A Discontinuous Galerkin Non-Hydrostatic Model with an Operator-Split Semi-Implicit Time Stepping Scheme

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Non-Hydrostatic (NH) Model Development: Basic Design

- Atmosphere: 3D Spherical domain (shell) with the vertical (radial direction) length scale $\mathcal{O}(10)$ km and the horizontal length scale $\mathcal{O}(10,000)$ km.

- The dynamics is governed by 3D compressible Euler/Navier-Stokes system of equations, based on conservation of mass, energy, momentum etc.

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) &= 0 \\
\frac{\partial \rho \mathbf{V}}{\partial t} + \nabla \cdot (\rho \mathbf{V} \otimes \mathbf{V}) &= -\nabla p' - (\rho - \bar{\rho})g\mathbf{k} - 2\rho \Omega \times \mathbf{V} + \mathbf{F}_M \\
\frac{\partial \rho \theta}{\partial t} + \nabla \cdot (\rho \theta \mathbf{V}) &= 0 \\
\frac{\partial \rho q_k}{\partial t} + \nabla \cdot (\rho q_k \mathbf{V}) &= 0
\end{align*}
\]

$\mathbf{V} = (u, v, w)$ 3D wind field, $\rho$ air density, $p$ pressure, $\theta$ potential temperature, $q_k$ moisture variables, $\Omega$ earth's rotation rate, $\mathbf{F}_M$ diffusive fluxes and forcing etc.
NH Model Development: Major Challenges

- **Why NH Models?**: Hydrostatic dynamics is not suitable or valid for horizontal resolution less than 10 KM \((1/8^\circ)\)
- Simulate atmospheric dynamics at ultra high-resolution (global cloud-system resolving model)

**Time Stepping: One of the most challenging aspects of NH modeling**

- **Implicit methods**: Large non-linear system (elliptic solvers), huge memory requirements, parallel efficiency at global scale?
- **Explicit methods**: Simple, parallel efficient but the CFL stability condition requires time-step reduction with grid refinement:

  \[
  CFL = \frac{C_{\text{max}} \Delta t}{\Delta h} \leq 1
  \]

  - \(C_{\text{max}}\) sound wave speed (340 m/s), \(\Delta h = \min(\Delta x, \Delta y, \Delta z)\). Typically \(\Delta x = O(100) \Delta z\)
  - Forcing the explicit time step \(\Delta t\) to be very small (fraction of a second!).

- **Semi-Implicit Methods**: A practical approach, successfully adopted by the community.
  - Operator-Split method: Split the Euler system into horizontal \((x,y\)-direction) and vertical \((z\)-direction) parts.
  - Solve implicitly the vertical (1D) component. The effective CFL is only limited to minimum horizontal grid spacing \((\Delta x, \Delta y)\).
  - **Horizontally Explicit and Vertically Implicit (HEVI)** approach, removes the dependence of vertical grid spacing by implicit time stepping.
High-Order Method Modeling Environment (HOMME)

Goal: Develop a NH dynamical core in HOMME framework
- Horizontal Grid system (Cubed-Sphere)
  - Equiangular central projection of a cube onto a sphere (Sadourny, 1972). Non-orthogonal curvilinear geometry (no polar singularities).
  - Quasi-uniform rectangular mesh, well suited for the element-based methods such as DG or SE methods.
  - Equations can be solved on the surface of a logical cube, in local \((x,y)\) coordinates.

