A Method for the Calculation of Invariant Manifolds in Hydrodynamic Stability Problems

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The investigation described in this report was undertaken to extend the method developed by Davey 1962, Davey, DiPrima and Stuart 1968, and Eagles 1971, and to increase our knowledge of the early onset of turbulence in the Taylor problem of flow between concentric rotating cylinders. This report is being written now to describe the results that have been obtained, because it appears that the limitations of the present method have been reached, and in that sense the investigation has been completed.

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The method developed by the authors referred to in the Preface has been improved in a number of minor respects, to make possible calculations at Taylor numbers up to about 10^6, for invariant manifolds up to dimension 14, and to suppress axial flow. The method was programmed for the Cray-1 computer and applied to two series of calculations, one for a narrow gap between the cylinders and one for a wide gap. The calculations exhibited the first few bifurcations and the corresponding modes of motion and determined the stability of those modes. No aperiodic motions were found. There is evidence for an additional subcritical bifurcation from the wavy vortex mode at a Taylor number \( \sim 1.148 \times 10^6 \) in the wide gap problem. Orbit calculations at \( 1.15 \times 10^6 \) failed to find further attractors and indicated an explosive transition to regions of the configuration space beyond the applicability of the present method.
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0. Introduction and Summary. The calculations of Davey 1962, of Davey, DiPrima and Stuart 1968, and of Eagles 1971, on the Taylor problem of flow between concentric rotating cylinders, are interpreted as the calculation of an invariant manifold - of one dimension in Davey's work and six in the other two - in the infinite-dimensional configuration space of the system, in fact as a calculation of the unstable manifold that emerges from the origin (the origin represents the laminar Couette flow) when the Reynolds number exceeds its critical value. The Taylor vortices and the wavy vortices are fixed points and closed orbits, respectively, of the motion in that manifold. The method is not a spectral method or eigenfunction expansion method, although certain of the eigenfunctions of the linearized problem play an important role. In contrast with such methods, the restrictions to a finite number of dimensions is not an approximation, because the manifold under study is rigorously invariant. (There are, of course, plenty of other approximations to worry about.)

In the interest of possible application to more general calculations, we have modified and extended the method in minor ways as follows:

1. The number of dimensions of the unstable manifold is determined by those eigenvalues of the linearized problem that lie in the right half of the complex plane, and it increases with the Reynolds number. Here it is arbitrary, not restricted to 6 as in the previous work. With the present program for the Cray-1 computer, calculations up to 14 dimensions are possible. By certain modifications of the procedure, the number of dimensions could probably be increased somewhat further.

2. The extensive algebraic calculations made by hand by the previous
authors, especially by Eagles, (which would be prohibitive by hand calculation in more than about six dimensions) have been mechanized and incorporated in the computer program.

3. Certain consequences of the cylindrical symmetry, which reduce the amount of calculation and storage by a factor around 3.0, and which were taken into account in the hand calculations referred to above, have been formulated as lemmas and also mechanized and incorporated in the computer program.

4. In order to be able to calculate the helical vortices a little more realistically, a modification has been introduced to inhibit net axial flow.

5. Multiple shooting or reorthonormalization has been introduced in the numerical solution of the two-point boundary problems, to deal with the stiffness of the radial ordinary differential equations at higher Reynolds numbers.

6. Numerous checks have been built into the program to monitor things like the accuracy of the numerical solution of the ordinary differential equations, the conservation laws, consequences of the symmetry, the biorthogonality of the eigenfunctions and adjoint eigenfunctions of the linearized problem, and certain other orthogonalities.

7. Orbit calculation procedures (based mainly on the Adams-Bashforth-Moulton method) have been provided, for the study of the resulting dynamical system in the invariant manifold, especially for looking for fixed points, closed orbits, and attractors generally.
With those modifications, the method was found suitable for study of the Taylor problem under a moderate range of parameters, though unfortunately not for high enough Reynolds numbers to permit study of the aperiodic phenomena found by Gollub and Swinney and other investigators. The main limitation comes from the number of dimensions of the invariant manifold. It appears we should have to go to at least 32 dimensions to reach the relevant Reynolds numbers. Unfortunately, the amount of computing and storage varies with a high power of the number of dimensions. It is also possible that at high Reynolds number the power series expansions that give the manifold and the dynamical system in it would have to be carried to higher-degree terms than has been done so far.

