Eddyless Channel Flows and Their Stability

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Numerical solutions are presented for steady flow in a zonal channel subjected to a meridionally varying zonal wind stress at the upper surface. The model used is adiabatic, hydrostatic, quasigeostrophic, and baroclinic with quadratic drag force at the lower boundary, vertical momentum diffusion in the interior, and a weak scale-selective horizontal momentum diffusion law. The solution technique is that of quasilinearization for the resulting non-linear, two-point boundary value problem. The steady solutions are also analyzed for their linearized stability to perturbations; they are found to be unstable for sufficiently strong wind stress and weak vertical momentum diffusion.
The study presented in this report is a component in a larger study of equilibrium geostrophic turbulence in a zonal channel. For intense forcing and weak frictional processes, the directly driven flows become unstable and generate turbulence, which itself then contributes to arriving at a state of statistical equilibrium. The present study allows one to identify the circumstances in which the turbulence can be expected to arise, by analyzing the stability of the steady, directly-driven solutions; these calculations can be made in much simpler models (i.e., zonally symmetric and time independent) than the one used for the numerical studies of turbulent equilibrium. This report also documents one of the contributions made by one of the authors (SJ) during his employment as a programming consultant to the Oceanography Section at NCAR.
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I. **Introduction**

This document is a description of steady solutions for a particular model of quasigeostrophic, wind-driven channel flows and their stability to infinitesimal amplitude perturbations. These issues arise in the context of an extended investigation of numerical calculations of turbulent channel flow by one of the authors (JM). Under circumstances of sufficiently weak wind-driving or sufficiently strong momentum diffusion the steady solutions are realizable and stable; for opposite circumstances they are not because turbulence forces the realizable time mean solutions to differ from the unstable steady solutions. The present study enables one to identify the point of departure between the two.

2. **Physical and Mathematical Formulation of the Problem**

McWilliams, Holland, and Chow (1978) presented a particular equilibrium turbulent solution for steadily driven currents in a quasigeostrophic, hydrostatic, adiabatic, baroclinic, $\beta$-plane, zonal channel, numerical model with the primary dissipation occurring as a drag force at the bottom boundary. This solution was felt to be sufficiently interesting to warrant a more extensive investigation of channel solutions in, however, a somewhat altered model. This more extensive investigation is currently in progress and will be reported on in McWilliams (1979).

The altered model, which also conforms to the physical description in the previous paragraph, has the following equations:

\[ D_1 q_1 = F_1 \]  \hspace{1cm} (2.1)
for \( i = 1, 3, 5 \). The index \( i \) labels the three layers in the model; \( q_i \) is the potential vorticity in each layer,

\[
q_1 = f_o + \beta(y-y_o) - \frac{fo^2}{g'_2H_1} (\psi_1-\psi_3) + \nabla^2 \psi_1
\]

\[
q_3 = f_o + \beta(y-y_o) - \frac{fo^2}{g'_2H_3} (\psi_1-\psi_3) - \frac{fo^2}{g'_4H_3} (\psi_3-\psi_5) + \nabla^2 \psi_3 \quad (2.2)
\]

\[
q_5 = f_o + \beta(y-y_o) + \frac{fo^2}{g'_4H_5} (\psi_3-\psi_5) + \nabla^2 \psi_5 + \frac{fo}{H_5} B
\]

\( \psi_i \) is the streamfunction in each layer (i.e., the horizontal velocities are \( u = -\psi_y \) and \( v = \psi_x \)); \( f_o \) and \( \beta \) are mid-channel values of respectively the Coriolis parameter and its \( y \)-derivative; \( H_i \) is the mean depth in each layer; \( g'_2 \) and \( g'_4 \) are reduced gravitational constants at the interfaces between respectively layers 1 and 3 and layers 3 and 5; \( B(x,y) \) is the elevation above a mean level of the lower boundary of the fluid (i.e., the bottom of layer 5); \( D_i \) is an advective time derivative,

\[
D_i = \partial_t + J_{x,y}(\psi_i, \cdot) \quad (2.3)
\]
\( \mathcal{F}_1 \) is the non-conservative potential vorticity forcing in each layer,

\[
\mathcal{F}_1 = -\frac{1}{H_1} \frac{\partial \tau(x)}{\partial y} (y) - K_4 \nabla^2 \psi_1 - \frac{A_2}{H_1} (\psi_1 - \psi_3),
\]

\[
\mathcal{F}_3 = -K_4 \nabla^2 \psi_3 + \frac{A_2}{H_3} \nabla^2 (\psi_1 - \psi_3) - \frac{A_4}{H_3} \nabla^2 (\psi_3 - \psi_5),
\]

\[
\mathcal{F}_5 = -K_4 \nabla^2 \psi_5 + \frac{A_4}{H_5} \nabla^2 (\psi_3 - \psi_5) - \varepsilon \nabla \cdot (|\nabla \psi_5 | \nabla \psi_5);
\]

\( \tau(x)(y) \) is the zonal wind-stress at the surface divided by a fluid density; \( K_4 \) is a lateral diffusivity; \( A_2 \) and \( A_4 \) are vertical momentum diffusion coefficients (n.b., one can interpret \( A_\alpha \) as a Newtonian kinematic viscosity \( \nu_\alpha \) at the interface \( \alpha \) by the relation

\[
A_\alpha = \frac{2}{H_{\alpha+1} + H_{\alpha-1}} \nu_\alpha
\]

for \( \alpha = 2, 4 \); and \( \varepsilon \) is a coefficient of bottom drag, which is related to a classical drag coefficient \( C_D \) by

\[
\varepsilon = \frac{C_D}{H_5}.
\]
Boundary conditions for the model are periodicity on a zonal scale \( L_X \), plus

\[
\nabla^2 \psi_i = \nabla^i \psi_i = 0 \quad \text{at} \quad y = 0, L_Y, \tag{2.7}
\]

plus a set of auxiliary conditions which will be recorded below only in a simplified form appropriate to the steady solutions of \( 2.1 \).