- Vertical Grid system (\(z\)-coordinate)
  - Terrain-following height-based vertical \(z\) coordinate, \(z \in [h_s, z_T]\).
  - Vertical coordinate transformation (Gal-Chen & Somerville, JCP 1975)
    \[
    \zeta = z_T \frac{z - h_s}{z_T - h_s}, \quad \zeta \in [0, z_T]
    \]
We consider a simplified 2D compressible Euler/Navier-Stokes system in vector form:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

\[
\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I}) = -\rho \mathbf{g} \cdot \hat{k} + \nabla \cdot (\nu \rho \nabla \mathbf{u})
\]

\[
\frac{\partial \rho \theta}{\partial t} + \nabla \cdot (\rho \theta \mathbf{u}) = 0 + \nabla \cdot (\nu \rho \nabla \theta)
\]

where \( \rho \) the density, \( \mathbf{u} = (u, w) \) the velocity vector such that \( w = \mathbf{u} \cdot \hat{k} \), and is \( \theta \) the potential temperature. The pressure \( p \) and \( \theta \) are related through the equation of state:

\[
p = p_0 \left( \frac{\rho \theta R_d}{p_0} \right)^{c_p/c_v}; \quad p_0 = 10^5 \text{Pa}, \quad R_d = 287 \text{J/(kgK)}
\]

The 2D Euler system in the \( x-z \) Cartesian geometry can be written in flux-form:

\[
\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho u \\ \rho w \\ \rho \theta \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u w \\ \rho u \theta \end{bmatrix} + \frac{\partial}{\partial z} \begin{bmatrix} \rho w \\ \rho w u \\ \rho w^2 + p \\ \rho w \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\rho \mathbf{g} \\ 0 \end{bmatrix}.
\]
Compressible 2D Euler System: [Computational Form]

- Decompose \( \rho, \theta \) and \( p \) as the sum of a mean-state \( \bar{(.)} \) and perturbation \((.)'\)
such that \( \rho = \bar{\rho} + \rho' \), \( \theta = \bar{\theta} + \theta' \), \( p = \bar{p} + p' \), \( (\rho \theta) = \bar{\rho \theta} + (\rho \theta)' \).

- The mean-state maintains hydrostatic balance \( \frac{dp}{dz} = -\bar{\rho}g \).

- Hydrostatically balanced mean-state is ‘removed’ from the Euler system, resulting in:

\[
\frac{\partial}{\partial t}\begin{bmatrix}
\rho' \\
\rho u \\
\rho w \\
(\rho \theta)'
\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}
\rho u \\
\rho u^2 + p' \\
\rho w \\
\rho u \theta
\end{bmatrix} + \frac{\partial}{\partial z}\begin{bmatrix}
\rho w \\
\rho w u \\
\rho w^2 + p' \\
\rho w \theta
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
-\rho'g \\
0
\end{bmatrix}.
\]

- The corresponding compact form:

\[
\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}) \quad \Rightarrow \quad \frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U})
\]

- \( \mathbf{U} = [\rho', \rho u, \rho w, (\rho \theta)']^T \) is the state vector, \( \mathbf{F} = (\mathbf{F}_x, \mathbf{F}_z) \) is the flux vector and \( \mathbf{S} \) is the vector containing source terms.

[Skamarock & Klemp (2008), Giraldo & Restelli, JCP (2008), Norman et al., JCP (2010)]
Terrain-Following \( z \)-Coordinates: [Vertical Coordinate Transformation]

- If \( h = h(x) \) is the prescribed mountain profile and \( z_T \) is the top of the model domain, then the \((x,z)\) height coordinates can be transformed to \((x,\zeta)\) coordinates using the following relation, where \( \zeta \) is monotonic.

\[
\zeta = z_T \frac{z - h}{z_T - h}, \quad z(\zeta) = h(x) + \zeta \frac{z_T - h}{z_T}; \quad h(x) \leq z \leq z_T
\]

### Physical Grid \((x,z)\)

![Physical Grid](image1.png)

### Computational Grid \((x,\zeta)\)

![Computational Grid](image2.png)
Terrain-Following $z$-Coordinates [2D Euler System]