The first four sections give a somewhat abstract formulation of the method, the next seven give details connected with the Navier-Stokes equations and the cylindrical symmetry, and the remaining sections describe briefly some of the modifications mentioned above and some of the results.

1. **Evolutionary equations of Navier-Stokes type in a Hilbert space.** Let the Navier-Stokes equations together with boundary conditions be written schematically as an equation of evolution

\[ Mv - Lv + B(v, v) = 0 \tag{1.1} \]

for a point \( v = v(t) \) in a Hilbert space \( \mathcal{H} \), where \( M \) and \( L \) are linear operators and \( B \) is a bilinear one. (Later, \( L \) will be written as \( \frac{\partial}{\partial r} - A \), where \( r \) is the cylindrical radial coordinate and \( A \) contains \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial \theta} \); \( B(v, v) \) will be written as \( K(v)v \), where \( K \) is a linear-operator-valued linear function of \( v \).) Further, \( L \) depends on a parameter \( \mu \) related to the Reynolds number. The point \( v \) in \( \mathcal{H} \) represents the departure of the fluid motion from the basic
Couette flow, and the functions that describe the Couette flow appear in the coefficients of the operator $L$.

We assume that the initial-value problem of (1.1) is well posed, so that the solutions determine a semiflow $\varphi^t (t \geq 0)$ in $\mathcal{H}$, $\varphi^t(v)$ being the solution with starting point $\varphi^0(v) = v$ in $\mathcal{H}$.

Normal modes of the fluid motion are represented by infinitesimal solutions of the form $\lambda t v$, where $\lambda$ and $v$ are an eigenvalue and eigenvector (eigenfunction) of the problem

$$\lambda Mv - Lv = 0. \quad (1.2)$$

We assume that this problem has a complete set of eigenfunctions $v^{(j)}$, $j = 1, 2, \ldots$, with corresponding eigenvalues $\lambda_j$, so ordered that $\Re \lambda_j \geq \Re \lambda_{j+1}$.

Then,

$$\lambda_j Mv^{(j)} - Lv^{(j)} = 0 \quad (j = 1, 2, \ldots). \quad (1.3)$$

Let $v^{(1)}$ be the corresponding solutions of the adjoint problem, so that

$$\lambda_1 M^* v^{†(1)} - L^* v^{†(1)} = 0 \quad (1 = 1, 2, \ldots).$$

We assume that they form a set biorthogonal to the functions $v^{(j)}$, in the sense that

$$(v^{†(1)}, Mv^{(j)}) = \delta_{1,j}. \quad (1.4)$$

It is assumed that for $\mu < 0$ all normal modes are stable, i.e. the eigenvalues $\lambda_j$ are all in the half plane $\Re \lambda < 0$, while for $\mu > 0$ (but not too large), a finite number of the normal modes $v^{(j)}$, $j = 1, \ldots, J$, have eigenvalues $\lambda_j$ in $\Re \lambda > 0$. 
2. The unstable manifold $\mathcal{U}$: For given $\mu > 0$, the unstable manifold is a $J$-dimensional manifold $\mathcal{U}$ in $\mathcal{H}$ tangent at the origin to the linear manifold $\mathcal{M}_0$ spanned by the vectors $v^{(1)}, \ldots, v^{(J)}$. $\mathcal{U}$ is invariant under $\phi^t$ (orbits that start in $\mathcal{U}$ remain in $\mathcal{U}$) and is attracting. These properties fix $\mathcal{U}$ and lead to the method of calculating it.

For theorems on the existence and properties of the unstable manifold, the reader is referred to Smale 1967 and to Abraham and Robbin 1967.