A set of parameter values was chosen for general similarity to the Antarctic Circumpolar Current [see the discussions in McWilliams, et al. (1978) and McWilliams (1979)]. They are

\[
\begin{align*}
H_1 &= 500 \text{ m} & f_0 &= -1.1 \times 10^{-4} \text{ s}^{-1} \\
H_3 &= 1250 \text{ m} & \beta &= 1.4 \times 10^{-11} \text{ m}^{-1} \text{s}^{-1} \\
H_5 &= 3250 \text{ m} & y_o &= 0.5 \times 10^6 \text{ m} \\
g_2' &= 0.01 \text{ ms}^{-2} & L_X &= 2 \times 10^6 \text{ m} \\
g_4' &= 0.01 \text{ ms}^{-2} & L_Y &= 10^6 \text{ m} \\
K_4 &= 10^{10} \text{ m s}^{-1} & \varepsilon &= 4 \times 10^{-7} \text{ m}^{-1} \\
A_2 &= 1.14 \times 10^{-7} \text{ ms}^{-1} & A_4 &= 0.44 \times 10^{-7} \text{ ms}^{-1}
\end{align*}
\tag{2.8}
\]

In addition we shall here consider only flat lower boundaries \((B = 0)\) and sinusoidal wind stresses,

\[
\tau^{(x)}(y) = \tau_0 \sin\left( \frac{\pi y}{L_Y} \right) \tag{2.9}
\]
A standard value for $\tau_0$ is $10^{-4}$ m$^2$s$^{-2}$.

Time-independent solutions of eqs. (2.1) that are also independent of $x$ can be found. In this case the equations governing the problem may be written as follows:

\[ \mathcal{G}_1 = -\frac{1}{H_1} \frac{d\tau}{dy} (x) - K_4 \frac{d^6 \psi_1}{dy^6} - \frac{A_2}{H_1} \frac{d^2}{dy^2} (\psi_1 - \psi_3) = 0 , \]

\[ \mathcal{G}_3 = -K_4 \frac{d^6 \psi_3}{dy^6} - \frac{A_2}{H_3} \frac{d^2}{dy^2} (\psi_3 - \psi_1) - \frac{A_4}{H_3} \frac{d^2}{dy^2} (\psi_3 - \psi_5) = 0 , \quad (2.10) \]

\[ \mathcal{G}_5 = -K_4 \frac{d^6 \psi_5}{dy^6} - \frac{A_4}{H_5} \frac{d^2}{dy^2} (\psi_5 - \psi_3) + 2\epsilon \frac{d\psi_5}{dy} \frac{d^2 \psi_5}{dy^2} = 0 , \]

where we have made the additional assumption that the flow is always in the positive $x$-direction; i.e., that $\frac{d\psi_5}{dy} < 0$ for all $y$.

The boundary conditions [eqs. (2.7)] become

\[ \frac{d^2 \psi_1}{dy^2} = \frac{d^4 \psi_1}{dy^4} = 0 \quad \text{at} \quad y = 0, L \quad . \quad (2.11) \]

(hereafter for conciseness we denote $L^V$ by $L$.) The auxiliary conditions referred to above are satisfied by (1) first integrals of the fundamental equations (2.10),
\[ \frac{d^5\psi_1}{dy^5} + \frac{A_2}{H_4K_4} \frac{d}{dy} (\psi_1 - \psi_3) = -\frac{1}{H_4K_4} \tau_0 \sin \frac{\pi y}{L} , \]
\[ \frac{d^5\psi_3}{dy^5} + \frac{A_2}{H_4K_4} \frac{d}{dy} (\psi_3 - \psi_1) + \frac{A_4}{H_4K_4} \frac{d}{dy} (\psi_3 - \psi_5) = 0 , \] (2.12)
\[ \frac{d^5\psi_5}{dy^5} + \frac{A_4}{H_4K_4} \frac{d}{dy} (\psi_5 - \psi_3) - \frac{\varepsilon}{K_4} \left( \frac{d\psi_5}{dy} \right)^2 = 0 ; \]

(2) mass conservation integrals,
\[ \int_0^L (\psi_1 - \psi_3) \, dy = \int_0^L (\psi_3 - \psi_5) \, dy = 0 ; \] (2.13)

and (3) an arbitrary normalization condition,
\[ H_1\psi_1 + H_3\psi_3 + H_5\psi_5 = 0 \quad \text{at} \quad y = L . \] (2.14)

For convenience in the mathematical formulation and numerical solution of the problem we make use of the following transformation to dimensionless variables:
\[ \xi = \frac{\gamma}{L}, \quad S = L \sqrt{\frac{\tau_o}{\epsilon H_5}}, \]

\[ \psi_i = \frac{\psi_{2i-1}}{S}, \quad h_i = \frac{H_{2i-1}}{L}, \quad (i = 1, 2, 3), \]

\[ \alpha = \frac{A_2 L^4}{H_1 K_4}, \quad \beta = \frac{A_2 L^4}{H_2 K_4}, \quad \gamma = \frac{A_4 L^4}{H_3 K_4}, \]

\[ \delta = \frac{A_4 L^4}{H_3 K_4}, \quad \varepsilon' = \frac{\varepsilon L S}{K_4}, \quad \eta = \frac{\tau_o L^5}{H_1 K_4 S} \]