- In the transformed $(x, \zeta)$ coordinates, the Euler 2D system becomes:

$$
\begin{align*}
\frac{\partial}{\partial t} & \begin{bmatrix} \sqrt{G}\rho' \\ \sqrt{G}\rho u \\ \sqrt{G}\rho w \\ \sqrt{G}(\rho \theta)' \end{bmatrix} + \frac{\partial}{\partial x} & \begin{bmatrix} \sqrt{G}\rho u \\ \sqrt{G}(\rho u^2 + p') \\ \sqrt{G}\rho u \theta \\ \sqrt{G}(\rho \theta)' \end{bmatrix} + \frac{\partial}{\partial \zeta} & \begin{bmatrix} \sqrt{G}\rho \tilde{w} \\ \sqrt{G}(\rho \tilde{w} + G^{13} p') \\ \sqrt{G}\rho \tilde{w} \theta \\ \sqrt{G}(\rho \theta)' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\sqrt{G}\rho' g \\ 0 \end{bmatrix}.
\end{align*}
$$

- Where the metric terms (Jacobians) and new vertical velocity $\tilde{w}$ are

$$
\sqrt{G} = \frac{dz}{d\zeta}, \ G^{13} = \frac{d\zeta}{dx}; \quad \tilde{w} = \frac{d\zeta}{dt} = \frac{1}{\sqrt{G}} (w + \sqrt{GG^{13}} u)
$$

- The metric terms are time-independent (one time computation)

- Alternative formulations are also possible [e.g., Schär (2002), Klemp (2011)] for $\zeta$, but the system of equations remains in flux-from.

$$
\frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{S}(\boldsymbol{U}), \quad \boldsymbol{U} = [\sqrt{G}\rho', \sqrt{G}\rho u, \sqrt{G}\rho w, \sqrt{G}(\rho \theta)']^T
$$
Discontinuous Galerkin (DG) Methods in 2D

- DG Method is an ideal candidate for atmospheric model discretization, due to its inherent conservation property, geometric flexibility & parallel efficiency etc.

2D Scalar conservation law:

\[
\frac{\partial U}{\partial t} + \nabla \cdot F(U) = S(U), \quad \text{in} \quad (0,T) \times \mathcal{D}; \quad \forall (x,\zeta) \in \mathcal{D},
\]

where \( U = U(x,\zeta,t), \nabla \equiv (\partial/\partial x, \partial/\partial \zeta), \quad F = (F,G) \) is the flux function, and \( S \) is the source term.

- The domain \( \mathcal{D} \) is partitioned into non-overlapping elements \( \Omega_{ij} \)
- Element edges are discontinuous
- Problem is locally solved on each element \( \Omega_{ij} \)
DG-2D Spatial Discretization for an Element $\Omega_e$ in $\mathcal{D}$

- Approximate solution $U_h$ belongs to a vector space $\mathcal{V}_h$ of polynomials $\mathcal{P}_N(\Omega_e)$.

- The Galerkin formulation: Multiplication of the basic equation by a test function $\varphi_h \in \mathcal{V}_h$ and integration over an element $\Omega_e$ with boundary $\Gamma_e$,

  \[ \int_{\Omega_e} \left[ \frac{\partial U_h}{\partial t} + \nabla \cdot \mathbf{F}(U_h) - S(U_h) \right] \varphi_h d\Omega = 0 \]

- Weak Galerkin formulation: Integration by parts (Green's theorem) yields:

  \[ \frac{\partial}{\partial t} \int_{\Omega_e} U_h \varphi_h d\Omega - \int_{\Omega_e} \mathbf{F}(U_h) \cdot \nabla \varphi_h d\Omega + \int_{\Gamma_e} \mathbf{F}(U_h) \cdot \vec{n} \varphi_h d\Gamma = \int_{\Omega_e} S(U_h) \varphi_h d\Omega \]

- The analytic flux $\mathbf{F}(U_h) \cdot \vec{n}$ must be replaced by a numerical flux such as the local Lax-Friedrichs (Rusanov) Flux:

  \[ \mathbf{F}(U_h) \cdot \vec{n} = \frac{1}{2} \left[ (\mathbf{F}(U^-_h) + \mathbf{F}(U^+_h)) \cdot \vec{n} - \alpha(U^+_h - U^-_h) \right] . \]

- For the Euler system, $\alpha$ is the upper bound on the absolute value of eigenvalues of the flux Jacobian $\mathbf{F}'(U)$.

