A flux-form version of the conservative semi-Lagrangian multi-tracer transport scheme (CSLAM)

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The current state of global models

- Massively parallel systems are the future of computing

- Traditional “regular latitude-longitude” computational grids for global weather and climate models have strong singularities at the poles. This currently requires non-local filtering, which is slow on parallel machines

- New grids are needed; and perhaps new numerical schemes as well

From MITgcm.org
The cubed-sphere grid

- Projects an inscribed cube onto the sphere, giving us six locally-cartesian grids to work with

- Grid spacing is nearly the same everywhere

- Weak singularities at the corners, which are less troublesome than on the regular latitude-longitude grid

- A fully 2D numerical method is likely needed for accuracy
CSLAM

- CSLAM is a “semi-Lagrangian” scheme, which solves the advection equation by evaluating the mass in an upstream cell, defined by back trajectories from the previous timestep.

- Each cell $A_i$ has a subgrid reconstruction $f_i$ fitted to cell-average solutions. These piecewise-defined reconstructions are then integrated over the area of the upstream cell.

\[
\frac{D\phi}{Dt} = 0
\]

\[
\phi_{k}^{n+1} = \frac{1}{\delta a_k} \sum_{\ell=1}^{L_k} \int \int_{a_{k\ell}} f_{\ell}(x, y) \, dx \, dy
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  **upstream** cell, defined by back trajectories from the previous timestep.

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CSLAM Advantages

- The upstream cells cover the whole sphere exactly; so the method is exactly mass-conservative

- The integration is actually a multiplication of area weights and the reconstruction coefficients. The weights do NOT depend on the data $\phi$; so they can be reused for each tracer. Adding more tracers is relatively cheap!

$$\frac{D\phi}{Dt} = 0$$

$$\phi_{k}^{n+1} = \frac{1}{\delta a_k} \sum_{\ell=1}^{L_k} \int \int_{a_{k\ell}} f_{\ell}(x, y) \, dx \, dy$$
Flux-form methods

- Integrate over the area swept out by the trajectories

- This gives us the integrated flux through each face of a cell. The change in the cell’s mass is then the divergence of the fluxes

  - Flux-form and semi-Lagrangian solutions are identical in the absence of limiting

  - Flux-form schemes are automatically mass conserving, and also allow us to use flux-corrected or flux-limited methods
Test 1: Cosine Bell in Solid Body Rotation

- Advect a (mostly) smooth blob of tracer around the sphere

- The bell goes through the singularities in the cubed sphere

- Each face has 48 grid points across it, and the timestep is 1800 seconds

- Numerical scheme is (theoretically) third-order accurate
Solution is identical to that of the semi-Lagrangian method; ripples occur in both s-L and flux-form solutions.
Filters and limiters

- Ripples ("undershoots and overshoots") are a serious problem if your tracer is a chemical species or a microphysical phase

  - These appear wherever the solution is not $C^\infty$ smooth

- We can **filter** the reconstruction (before computing the integrals) to ensure that the solution is "monotone" or free of ripples.

- Alternately, we can apply a **flux limiter**, which alters the flux to get monotonicity. This **requires** a flux-form method.

- Both methods tend to erroneously damp smooth extrema compared to the original, "unlimited" solution
Exact Solution

Error: Unlimited

Error: Zalesak Limiter

Error: Monotone Filter
East-west cross-section

Exact
Unlimited
Mono Filter
Zalesak Limiter

Error
Convergence rates: $l_2$

<table>
<thead>
<tr>
<th>Method</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unlimited</td>
<td>2.3</td>
</tr>
<tr>
<td>Zalesak</td>
<td>2.4</td>
</tr>
<tr>
<td>Monotone</td>
<td>2.3</td>
</tr>
</tbody>
</table>

Cosine bell, 45 deg angle, scaled dt, $l_2$ error
Convergence rates: $l_\infty$

- Unlimited: 2.2
- Zalesak: 1.8
- Monotone: 1.6
Test 2: Slotted cylinder

- Classic problem in numerical analysis: how to best handle a discontinuous initial condition

- Severe ripples will occur without some sort of limiting/filtering
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- Classic problem in numerical analysis: how to best handle a discontinuous initial condition

- Severe ripples will occur without some sort of limiting/filtering
Cross section:

East-west cross-section

Exact
Unlimited
Monotone filter
Zalesak Limiter
Cross-sections: double resolution

East-west cross-section

- Exact
- Unlimited
- Monotone filter
- Zalesak Limiter
Possible improvement: selective limiting

• Few 2D limiters and filters presently exist; most methods are for 1D “dimensionally split” schemes.

• Modifications to the existing limiters and filters may reduce their diffusivity. One method is selective limiting (Blossey and Durran 2008), which evaluates the smoothness of the solution.

  • If the solution is “smooth enough”, do not do any limiting. This avoids erroneously damping smooth extrema.
Examples of selective limiting

East-west cross-section

Error

Error
Conclusion

- A flux-form version of the CSLAM scheme (FF-SLAM?) has been introduced to a cubed-sphere model.

- Flux-form allows us to use flux limiting as well as filtering to reduce ripples in the solution.

- The Zalesak flux limiter was found to be less diffusive than the monotone filter.
  
  - The existing semi-Lagrangian scheme would restrict us to the less-accurate filter.

- Could a more efficient, less diffusive 2D filter be devised? Selective filtering could show the way.
Possible improvements

• Few 2D limiters and filters presently exist; most methods are for 1D “dimensionally split” schemes.

• Modifications to the existing limiters and filters may reduce their diffusivity
  
  • Zalesak’s limiter can be made slightly more accurate (but slower) by either applying the process multiple times, or by using a “higher order” monotone solution

• Applying the limiter or filter only where needed will also reduce diffusivity, while possibly speeding up the process
Examples of selective limiting

East-west cross-section

Exact
Unlimited
Monotone filter
Zalesak Limiter
BD Selective filter
Cosine bell, 45 deg angle, scaled dt, $l_2$ error

- unlimited
- Zalesak limiter
- Monotone filter
- BD Selective filter
- 2nd/3rd Order
Converting from semi-Lagrangian to flux-form

- Semi-Lagrangian cells only come in one simple shape.

- Flux areas can take all sorts of interesting shapes (even zero-area areas) and both clockwise and counter-clockwise orientations, which need to be accounted for in the model.

- Flux areas are also more sensitive to crossing trajectories, and can become unstable at high resolutions.