We also introduce the negative of the dimensionless zonal velocity

\[ u_i = \frac{d\psi_i}{d\xi} , \quad \text{for } i = 1, 2, 3 . \]

With the above transformations, eqs. (2.11)-(2.12) may be rewritten as the following 12th-order, nonlinear, two-point boundary-value problem:

\[ \left( \frac{d^4}{d\xi^4} + \alpha \right) u_1 - \alpha u_2 = -\eta \sin \pi \xi , \]

\[ -\beta u_1 + \left( \frac{d^4}{d\xi^4} + \beta + \gamma \right) u_2 - \gamma u_3 = 0 , \]
\[-\delta u_2 + \left( \frac{d^4 u}{d\xi^4} + \delta \right) u_3 - \varepsilon' u_3^2 = 0,\]

with

\[\frac{d u_i}{d\xi} = \frac{d^3 u_i}{d\xi^3} = 0 \text{ at } \xi = 0,1 \text{ for } i = 1,2,3.\]  \hfill (2.18)

The dimensionless streamfunctions are found from the solution \(u_i(\xi)\) of the boundary-value problem by quadratures of eq. (2.16), with constants of integration chosen to satisfy eqs. (2.13)-(2.14).

From the symmetry of the boundary-value problem, it follows that, if \(u_i = f_i(\xi)\) is a solution, then \(u_i = f_i(1-\xi)\) is also a solution. If the solution of the boundary-value problem is unique, then it must be symmetric about the axis \(\xi = 1/2\). In this case we need only to solve the problem numerically over the interval \(0 \leq \xi \leq 1/2\) with the boundary conditions,

\[\frac{d u_i}{d\xi} \frac{d^3 u_i}{d\xi^3} = 0 \text{ at } \xi = 0, 1/2 \text{ for } i = 1,2,3.\]  \hfill (2.19)

Questions concerning the uniqueness of the solutions of nonlinear, two-point boundary-value problems are in general difficult to answer. Here we have
not tried to show whether the boundary-value problem studied in this investigation does or does not have a unique solution; instead, we restrict our attention to the symmetric solution.

The system of equations (2.17) can be rewritten in canonical form as the following system of 12 first-order differential equations:

\[
\frac{du_1}{d\zeta} = u_{i+3} \quad \text{for } i = 1, 2, 3, \quad (2.20)
\]

\[
\frac{du_{10}}{d\zeta} = -\alpha u_1 + \alpha u_2 - \eta \sin \pi \xi, \quad (2.21)
\]

\[
\frac{du_{11}}{d\zeta} = \beta u_1 - (\beta + \gamma) u_2 + \gamma u_3, \quad (2.22)
\]

\[
\frac{du_{12}}{d\zeta} = \delta u_2 - \delta u_3 + \varepsilon' u_3^2, \quad (2.23)
\]

and the boundary conditions (2.19) become

\[
u_1 = 0 \text{ at } \zeta = 0, 1/2 \text{ for } i = 4, 5, 6, 10, 11, 12. \quad (2.24)
\]

3. The Numerical Method

To obtain a numerical solution of the nonlinear two-point boundary-value problem we use the method of quasilinearization, which is discussed by
Roberts and Shipman (1972, Chap. 5), together with SUPORT, a general-purpose computer code for solving linear two-point boundary-value problems written by Scott and Watts (1975).

In quasilinearization the nonlinear two-point boundary-value problem is solved by first linearizing the differential equations about some trial solution which satisfies the differential equations only approximately but which satisfies the boundary conditions exactly and then solving an iterative sequence of linear two-point boundary-value problems. For the first iteration a trial solution must be guessed which is close enough to the true solution to ultimately ensure convergence; after that the solution from the previous iteration is taken as the trial solution. When the solutions from two consecutive iterations agree within some specified tolerance, the iterations are stopped and the solution from the last iteration is taken as the solution of the nonlinear two-point boundary-value problem.

In the boundary-value problem defined by eqs (2.20)-(2.24) the only nonlinear term is the last term in eq. (2.23). If we let \( u_3^{(0)}(\xi) \) denote the trial solution, then under quasilinearization the nonlinear term becomes

\[
\varepsilon' u_3^2 = \varepsilon' [u_3^{(0)} + (u_3 - u_3^{(0)})]^2 \simeq 2\varepsilon' u_3^{(0)} u_3 - \varepsilon' u_3^{(0)^2} \quad (3.1)
\]

where the square of \( u_3 - u_3^{(0)} \) is neglected. Thus, since eqs. (2.20)-(2.22) are already linear, the only equation affected by quasilinearization is eq. (2.23), which becomes
The initial trial solution was always taken to be

\[ u_3^{(o)}(\xi) = -1; \quad u_1^{(o)}(\xi) = 0, \quad i \neq 3, \]  \hspace{1cm} (3.3)

which satisfies the boundary conditions, eqs. (2.24). This guess for \( u_3^{(o)} \) was obtained from a solution of eqs. (2.17) at \( \xi = 1/2 \), neglecting in each equation the fourth derivative terms (which are generally small away from \( \xi = 0 \) or 1). There are, of course, non-zero \( u_1 \) and \( u_2 \), values associated with \( u_3 = -1 \), but they need not be included in \( u_1^{(o)} \) since the quasilinearization only involves \( u_3^{(o)} \).