  \[ \alpha \to \max\{ |v^-| + c^-, |v^+| + c^+ \}, c = \sqrt{\gamma R \alpha T}, \quad v^\pm = u^\pm \cdot \vec{n} \]
DG-2D: Nodal Spatial Discretization

- Every element $\Omega_e$ is mapped onto a unique reference element $[-1, 1]^2$, with local coordinates $(\xi, \eta) \in [-1, 1]$.
- Construct a nodal basis set using a tensor-product of Lagrange polynomials $h_i(\xi)$, with roots at Gauss-Lobatto-Legendre (GLL) or Gauss-Legendre (GL) quadrature points $\{\xi_i\}$.
- The nodal basis set is $\{h_i(\xi) \ast h_j(\eta)\}$ with
  \[
  h_i(\xi)|_{\text{GLL}} = \frac{(\xi^2 - 1) P_N'(\xi)}{N(N+1) P_N(\xi_i)(\xi - \xi_i)} \quad \text{OR} \quad h_i(\xi)|_{\text{GL}} = \frac{P_{N+1}(\xi)}{P_{N+1}(\xi_i)(\xi - \xi_i)}
  \]
- $P_N(\xi)$ is the $N^{th}$ degree Legendre polynomial.
- The approximate solution and test functions are expressed in terms of basis function:
  \[
  U_h(\xi, \eta) = \sum_{i=0}^{N} \sum_{j=0}^{N} U_{ij} h_i(\xi) h_j(\eta) \quad \text{for} \quad -1 \leq \xi, \eta \leq 1
  \]
- Final form for the discretization leads to a system of ODEs:
  \[
  \frac{\partial U}{\partial t} + \nabla \cdot F(U) = S(U) \quad \Rightarrow \quad \frac{d}{dt} U_h(t) = \mathcal{L}(U_h)
  \]
High-Order Nodal Spatial Discretization [GL or GLL ?]

Pros & Cons

- The GL quadrature rule is exact for polynomials of degree up to $2N + 1$, but the GLL quadrature rule is exact for polynomials of degree up to $2N - 1$.
- For a given d.o.f, the GL quadrature is more accurate as opposed to GLL. For relatively low-order approximations ($N \leq 4$), the GL quadrature is desirable.
- GL grid requires an interpolation of solution at the flux points ($\xi, \eta = \pm 1$), additional computational overhead.
- GLL results in inexact integration, but easy to implement and efficient. For high-order ($N > 5$), GLL might be a better choice as the loss of accuracy is not significant.
DG-2D: Diffusion Process

Local Discontinuous Galerkin (LDG) method

- Bassi and Rebay (JCP, 1997) introduced a scheme for treating diffusion (viscous flux) terms in DG discretization of the compressible Navier-Stokes equations.
- Cockburn & Shu (1998) generalized this approach known as the LDG method.

Consider the following advection-diffusion equation on an element $\Omega$, with known (constant) diffusion coefficient $\nu$.

$$\frac{\partial U}{\partial t} + \nabla \cdot F(U) = \nu \nabla^2 U$$

- The key idea of LDG approach is the introduction of a local auxiliary variable $q = \nu \nabla U$, and rewrite the above problem as a first-order system:

\[
\begin{align*}
q - \nu \nabla U &= 0 \\
\frac{\partial U}{\partial t} + \nabla \cdot F(U) - \nabla \cdot q &= 0
\end{align*}
\]

- A robust approach, however, computationally expensive (uses wider stencil), and prohibitive for high-order diffusion (hyper-viscosity).
- More efficient approach based on Flux Reconstruction method may be desirable (Huynh, 2009).
The Diffusion Process: Flux Reconstruction (FR) method

- Introduced by Huynh (2007, 2009)

DG/FR shallow-water model in HOMME (quadrature-free implementation)

[Nair (2014)]

Differential Form:

\[
\frac{\partial U}{\partial t} + \nabla \cdot F(U) = S(U)
\]

