A Third-Order Non-Oscillatory Transport Scheme

Kiran K. Katta
(kkekatta@miners.utep.edu)

SIParCS 2010 Internship
National Center for Atmospheric Research
Boulder, Colorado

Advisor:
Ram Nair(NCAR)
Outline

- Introduction
- Overview of Godunov Type of Schemes
- A Third-Order Semi-Discrete Non-Oscillatory Scheme
- Linear Reconstruction
- Accuracy Tests
- Results
- Conclusions and Future Work
Objective is to solve transport (advection) equations on a Cartesian plane.

Achieve non oscillatory solution with conservation.

A third-order semi-discrete genuinely multidimensional central scheme constructed by Kurganov(2001) et al. is considered.

Advantages include:

- No Riemann Solver
- Can be used as a black box solver
- One can treat the system component wise
- Uses compact computational stencils
Overview of Godunov-type schemes

Godunov-Type Schemes are Projection-Evolution Methods

- Reconstruction
- Evolution
- Projection
  a. *Upwind*
    MUSCL, PPM, WENO
  b. *Central*
    NT, KT
Third-Order Semi-Discrete Non-Oscillatory Scheme in 1D

- 1D Scalar Conservation Law:

\[ u_t + \nabla_x \cdot f(u) = 0, \quad x \in \mathbb{R}^d, \]

Initial Data Subject to

\[ u(x, 0) = u_0(x). \]

- E.g., \( F(U) = c \ U \) (Linear Advection)

- The Domain \( \Omega \) (Periodic) is partitioned into \( N_x \) Non-Overlapping cells.

\[ I_j = [x_{j-1/2}, x_{j+1/2}] \]

\[ \Delta x_j = (x_{j+1/2} - x_{j-1/2}) \quad j = 1, \ldots, N_x \]
Linear Reconstruction

- Definition Sliding Average of $u(\cdot, t)$,

$$\tilde{u}(x, t) := \frac{1}{\Delta x} \int_{I(x)} u(\xi, t) \, d\xi,$$

$$I(x) = \{ \xi : |\xi - x| < \frac{\Delta x}{2} \}$$

- At Time Level $t = t_n$, a Piecewise Polynomial Function is Reconstructed

$$\tilde{u}(x, t^n) = p_j^n(x), \quad x_{j-1/2} < x < x_{j+1/2}, \quad \forall j,$$

- Reconstruction satisfies the property:

$$\bar{U}_j = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} u_j(x) \, dx,$$
Reconstruction 1D case

- The semi discrete scheme for the system

\[ \frac{d}{dt} \bar{u}_j(t) = \lim_{\Delta t \to 0} \frac{\bar{u}^n_{j+1} - \bar{u}^n_j}{\Delta t} \]

- The \( r^{th} \) order semi discrete scheme

\[ \frac{d}{dt} \bar{u}_j(t) = -\frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x} \]

- Numerical Flux is given by

\[ H_{j+\frac{1}{2}}(t) := \frac{f(u^+_{j+\frac{1}{2}}(t)) + f(u^-_{j+\frac{1}{2}}(t))}{2} \]
\[ -\frac{a_{j+\frac{1}{2}}(t)}{2} \left[ u^+_{j+\frac{1}{2}}(t) - u^-_{j+\frac{1}{2}}(t) \right] \]

- With

\[ u^+_{j+\frac{1}{2}} := p_{j+1}(x_{j+\frac{1}{2}}), \quad u^-_{j+\frac{1}{2}} := p_j(x_{j+\frac{1}{2}}) \]
Strong stability preserving Third-Order Runge-Kutta (SSP-RK) scheme is employed.

\[
U^{(1)} = U^n + \Delta t \mathcal{L}(U^n)
\]
\[
U^{(2)} = \frac{3}{4} U^n + \frac{1}{4} U^{(1)} + \frac{1}{4} \Delta t \mathcal{L}(U^{(1)})
\]
\[
U^{n+1} = \frac{1}{3} U^n + \frac{2}{3} U^{(2)} + \frac{2}{3} \Delta t \mathcal{L}(U^{(2)}).
\]

Where the superscripts \( n \) and \((n + 1)\) denote time levels \( t \) and \((t + \Delta t)\), respectively.