The computer code SUPORT of Scott and Watts (1975), which we use to solve our iterative sequence of linear two-point boundary-value problems, is based on a shooting method; that is, it uses an initial-value technique to solve a two-point boundary-value problem. More specifically SUPORT uses superposition; that is, the solution is expressed as a linear combination of linearly independent solutions of the homogeneous system plus a particular solution of the nonhomogeneous system. The code exploits the fact that by suitably choosing the initial conditions for these solutions it is possible to reduce the number of solutions of the homogeneous system that are needed to the number of "missing initial conditions."
The code carries out the numerical integration of the appropriate initial value problem with a package of subroutines using a variable-step Runge-Kutta-Fehlberg method. The linear independence of the solutions is assured by keeping them nearly mutually orthogonal. Whenever the solutions start to lose their numerical linear independence they are reorthonormalized. The coefficients in the linear combination are determined from the final boundary conditions. SUPORT then obtains the desired solution by piecing together the solutions on the various subintervals defined by the reorthonormalization points. The theoretical basis of a shooting method practically identical to one used in SUPORT is discussed by Roberts and Shipman (1972, Chap. 4) as the "method of complementary functions." For brevity we shall refer to the method of using SUPORT with quasilinearization as the shooting method.

In order to solve the steady channel-flow problem posed by eqs. (2.10)-(2.14), we have written a control program, subroutine CHANNEL, to call the package SUPORT. Subroutine CHANNEL directs the solution of the steady channel-flow problem for any specified values of the parameters \( L, \varepsilon, \tau, k, A_2, A_4, H_1, H_3, \) and \( H_5 \). This program (1) converts back and forth between dimensional (m.k.s.) and dimensionless variables, (2) sets up the input to subroutine SUPORT, (3) controls the iterations required by the quasilinearization method, (4) calculates the streamfunctions, and (5) produces the desired output. We specified a tolerance that should yield about five significant figures of accuracy. No case studied required more than six iterations to converge. An outline of subroutine CHANNEL is given in the Appendix.

In addition to the control program we must also supply two subroutines to be used by the package SUPORT to evaluate the homogeneous and nonhomogeneous parts, respectively, of the derivative formulas given by
eqs. (2.20)-(2.22) and (3.2). In order to be able to evaluate the derivatives for any value of x as required by SUPORT, which uses a variable step of integration, we use the package CURV described by Adams et al. (1976) to interpolate for values of \( u_3^{(o)}(\xi) \) between the gridpoints on which it is stored. CURV performs the interpolation by the method of splines under tension. It is also used to perform the numerical quadratures needed to evaluate the streamfunctions from eq. (2.16).

4. The Stability Analysis

Once a steady solution \( \overline{\psi}_j(y) \) has been calculated from eqs. (2.10)-(2.14), a stability analysis of this flow can be performed by solving the eigenvalue problem associated with small amplitude streamfunction perturbations of the form

\[
\psi_j = \text{Re} \left\{ f_j(y) e^{(ikx + \sigma t)} \right\}, \quad j = 1, 3, 5.
\]

The equations for this eigenvalue problem arise from a linearization of eqs. (2.1) about the reference solution \( \overline{\psi}_j \). The computational techniques by which this eigenvalue problem has been solved are quite similar to those described in Haidvogel and Holland (1978) for a set of model equations similar to (2.1). The results which we shall show are restricted to the most unstable eigenmode: for a given set of parameters and the corresponding \( \overline{\psi}_j \), we select the mode which has the greatest positive value for \( \text{Re}[\sigma(k)] \), either for all k or a particular k. The computer code for the stability calculation is based upon the values of \( \overline{\psi}_j \) on a uniform grid of points across the channel. We used the solution returned by the
steady-channel-flow code on a uniform grid of 53 points as input to the stability code.

5. Parameter Studies

A standard case, case S, is defined by the parameter values given in eqs. (2.8) plus the value $\tau_0 = 10^{-4} \text{m}^2\text{s}^{-2}$. We have solved the steady channel-flow problem for two parameter sequences, each beginning with case S. In the first sequence (the $\tau$-sequence) all the parameters were fixed at the case-S values except the wind-stress amplitude $\tau_0$, which was decreased in a geometric progression from $10^{-4} \text{m}^2\text{s}^{-2}$ (case S) to $10^{-10} \text{m}^2\text{s}^{-2}$. In the second sequence (the $A_2$-sequence) all the parameters were fixed at the case-S values except the two interfacial frictional coefficients $A_2$ and $A_4$, which were simultaneously increased in a geometric progression from the case-S values of $1.14 \times 10^{-7}$ and $4.44 \times 10^{-8} \text{ms}^{-1}$, respectively, to values of $1.14 \times 10^{-2}$ and $4.44 \times 10^{-3} \text{ms}^{-1}$, respectively. Thus, in both sequences the ratio $A_2/A_4$ remains fixed at the case-S value.

For all cases, the velocities decrease systematically from the top layer, which is driven by the wind, to the bottom layer, which is slowed down by drag against the bottom. The highest velocity for each layer always occurs at the center of the channel, where the wind is the strongest. Thus the maximum velocity in the channel always occurs at the center of the top layer. For all the cases studied, the flow everywhere is in the direction of the wind (i.e., in the positive $x$-direction) which is an a posteriori justification of the assumption made in deriving the last of eqs. (2.10). Figure 1 shows that the maximum velocity in the upper two layers is nearly proportional to the wind-stress amplitude, while that in the lower layer is nearly proportional to $\tau_0^{1/2}$. Figure 2 shows that the maximum velocity in the upper two layers decreases monotonically with the
interfacial-friction, while that in the lower layer remains nearly constant.