Weak Galerkin Form (SE/DG methods):

\[
\frac{\partial}{\partial t} \int_{I_{i,j}} U_h \varphi_h \, ds - \int_{I_{i,j}} F(U_h) \cdot \nabla \varphi_h \, ds
\]

\[
+ \int_{\partial I_{i,j}} F(U_h) \cdot \vec{n} \varphi_h \, d\Gamma = \int_{I_{i,j}} S_h \varphi_h \, ds
\]

- Solve the differential form via spectral differencing (SD)
- At the element edges, continuity of fluxes are maintained by FR procedure
The Diffusion Process: Flux Reconstruction (FR) method

- For a simple conservation law, with a solution polynomial \( U_i(x) \) of degree \( N \)

\[
\frac{\partial U_i}{\partial t} + \frac{\partial F_i(x)}{\partial x} = 0 \quad \text{on} \quad I = [x_{i-1/2}, x_{i+1/2}]
\]

- Reconstructed new flux,

\[
\tilde{F}_i(x) = F_i(x) + \left[ \hat{F}_{i-1/2} - F(x_{i-1/2}) \right] G_L(x) + \left[ \hat{F}_{i+1/2} - F(x_{i+1/2}) \right] G_R(x),
\]

where \( G_L \) and \( G_R \) are the left and right correction functions (Radau polynomials) of degree \( N + 1 \).

- DG or SD methods can be recovered by choosing suitable \( G_L, G_R \) (Huynh, 2007):

\[
\frac{\partial U_{i,k}}{\partial t} + \frac{\partial F_{i,k}}{\partial x} + \left[ \hat{F}_{i-1/2} - F(x_{i-1/2}) \right] G_L'(x_{i,k}) + \left[ \hat{F}_{i+1/2} - F(x_{i+1/2}) \right] G_R'(x_{i,k}) = 0,
\]

For diffusion problem: \( U_t = U_{xx} \)

- Find the common value at the interface \( \hat{U}_{i+1/2} = (U_{j+1/2}^- + U_{j+1/2}^+) / 2 \)

- Find the common derivative the cell interface \( x_{i+1/2} \), using FR procedure on elements \( I_i, I_{i+1} \)

\[
[U_x]_{i+1/2}^{\text{com}} = ([U_i]_x^R + [U_{i+1}]_x^L) / 2. \quad \text{Find the second derivative} \ U_{xx}.
\]

- The process requires only 3-element wide stencil \( (I_{i-1}, I_i, I_{i+1}) \); i.e., one parallel communication.
Time Stepping Challenges for the ODE system

For the resulting ODE systems:

\[
\frac{dU_h}{dt} = L(U^h), \quad t \in (0, t_T)
\]

where \( L \) is the DG spatial discretization operator.

Options & Challenges

- **Explicit time integration** efficient and easy to implement.
  Stringent CFL constraint \( \Rightarrow \) tiny \( \Delta t \), limited practical value.

\[
\frac{C\Delta t}{\bar{h}} < \frac{1}{2N + 1}
\]

- \( C = \max\{|u| + c, |w| + c\}, \ c = \sqrt{\gamma R_d T} \), dominated by fast moving acoustic waves and gravity wave.

- Minimum grid spacing \( \bar{h} = \min\{\Delta x, \Delta z\} \), where \( \Delta z \ll \Delta x \).

- \( P^N\)-DG, choose \( N = \{2, 3, 4\} \), moderate order.

- Strong Stability-Preserving (SSP)-RK.

- **Implicit time integration**, unconditionally stable but generally expensive to solve. **Overall efficiency at a global scale is not known.**

- Semi-implicit time integration
  - Implicit solver for Linear part and explicit solver for nonlinear parts. Needs efficient Helmholtz solver.
  - HEVI: horizontal explicit and vertical implicit.
For the resulting ODE system

\[ \frac{dU_h}{dt} = L(U_h), \quad \text{with} \quad \frac{C\Delta t}{\bar{h}} < \frac{1}{2N+1} \]

To overcome \( \bar{h} = \min\{\Delta x, \Delta z\} \), treat the vertical time discretization (z-direction) in an implicit manner.