For the Linear case, CFL limit is 1.
1D Scheme:
Results (Linear Advection)

Figure 1. Limiter Removes Spurious Oscillations
Accuracy test for linear advection problem 1D case

\[ u_t + u_x = 0, \quad x \in [0, 2\pi] \]
\[ u(x, 0) = \sin x \]

- Limiter preserves the smoothness of the solution

### Table

<table>
<thead>
<tr>
<th>N(# of cells)</th>
<th>L_1 error</th>
<th>rate</th>
<th>L_∞ error</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>2.57459728253895889E-003</td>
<td></td>
<td>2.02373253303633760E-003</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>3.23984067162432496E-004</td>
<td>3</td>
<td>2.54507246700952372E-004</td>
<td>3</td>
</tr>
<tr>
<td>160</td>
<td>4.16561979646968363E-005</td>
<td>3</td>
<td>3.27189791449189471E-005</td>
<td>2.7</td>
</tr>
<tr>
<td>320</td>
<td>6.33817825783723357E-006</td>
<td>2.8</td>
<td>4.98023693862315042E-006</td>
<td>2.7</td>
</tr>
</tbody>
</table>
2D Semi-Discrete Scheme

This technique can also be directly applied to multidimensional problems if one uses the so-called ‘**Dimension-by-Dimension**’ approach.

**Semi Discrete Scheme for a 2D System** \( u_t + f(u)_x + g(u)_y = 0 \)

is

\[
\frac{d}{dt} \bar{u}_{j,k}(t) = -\frac{H_{j+\frac{1}{2},k}^x(t) - H_{j-\frac{1}{2},k}^x(t)}{\Delta x} - \frac{H_{j,k+\frac{1}{2}}^y(t) - H_{j,k-\frac{1}{2}}^y(t)}{\Delta y}.
\]

where

\[
H_{j+\frac{1}{2},k}^x(t) := \left\{ \begin{array}{l}
 f(u_{j+1,k}^{NW}(t)) + f(u_{j,k}^{NE}(t)) + 4(f(u_{j+1,k}^{W}(t)) + f(u_{j,k}^{E}(t))) \\
 + f(u_{j+1,k}^{SW}(t)) + f(u_{j,k}^{SE}(t)) \end{array} \right\} \cdot \left\{ \frac{12}{12} - \frac{a_{j+\frac{1}{2},k}(t)}{12} \right\} \\
\times \left[ u_{j+1,k}^{NW}(t) - u_{j,k}^{NE}(t) + 4(u_{j,k+1}^{W}(t) - u_{j,k}(t)) \\
+ u_{j+1,k}^{SW}(t) - u_{j,k}^{SE}(t) \right],
\]
2D Semi-Discrete Scheme continued...

Figure 2. (Left) Reconstruction in x and y Direction. (3X 3 Stencil) (Right) Reconstruction in diagonal directions.

$$H_{j+rac{1}{2},k}^x(t) := \left\{ \begin{array}{l} f(u_{j+1,k}^{NW}(t)) + f(u_{j,k}^{NE}(t)) + 4(f(u_{j+1,k}^W(t)) + f(u_{j,k}^E(t))) \\ + f(u_{j+1,k}^{SW}(t)) + f(u_{j,k}^{SE}(t)) \end{array} \right\} / \{12\} - \frac{\alpha_{j+\frac{1}{2},k}(t)}{12} \times \begin{array}{l} u_{j+1,k}^{NW}(t) - u_{j,k}^{NE}(t) + 4(u_{j,k+1}^W(t) - u_{j,k}^E(t)) \\ + u_{j+1,k}^{SW}(t) - u_{j,k}^{SE}(t) \end{array},$$
Initial Data Cosine-Cone Square Block

Solid Body Rotation Cosine-Cone Square Block After One Revolution Without Limiting
Initial Data Cosine-Cone Square Block

Solid Body Rotation Cosine-Cone Square Block After One Revolution with Limiting
Initial Data Considered Deformational Flow Test

Solution After 250 Time Steps

Accuracy Test 2D Case

M_Error vs. # of Time Steps (Gaussian Hill at Centre as Initial Data)

L1 and L∞ Errors vs. No. of Cells

Table 1: Errors (Gaussian Hill at the Center of the Domain as Initial Data)
Conclusions & Forthcoming Work

➢ Scheme is
  ➢ O(3) Accurate
  ➢ Non-Oscillatory & Conservative
  ➢ Potential to be Computationally Efficient

➢ Upcoming Work Includes
  ➢ The implementation of Diagonal Elements.
  ➢ Apply this scheme on to a Cubed Sphere.
References


Thank you !!!
Kiran Katta

Questions ???