The stability of the steady flows obtained for each member of the two sequences was tested with the stability code described in the previous section. Case S is strongly unstable, but the growth rate of instability declines along each sequence until eventually stable steady flows are reached. Figures 3 and 4 show the growth rate as a function of $\tau_0$ and $A_2$, respectively. For strong wind forcing the growth rate is very nearly proportional to $\tau_0$, and for weak interlayer friction the growth rate is very nearly inversely proportional to $A_2$. However, these proportionalities do not hold near the point of marginal stability.

The search for unstable modes was made over the range $1 \leq k \leq 13$, $\Delta k = 1/2$, for case S and for the two members of each sequence bracketing the point of marginal stability. For other members of the two sequences the search was made in a more restricted range. Here $k$ is in units of cycles per $2 \times 10^6$ m; $2 \times 10^6$ m was a standard value for the channel length in the numerical studies to be reported in McWilliams (1979). The wave number of the mode with the greatest growth rate for each unstable member of the two sequences is presented in Tables I and II. It is always of the order of 6 or 7. Figure 5 shows the spectrum of growth rates for case S (i.e., the largest growth rate for each wave number $k$). The values in Fig. 5 for Re{$\sigma$} are quite large ($< 10^{-2}$ s$^{-1}$), so large in fact as to be inconsistent with our original assumption of quasigeostrophy (where we require $\sigma < f \sim 10^{-4}$ s$^{-1}$). This is to be interpreted as indicating the unrealizability of the steady solution for case S, since it is many orders of magnitude away from marginal stability.
The steady-flow and stability calculations are summarized in Figs. 6 - 23. Results are presented for case S (unstable), for the unstable member of the $\tau$-sequence nearest marginal stability, $\tau_0 = 1.911 \times 10^{-9} \text{ m}^2\text{s}^{-2}$, for the last member (stable) of the $\tau$-sequence, $\tau_0 = 10^{-10} \text{ m}^2\text{s}^{-2}$, for the unstable member of the $A_2$-sequence nearest marginal stability, $A_2 = 3.874 \times 10^{-3} \text{ ms}^{-1}$ and $A_4 = 1.509 \times 10^{-3} \text{ ms}^{-1}$, and for the last member (stable) of the $A_2$-sequence, $A_2 = 1.14 \times 10^{-2} \text{ ms}^{-1}$ and $A_4 = 4.44 \times 10^{-3} \text{ ms}^{-1}$. The figures show profiles of the streamfunctions and velocities across the channel in the top (A), middle (B), and bottom (C) layers and contour maps of the perturbation to the streamfunction in the top layer for the mode with the greatest growth rate. For case S the profiles of the first three (spatial) derivatives of the velocities are also shown as well as contour maps of the perturbation to the streamfunctions in the middle and bottom layers. The velocity derivatives for the whole $\tau$-sequence (not presented) are qualitatively similar to case S; however, in the $A_2$-sequence for large interlayer friction the first derivatives of velocity (also not presented) show pronounced boundary-layer behavior near the edges of the channel with the broad maxima for case S becoming more spike-like and shifted toward the edges.

6. **Finite-Difference Method**

To compare the accuracy of the shooting method with one more like the numerical method used in time-dependent numerical models, the steady-state channel flow two-point boundary-value problem was also solved by a finite-difference method.
First a general subroutine was written to solve linear two-point boundary-value problems consisting of several (say M) coupled second-order differential equations of the form

\[
\frac{d^2 y_i}{dx^2} + \sum_{j=1}^{M} u_{ij}(x) \frac{dy_j}{dx} + \sum_{j=1}^{M} v_{ij}(x)y_j = w_i(x),
\]

for \( i = 1, 2, \ldots, M \) and \( a \leq x \leq b \), \hspace{1cm} (6.1)

for \( i = 1, 2, \ldots, M \) and \( a \leq x \leq b \),

with the values of the solution specified at the endpoints; i.e.,

\[
y_i = c_i \quad \text{at } x = a
\]

\[
y_i = d_i \quad \text{at } x = b,
\]

for \( i = 1, 2, \ldots, M \). \hspace{1cm} (6.2)

The derivatives in eqs. (6.1) are approximated by centered difference formulas; viz.,

\[
\frac{dy_i}{dx} = \frac{y_i(x_{n+1}) - y_i(x_{n-1})}{2h},
\]

\hspace{1cm} (6.3)

and

\[
\frac{d^2 y_i}{dx^2} = \frac{y_i(x_{n+1}) - 2y_i(x_n) + y_i(x_{n-1})}{h^2},
\]

\hspace{1cm} (6.4)
on each of the interior grid points $x_n$, where

$$a = x_1 < x_n < x_N = b \quad \text{for } n = 2, 3, \ldots, N-1 \quad \text{(6.5)}$$

The grid spacing is uniform; thus

$$h = \frac{b-a}{N-1} = x_{n+1} - x_n \quad \text{for } n = 1, 2, \ldots, N-1 \quad \text{(6.6)}$$

For each specific linear two-point boundary-value problem a subroutine must be supplied to evaluate the matrices $u_{ij}(x)$ and $v_{ij}(x)$ and the vector $w_i(x)$ on the gridpoints $x_n$. The system of linear algebraic equations resulting from writing eqs. (6.1)-(6.2) in finite-difference form has a coefficient matrix that is banded in blocks about the principal diagonal, and this feature is exploited in solving these equations.