- **Benefit**: The effective Courant number is only limited by the minimum horizontal grid-spacing \( \min\{\Delta x, \Delta y\} \).
- **Bonus**: The HOMME hydrostatic dynamical core relies on explicit time-stepping with excellent parallel scalability (\( \mathcal{O}(10^5) \) processors). The ‘HEVI’ split approach might retain its parallel efficiency for NH equations too.
- Horizontal part and vertical part connected by **Strang-type** time splitting, permitting \( \mathcal{O}(\Delta t^2) \) accuracy.
- **Remarks of HEVI**:  
  - Particularly useful for 3D NH modeling (\( \Delta z : \Delta x = 1 : 1000 \)).  
  - Global NH models adopt the HEVI philosophy, NICAM\(^1\), MPAS\(^2\) etc.  
  - Recent high-order FV-NH\(^3\) models based on operator-split method.

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\(^1\)Satoh et al. 2008  
\(^2\)Skamarock et al. 2012  
\(^3\)Norman et al. (JCP, 2011), Ulrich et al. (MWR, 2012)
The Euler system for \( \mathbf{U} = (\sqrt{G}\rho', \sqrt{G}u, \sqrt{G}\rho\tilde{w}, \sqrt{G}(\rho\theta')^T \) is split into horizontal \((x)\) and vertical \((\zeta \text{ or } z)\) components:

\[
\begin{align*}
\text{(Euler sys)} \quad & \frac{\partial \mathbf{U}}{\partial t} + \frac{\mathbf{F}_x(\mathbf{U})}{\partial x} + \frac{\mathbf{F}_z(\mathbf{U})}{\partial z} = \mathbf{S}(\mathbf{U}) \\
\text{(H-part)} \quad & \frac{\partial \mathbf{U}}{\partial t} + \frac{\mathbf{F}_x(\mathbf{U})}{\partial x} = \mathbf{S}_x(\mathbf{U}) = (0, 0, 0, 0)^T \\
\text{(V-part)} \quad & \frac{\partial \mathbf{U}}{\partial t} + \frac{\mathbf{F}_z(\mathbf{U})}{\partial z} = \mathbf{S}_z(\mathbf{U}) = (0, 0, -\rho'g, 0)^T
\end{align*}
\]

One possible option is to perform “\( \text{H} - \text{V} - \text{H} \)” sequence of operations:

- Advance \( H \)-part by \( \Delta t/2 \) to get \( \mathbf{U}^* \), from the initial value \( \mathbf{U}^n \)
- Evolve \( V \)-part by a full time-step \( \Delta t \), to obtain \( \mathbf{U}^{**} \) from \( \mathbf{U}^* \)
- Advance \( H \)-part with \( \mathbf{U}^{**} \) by \( \Delta t/2 \), to get the new solution \( \mathbf{U}^{n+1} \)

Other possible option is “\( V - \text{H} - \text{V} \)” sequence.

The vertical part may be solved implicitly with DIRK (Diagonally Implicit Runge-Kutta) \(^4\).

HEVI can be viewed as an IMEX Runge-Kutta (RK) method (Giraldo et al. 2009)

For the implicit solver:

- inner linear solver uses Jacobian-Free GMRES.
- It usually takes 1 or 2 iterations for the outer Newton solver.

\(^4\)Durran, 2010
DGNH-P2: Inertia-Gravity Wave Test (\(\theta'\))

- The evolution of a potential temperature perturbation \(\theta'\) (K) in a channel having periodic lateral and no-flux top/bottom boundary conditions. [Skamarock & Klemp (1994)]

- Domain \([0, 300] \times [0, 10] \text{ km}^2\), simulated for \(T=3000\) s

- Widely used for testing time-stepping methods in NH models, and \(\Delta z << \Delta x\)
Potential temperature perturbation $\theta'$ after 3000 s.

