s\right)^2 - h_s^2;$$

and potential enstrophy,

$$\psi = \frac{(\zeta + f)^2}{2h}.$$  \hspace{1cm} (51)

It can be seen in Fig. 8 that there is a drift from machine precision conservation of mass, which has its root in the thresholding of the MWDG ELP operators. The normalized integral plots are similar in accuracy (Nair et al. 2005a); however, the initial sharp increase in potential enstrophy reported in Nair et al. (2005b) is smoothed for the ELP method.

We were able to simulate the DG RK4 up to a stable time step of 41 s. The MWDG ELP scheme remained stable up to a time step of 82 s. The wall-clock time of the DG RK4 method was 1.61 times that of MWDG ELP. Although the MWDG ELP scheme represents real gains over DG RK4, the increased time step is not in the range of an implicit time step for this case. It was reported in St-Cyr and Thomas (2005) that a 160-fold improvement to the explicit time-step spectral element method is possible using a semi-implicit operator-integrating factor-splitting method (Maday et al. 1990). The CPU time performance is not reported and would naturally damp the gains in time-step size since this method required a RK4 substepping method for the advection operator and an iterative solve.
f. Cross-polar flow

The next test that we explore in this paper is the inviscid variant of the test proposed in McDonald and Bates (1989), where a cross-polar flow with a geostrophically balanced initial state is simulated by solving (2) on the surface of a sphere, with initial conditions given by the analytic $h$ field,

$$gh = gh_0 + 2a\Omega u_0 \sin \theta \cos \theta \sin \lambda,$$

$$h_s = 0,$$  \hspace{1cm} (52)

with the advecting wind given by

$$\mathbf{v} = u_0 \begin{pmatrix} -\sin \theta \sin \lambda (4 \cos^2 \theta - 1) \\ \sin^2 \theta \cos \lambda \end{pmatrix},$$  \hspace{1cm} (53)

with $u_0 = 2\pi a/24$ days, $gh_0 = 5768$ m$^2$ s$^{-2}$, and the Coriolis parameter given in (51).

Figure 9 depicts the initial and final solutions at day 10 of example f with $n = 2$ and $k = 2$ using the MWDG ELP scheme. The normalized integrals can be seen in Fig. 10, where, as in the previous example, there is a drift from machine precision conservation of mass and a similar order of accuracy is observed for the total energy and potential enstrophy.

We were able to simulate the DG RK4 up to a stable time step of 43 s. The MWDG ELP scheme remained stable up to a time step of 83 s. The wall-clock time of the DG RK4 method was 1.59 times that of MWDG ELP.

g. Barotropic instability

The final test that we explore in this paper is the test proposed in Galewsky et al. (2004). The initial height is given strictly in terms of latitude and obtained by numerically integrating the balance equation:
where $h_0$ is such that the mean layer depth is 10 km and the Coriolis parameter is given in (51). The barotropic instability is initiated by perturbing the flow with a localized bump to the balanced height field with the function

$$h_s(\theta, \lambda) = \tilde{h} \cos(\theta) e^{(-\lambda/\alpha)^2} e^{-(\theta_2 - \theta/\beta)^2},$$

with longitude $-\pi < \lambda < \pi$, $\theta_2 = \pi/4$, $\alpha = 1/3$, $\beta = 1/15$, and $\tilde{h} = 120$ m. The advecting wind is also given strictly in terms of latitude as

$$v = \begin{bmatrix} u(\theta) \\ 0 \end{bmatrix},$$

where

$$u(\theta) = \begin{cases} 0 & \text{if } \theta \leq \theta_0, \\ \frac{u_{\text{max}}}{e_n} & \text{if } \theta_0 < \theta < \theta_1, \\ 0 & \text{if } \theta \geq \theta_1, \end{cases}$$

with $u_{\text{max}} = 80$ m s$^{-1}$, $\theta_0 = \pi/7$, $\theta_1 = \pi/2 - \theta_0$, and $e_n = e^{-4(\theta_1 - \theta_0)^2}$.

Figure 11a depicts the vorticity at day 6 of example $f$ with $n = 3$ and $k = 9$ using the MWDG ELP scheme. We were able to simulate the DG RK4 up to a stable time step of 40 s. Figure 11b depicts the vorticity at day 6 of example $f$ with $n = 4$ and $k = 9$ using the MWDG ELP scheme. We were able to simulate the DG RK4 up to a stable time step of 18 s. A moderate (suboptimal) time step of 6 s was reported in Nair (2009) using a DG scheme on the cube sphere that had approximately the same degrees of freedom, which was an equivalent equator resolution of 0.64$^\circ$ as compared to 0.625$^\circ$ for this example. The MWDG ELP scheme remained stable up to a time step of 38 s. The wall-clock time of the DG RK4 method was 1.61 times that of MWDG ELP.

7. Conclusions

This research has demonstrated that significant increases in time-step length are possible for standard shallow-water test case problems on the sphere by using the ELP MWDG method as compared to DG RK4, and these increases result in real improvements in wall-clock simulation time. A 60-fold increase in time step is achieved for the first test in the standard suite developed by the climate modeling community (Williamson et al. 1992) in the most-challenging advection direction for the cubed-sphere geometry, which translates to about a 25-fold improvement in wall-clock simulation time. The results were more modest for nonlinear test cases, which showed an
improvement of approximately 50% in wall-clock simulation time.

The introduction of multiscale methods to earth system models will not only improve simulation time in the dynamical core, but also provide new tools to the user community. Multiscale methods have the potential to advance the understanding of the coupled interactive physical and chemical processes that modulate the climate system. For instance, multiscale representations will allow the community to analyze and correlate the modes of forcing and coupled surface interaction that drive the decadal and regional climate patterns. Multiscale methods may facilitate a broad advance in the area of downscaling and two-way coupling that could change the way regional hydrology and air quality forecasts are produced.

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**APPENDIX**

**Error Tolerance Thresholding for Multiwavelets**

The thresholding technique explained in Beylkin et al. (2008) for a given tolerance ε is explained in this appendix.
Without loss of generality, we describe this technique in one dimension.

Suppose that the multiwavelet expansion (14) is known for the function, \( f(x) \), on the intervals \( I^n_l = [2^n l, 2^n(l + 1)] \) for \( l = 0, \ldots, 2^n - 1 \). For each interval \( I^n_l \), refinement to \( I^n_{2l} \) and \( I^n_{2l+1} \) occurs if

\[
\max_{x \in I^n_l} |f(x) - \bar{f}(x)| > \epsilon, \tag{A1}
\]

for \( l = 0, \ldots, 2^n - 1 \).

**FIG. 10.** Section f normalized integrals for mass conservation, total energy, and potential enstrophy.

**FIG. 11.** (a) The numerical solution of vorticity at day 6 for the problem in section 6g with \( n = 3 \) and \( k = 9 \) using MWDG ELP with a time step of 80 s, and (b) \( n = 4 \) and \( k = 9 \) using MWDG ELP with a time step of 80 s. The equivalent resolutions at the equator for this set of multiwavelet parameters on the cube sphere are 1.25° and 0.625°, respectively.
where $I_k^n$ is the $k$th-order Gaussian nodes for $I^n_r$. Refinement continues until no intervals satisfy (A1).

REFERENCES