This subroutine for solving linear two-point boundary-value problems with a finite-difference technique was incorporated into a computer code for checking the solution obtained by the shooting method. The code uses quasi-linearization to solve the 18th-order nonlinear system of differential equations given by eqs. (2.10). Twelve of the boundary conditions are given by eqs. (2.11), and for the other six the values of the streamfunctions at the endpoints were specified to be the same as those obtained by the shooting method. Equations (2.10) were written as a system of nine second-order
differential equations and then linearized to make them amenable to solution with the above mentioned finite-difference subroutine for linear boundary-value problems. A grid of 53 points, evenly spaced across the width of the channel, was chosen to make the method equal in resolution to time-dependent numerical models. The initial trial solution for the iterations in the quasilinearization method was taken as the solution obtained by the shooting method. Thus the solution from the shooting method was allowed to relax on a finite-difference grid.

The results for the maximum velocity obtained by the shooting method and the finite-difference method are compared in Table 3. The differences are quite small except near the edges of the channel, where the difference varies and approximately 0.1% for case S and the entire $\tau$ sequence to approximately 4% for the last unstable model in the $A_2$ sequence.

Near each edge of the channel there is a boundary layer. The uniform grid used in the finite-difference method is rather coarse and puts only two or three points inside each boundary layer. On the other hand, the shooting method uses sophisticated subroutines for solving initial value problems that can vary the number of integration steps between output gridpoints so as to keep the error made for each integration step below some specified tolerance. Thus the shooting method can be expected to put in as many steps as needed to keep the error small.

7. **Summary**

A technique of solution for steady channel flow has been described in this paper, and the resulting solutions have been analyzed for their stability to infinitesimal amplitude perturbations. These solutions are stable for
sufficiently weak wind forcing or sufficiently large interfacial friction. On the other hand, for values of these quantities selected as appropriate for the Antarctic Circumpolar Current, the steady solutions are extremely unstable, hence unrealizable.
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APPENDIX

Outline of Subroutine CHANNEL

1. Specify the parameters $L$, $c$, $T$, $K_4$, $A_2$, $A_4$, $H_1$, $H_3$, $H_5$.
2. Calculate the constants $a$, $b$, $\gamma$, $\delta$, $\varepsilon'$, $\eta$.
3. Define a grid of 27 evenly spaced points on the interval $0 \leq \xi \leq 1/2$.
4. Store the initial trial solution, $u_i^{(o)}(\xi)$, eqs. (3.3), on the gridpoints.
5. Define the boundary conditions, eqs. (2.24).
6. Specify the desired accuracy of solution.
7. Set the iteration count $k = 1$.
8. Set up an interpolation formula for finding the trial solution between gridpoints.
9. Call SUPORT to find the solution $u_1(\xi)$ of the linear two-point boundary-value problem, eqs. (2.20)-(2.22), (3.2), and (2.24) on the gridpoints.
10. Did SUPORT find a solution? If no, stop. If yes, proceed to the next step.
11. Has the solution converged? Does it agree with the trial solution within the prescribed tolerance? If yes, go to step 15. If no, proceed to the next step.
12. Is the iteration count $k > 10$? If yes, stop. If no, proceed to the next step.
13. Save the solution to use as the trial solution for the next iteration. Set $u_i^{(o)}(\xi) = u_1(\xi)$.
14. Increment the iteration count (set $k = k+1$) and go to step 8.

15. Calculate the dimensionless streamfunctions, normalized to zero at the center of the channel, by numerical quadratures. Because of symmetry this automatically satisfies eqs. (2.13).

16. Calculate and add a constant to all three streamfunctions so that they will satisfy the normalization condition, eq. (2.14).

17. Evaluate the streamfunctions and velocities in mks units on a grid of 53 evenly spaced points covering the full channel width.

18. Print out, punch, and plot the streamfunctions and velocities and return to the calling program.
TABLE I
Zonal Wave Numbers and Growth Rates of Maximum Instability
and Maximum Steady-State Velocities in the $\tau$ Sequence

<table>
<thead>
<tr>
<th>$\tau$ [$m^2 s^{-2}$]</th>
<th>$k_{\text{max}}$ [c.p. $2 \times 10^6 m$]</th>
<th>$\sigma_{\text{max}}$ [$s^{-1}$]</th>
<th>$u_{\text{max}}$ [$ms^{-1}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-5}$ (Case S)</td>
<td>6</td>
<td>$1.213 \times 10^2$</td>
<td>$2.827 \times 10^3$</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>6</td>
<td>$1.213 \times 10^{-3}$</td>
<td>$2.828 \times 10^{-2}$</td>
</tr>
<tr>
<td>$10^{-7}$</td>
<td>6</td>
<td>$1.212 \times 10^{-4}$</td>
<td>$2.830 \times 10^{-3}$</td>
</tr>
<tr>
<td>$10^{-8}$</td>
<td>6</td>
<td>$1.207 \times 10^{-5}$</td>
<td>$2.835$</td>
</tr>
<tr>
<td>$5.623 \times 10^{-9}$</td>
<td>6.5</td>
<td>$1.131 \times 10^{-6}$</td>
<td>$2.852 \times 10^{-1}$</td>
</tr>
<tr>
<td>$3.162 \times 10^{-9}$</td>
<td>7</td>
<td>$5.824 \times 10^{-7}$</td>
<td>$1.608 \times 10^{-2}$</td>
</tr>
<tr>
<td>$2.371 \times 10^{-9}$</td>
<td>7</td>
<td>$2.473 \times 10^{-7}$</td>
<td>$9.077 \times 10^{-3}$</td>
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<tr>
<td>$2.054 \times 10^{-9}$</td>
<td>7.5</td>
<td>$1.176 \times 10^{-8}$</td>
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</tr>
<tr>
<td>$1.911 \times 10^{-9}$</td>
<td>7</td>
<td>$4.522 \times 10^{-9}$</td>
<td>$5.916 \times 10^{-2}$</td>
</tr>
<tr>
<td>$1.778 \times 10^{-9}$</td>
<td>stable</td>
<td>$4.226 \times 10^{-10}$</td>
<td>$5.509 \times 10^{-2}$</td>
</tr>
<tr>
<td>$10^{-9}$</td>
<td>stable</td>
<td>---</td>
<td>$5.130 \times 10^{-2}$</td>
</tr>
<tr>
<td>$10^{-10}$</td>
<td>stable</td>
<td>---</td>
<td>$2.904 \times 10^{-3}$</td>
</tr>
<tr>
<td>$10^{-11}$</td>
<td>stable</td>
<td>---</td>
<td>$3.069 \times 10^{-3}$</td>
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TABLE II