- $\Delta t = 0.04$ s for explicit RK-DG & $\Delta t = 0.4$ for HEVI-DG
- $\Delta x = 500m$, $\Delta z = 50m$, i.e., $67 \times 201$ elements, $p^2$-GL grid.
- The magnitude of $\theta'$ is $O(10^{-3})$, difference fields for split-explicit RK is $O(10^{-7})$ and that with HEVI-DG is $O(10^{-6})$
The Courant number for HEVI-DG is only constrained by horizontal grid-spacing ($\Delta x$).

- $\Delta x = 10 \Delta z$
- HEVI-DG uses 10 times larger $\Delta t$ that allowed by explicit RK-DG

h-convergence for RK2 and HEVI:

Ref: Bao, Kloefkorn & Nair (MWR, 2014)
HEVI-DG: Convergence with large aspect ratio (1 : 100)

- $\Delta x = 100\Delta z$, i.e., 100 times larger $\Delta t$ for HEVI-DG
- Difference field $\theta'$ is $O(10^{-5})$.
- 2nd-order temporal convergence with a smooth test case.
Straka Density Current [Straka et al. (1993)]

To validate Diffusion Process in HEVI-DG with LDG/FCP

Thermal Bubble ($\theta'$)

- Domain \([-26.5, 26.5] \times [0, 6.4] km^2\). $\theta = \bar{\theta} + \Delta \theta$;
- Initially, $\bar{\theta} = 300K$, $\Delta \theta = -15K$, $u = w = 0$.
- Simulated for 900 s, with diffusion ($\nu = 75.0 m^2/s.$) added to the momentum and the potential temperature equations.
- No-flux boundary conditions ($u \cdot n = 0$) are used for all boundaries
- Due to the symmetry, only half of the domain is shown
Grid convergence: No noticeable changes in the fields at 100 m or higher resolutions

$\Delta t = 0.075 \text{ s}$ (both RK-DG and HEVI-DG), Diffusion Coeff $\nu = 75.0 m^2/s$. Handled by LDG/FR.
Straka Density Current: Diffusion LDG vs. FR

- Second-order diffusion with LDG and FR approach
- FR-Diffusion is found to be about 25% more efficient than that of LDG.

- Domain \([-26.5, 26.5] \times [0, 6.4] \text{km}^2\), simulated for \(T = 900\text{s}\).
- \(\Delta t = 0.05\text{s}, \Delta z = 100\text{m}, \Delta x = 100\text{m}\) with \(P^2\)-DG and
- FR approach uses narrow stencil (nearest neighbor), results are comparable with that of LDG.
Schär Mountain Test [Schär et al. (2002)]

- The mountain profile

\[ h(x) = h_0 \exp \left( -\frac{x^2}{a^2} \right) \cos^2 \left( \frac{\pi x}{\lambda} \right) \]

- Domain \([-25, 25] \times [0, 10] \) km, simulated for \( T = 10h \), and \( h_0 = 250\)m, \( \lambda = 4\)km, \( a = 5\)km, \( u = 10 \) m/s. (Non-reflecting BC)

- 50 × 25 elements, \( \Delta t = 0.125\)s, \( \Delta z = 210\)m, \( \Delta x = 250\)m with \( P^3\)-DG and RK-3 integrator.

Vertical Wind (m/s)

HEVI-DG

RK-DG
Summary

- DG-NH model (split and unsplit) performs well under benchmark test cases.
- HEVI effectively relaxes the CFL constraint to the horizontal dynamics only, and permits significantly larger time step as opposed to the fully explicit method.
- The diffusion process via FR method is efficient, and accuracy is comparable to that of LDG.
- Because of the compact stencil (one communication per Laplacian solve), FR hyper-diffusion scheme is an efficient option in HOMME.

Future Work

- Implementation of HEVI in 3D HOMME for more challenging benchmark tests (WIP).
- Improve the efficiency, for horizontal part: multi-rate time integration scheme, subcycling.
- Adopt efficient preconditioner for the implicit solver in the vertical part. Test Hybrid-DG method for the implicit part.

Thank You!

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