Zonal Wave Numbers and Growth Rates of Maximum Instability and Maximum Steady-State Velocities in the $A_2$ Sequence

<table>
<thead>
<tr>
<th>$A_2$ [ms$^{-1}$]</th>
<th>$k_{max}$ [c.p. 2x10$^6$ m]</th>
<th>$\sigma_{max}$ [s$^{-1}$]</th>
<th>$u_{1max}$ [ms$^{-1}$]</th>
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</thead>
<tbody>
<tr>
<td>1.14 x 10$^{-7}$ (Case S)</td>
<td>6</td>
<td>1.213 x 10$^{-2}$</td>
<td>2.827 x 10$^3$</td>
</tr>
<tr>
<td>1.14 x 10$^{-6}$</td>
<td>6</td>
<td>1.361 x 10$^{-3}$</td>
<td>3.128 x 10$^2$</td>
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<td>1.14 x 10$^{-5}$</td>
<td>6</td>
<td>1.368 x 10$^{-4}$</td>
<td>3.156 x 10$^1$</td>
</tr>
<tr>
<td>1.14 x 10$^{-4}$</td>
<td>6</td>
<td>1.355 x 10$^{-5}$</td>
<td>3.407</td>
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<td>1.14 x 10$^{-3}$</td>
<td>6</td>
<td>1.136 x 10$^{-6}$</td>
<td>5.903 x 10$^{-1}$</td>
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<td>2.027 x 10$^{-3}$</td>
<td>6</td>
<td>4.730 x 10$^{-7}$</td>
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<td>3.605 x 10$^{-3}$</td>
<td>6.5</td>
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<td>3.874 x 10$^{-3}$</td>
<td>6.5</td>
<td>1.489 x 10$^{-8}$</td>
<td>3.694 x 10$^{-1}$</td>
</tr>
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<td>4.163 x 10$^{-3}$</td>
<td>stable</td>
<td>---</td>
<td>3.631 x 10$^{-1}$</td>
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<tr>
<td>4.807 x 10$^{-3}$</td>
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<td>---</td>
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</tr>
<tr>
<td>5.551 x 10$^{-3}$</td>
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<td>---</td>
<td>3.416 x 10$^{-1}$</td>
</tr>
<tr>
<td>6.411 x 10$^{-3}$</td>
<td>stable</td>
<td>---</td>
<td>3.330 x 10$^{-1}$</td>
</tr>
<tr>
<td>1.14 x 10$^{-2}$</td>
<td>stable</td>
<td>---</td>
<td>3.086 x 10$^{-1}$</td>
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### TABLE III
Comparisons of Solutions from the Shooting (SM) and Finite Difference (FD) Methods

- **a) case S**

<table>
<thead>
<tr>
<th>y/L</th>
<th>u (SM)</th>
<th>u (FD)</th>
<th>% diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1.075 \times 10^3$ m/sec</td>
<td>$1.076 \times 10^3$ m/sec</td>
<td>0.121</td>
</tr>
<tr>
<td>0.038</td>
<td>$1.083 \times 10^3$ m/sec</td>
<td>$1.084 \times 10^3$ m/sec</td>
<td>0.119</td>
</tr>
<tr>
<td>0.500</td>
<td>$2.827 \times 10^3$ m/sec</td>
<td>$2.826 \times 10^3$ m/sec</td>
<td>0.038</td>
</tr>
</tbody>
</table>

- **b) Last unstable τ**

<table>
<thead>
<tr>
<th>y/L</th>
<th>u (SM)</th>
<th>u (FD)</th>
<th>% diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2.139 \times 10^{-2}$</td>
<td>$2.141 \times 10^{-2}$</td>
<td>0.116</td>
</tr>
<tr>
<td>0.038</td>
<td>$2.154 \times 10^{-2}$</td>
<td>$2.157 \times 10^{-2}$</td>
<td>0.114</td>
</tr>
<tr>
<td>0.500</td>
<td>$5.509 \times 10^{-2}$</td>
<td>$5.507 \times 10^{-2}$</td>
<td>0.037</td>
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- **c) Last stable τ**

<table>
<thead>
<tr>
<th>y/L</th>
<th>u (SM)</th>
<th>u (FD)</th>
<th>% diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1.272 \times 10^{-3}$</td>
<td>$1.273 \times 10^{-3}$</td>
<td>0.102</td>
</tr>
<tr>
<td>0.038</td>
<td>$1.280 \times 10^{-3}$</td>
<td>$1.281 \times 10^{-3}$</td>
<td>0.100</td>
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<tr>
<td>0.500</td>
<td>$3.069 \times 10^{-3}$</td>
<td>$3.068 \times 10^{-3}$</td>
<td>0.028</td>
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</table>

- **d) Last unstable $A_2$**

<table>
<thead>
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<th>u (SM)</th>
<th>u (FD)</th>
<th>% diff.</th>
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</thead>
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<tr>
<td>0</td>
<td>$6.843 \times 10^{-2}$</td>
<td>$7.106 \times 10^{-2}$</td>
<td>3.845</td>
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<td>0.038</td>
<td>$7.853 \times 10^{-2}$</td>
<td>$8.030 \times 10^{-2}$</td>
<td>2.251</td>
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<td>0.500</td>
<td>$3.694 \times 10^{-1}$</td>
<td>$3.693 \times 10^{-1}$</td>
<td>0.338</td>
</tr>
</tbody>
</table>

- **e) Last stable $A_2$**

<table>
<thead>
<tr>
<th>y/L</th>
<th>u (SM)</th>
<th>u (FD)</th>
<th>% diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$6.613 \times 10^{-2}$</td>
<td>$6.828 \times 10^{-2}$</td>
<td>3.246</td>
</tr>
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<td>0.038</td>
<td>$7.469 \times 10^{-2}$</td>
<td>$7.622 \times 10^{-2}$</td>
<td>2.051</td>
</tr>
<tr>
<td>0.500</td>
<td>$3.086 \times 10^{-1}$</td>
<td>$3.086 \times 10^{-1}$</td>
<td>0.031</td>
</tr>
</tbody>
</table>
Figure 1. Maximum velocity (ms$^{-1}$) in the top (A), middle (B), and bottom (C) layers as a function of the wind-stress amplitude $\tau_0$ (m$^2$s$^{-2}$).
Figure 2. Maximum velocity ($\text{ms}^{-1}$) in the top (A), middle (B), and bottom (C) layers as a function of the interfacial frictional coefficient $A_2$ ($\text{ms}^{-1}$). $A_2/A_4$ is constant.
Figure 3. Growth rate (s\(^{-1}\)) as a function of the wind-stress amplitude \(\tau_0\) (m\(^2\) s\(^{-2}\)).
Figure 4. Growth rate (s\(^{-1}\)) as a function of the interfacial frictional coefficient \(A_2\) (m/s\(^{-1}\)). \(A_2/A_4\) is constant.
Figure 5. The spectrum of maximum growth rates [$s^{-1}$] for case S.
Figure 6. Transchannel profiles of the streamfunctions (m s\(^{-1}\) vs. m) for case S.
Figure 7. Transchannel profiles of the velocities (ms$^{-1}$ vs. m) for case S.
Figure 8. Transchannel profiles of the first (spatial) derivatives of the velocities ($s^{-1}$ vs. m) for case S.
Figure 9. Transchannel profiles of the second (spatial) derivatives of the velocities (m\(^{-1}\)s\(^{-1}\) vs. m) for case S.
Figure 10. Transchannel profiles of the third (spatial) derivatives of the velocities (m\(^{-2}\) s\(^{-1}\) vs. m) for case S.
Figure 11. The perturbation to the streamfunction in the top layer for the "most unstable" mode for case S.
Figure 12. The perturbation to the streamfunction in the middle layer for the "most unstable" mode for case S.
Figure 13. The perturbation to the streamfunction in the bottom layer for the "most stable" mode for case S.
Figure 14. Transchannel profiles of the streamfunctions ($m^2 s^{-1}$ vs. m) for $\tau_o = 1.911 \times 10^{-9} \text{ m}^2 \text{s}^{-2}$ (slightly unstable).
Figure 15. Transchannel profiles of the velocities (m s\(^{-1}\) vs. \(m\)) for 
\[ \tau_o = 1.911 \times 10^{-9} \text{ m}^2\text{s}^{-2} \] (slightly unstable).
Figure 16. The perturbation to the streamfunction in the top layer for the "most unstable" mode for $\tau_0 = 1.911 \times 10^{-9} \text{ m s}^{-1}$ (slightly unstable).
Figure 17. Transchannel profiles of the streamfunctions $\frac{2}{m s^{-1}}$ vs. $m$ for $\tau_0 = 1 \times 10^{-10} m^2 s^{-2}$ (stable).
Figure 18. Transchannel profiles of the velocities ($\text{ms}^{-1}$ vs. m) for
$\tau_0 = 1 \times 10^{-10}$ m$^2$s$^{-2}$ (stable).
Figure 19. Transchannel profiles of streamfunctions (m²s⁻¹ vs. m) for $A_2 = 3.874 \times 10^{-3}$ ms⁻¹ and $A_4 = 1.509 \times 10^{-3}$ ms⁻¹ (slightly unstable).
Figure 20. Transchannel profiles of the velocities (ms$^{-1}$ vs m) for
$A_2 = 3.874 \times 10^{-3} \text{ ms}^{-1}$ and $A_4 = 1.509 \times 10^{-3}$ (slightly unstable).
Figure 21. The perturbation to the streamfunction in the top layer for the "most unstable" mode for $A_2 = 3.874 \text{ ms}^{-1}$ and $A_4 = 1.509 \times 10^{-3} \text{ ms}^{-1}$ (slightly unstable).
Figure 22. Transchannel profiles of the streamfunctions ($m^2 s^{-1}$ vs. $m$) for $A_2 = 1.14 \times 10^{-2} \text{ ms}^{-1}$ and $A_4 = 4.44 \times 10^{-3} \text{ ms}^{-1}$ (stable).
Figure 23. Transchannel profiles of the velocities (ms$^{-1}$ vs. m) for $A_2 = 1.14 \times 10^{-2}$ ms$^{-1}$ and $A_4 = 4.44 \times 10^{-3}$ ms$^{-1}$ (stable).