3. Formal calculation of $\mathcal{U}$ in the one-dimensional case. When $J = 1$, the tangent manifold $\mathcal{M}_0$ is simply the ray

$$\mathcal{M}_0 = \{ Av^{(1)} : A \in \Phi \} \quad (3.1)$$

in $\mathcal{H}$, and $\mathcal{U}$ is a curve in $\mathcal{H}$ tangent to $\mathcal{M}_0$ at the origin. We take $A$ as the coordinate of a point $v$ in $\mathcal{H}$ by projecting $v$ onto $\mathcal{M}_0$ using the eigenvectors $v^{(j)}$. Specifically, we assume that $\mathcal{U}$ is analytic, so that the point $v = v(A)$ in $\mathcal{H}$ can be represented by a power series

$$v(A) = \sum_{q=1}^{\infty} A^q v_q \quad (3.2)$$

where $v_1$ is equal to the first eigenfunction $v^{(1)}$ while the other $v_q$ can be expressed in terms of the other eigenfunctions, hence are orthogonal to $v^{(1)}$ in the sense that

$$(v^{(1)}, M_q v_q) = 0 \text{ for } q = 2, 3, \ldots. \quad (3.3)$$

In this way the coordinate $A$ is determined in $\mathcal{U}$.

The $v_q$ in (3.2) for $q = 2, 3, \ldots$ are determined by the requirement that $\mathcal{M}$ be invariant under the semiflow $\phi^t$ in $\mathcal{H}$, i.e. that an orbit
that starts in $\mathcal{W}$ lies in $\mathcal{W}$. If $A(t)$ is the coordinate of the orbit, then $\dot{A} = \frac{d}{dt} A(t)$, is a function of $A$, and we write it as

$$\dot{A} = \sum_{p=1}^{\infty} a_p A^p,$$  \hfill (3.4)$$

where the $a_p$ are numerical coefficients, to be determined along with the $v_q$ in (3.2) by the requirement that the function $v(t) = v(A(t))$ satisfy the evolutionary equation (1.1). If we differentiate (3.2) with respect to $t$ and use (3.4), the result is

$$\dot{v} = \sum_{(p,q)} q a_p A^{p+q-1} v_q$$

$$= \sum_{s=1}^{\infty} A^s \sum_{q=1}^{s} a_{q+1-q} v_q.$$ \hfill (3.5)$$

We put $v$ from (3.5) and $v$ from (3.2) into (1.1) and equate to zero the net coefficient of $A^s$ for each $s = 1, 2, ...$. We find for $s = 1$

$$(a_1 M - L)v_1 = 0$$ \hfill (3.6)$$

and for $s > 1$

$$(s a_1 M - L)v_s = \sum_{q=1}^{s-1} [-q a_{s+1-q} M v_q - B(v_q, v_{s-q})].$$ \hfill (3.7)$$

These equations determine the unknowns $a_s$ and $v_s$ inductively as follows:

Since $v_1$ is equal to the first eigenfunction $v^{(1)}$, (3.6) shows that $a_1$ is equal to the first eigenvalue $\lambda_1$. Then each equation (3.7) in succession for $s = 2, 3, ...$ contains two unknowns, $v_s$ and $a_s$; $a_s$ is determined first by taking the inner product of the equation with $v^{+1}$ and using the requirement that $(v^{+1}, M v_q)$ be $= 0$ for all $q > 1$. The left member of the resulting equation is
Therefore $a_3$ is determined by the equation

$$\left(v^{\dagger}(1), \sum_{q=1}^{s-1} [-q a_{s+1-q} M v_q - B(v_q, v_{s-q})] \right) = 0,$$

(3.9)

where it is the only unknown and where it appears (for $q = 1$) multiplied by the nonzero coefficient $-(v^{\dagger}(1), M v_1) = -1$. With $a_3$ known, $v_3$ is then determined by (3.7). For $\mu > 0$, the operator $s a_1 M - L$ in (3.7) is nonsingular, since $s a_1 > a_1$, hence (3.7) has a unique solution.

We note in passing that the procedure can be used also for $\mu = 0$, in which case $a_1 = 0$ and the operator $s a_1 M - L$ is singular, but we have satisfied the requirement of the Alternative Theorem that the right member of the equation be orthogonal to $v^{\dagger}(1)$, hence the solution exists, and it is made unique by the requirement that $(v^{\dagger}(1), M v_s) = 0$. It was this consideration that led Davey in 1962 to determine the coordinate $A$ in the manifold $\mathcal{M}$ by the requirement (3.3).

In this way the manifold $\mathcal{M}$ is determined as a curve in $\mathcal{H}$ in that neighborhood of the origin in which the power series (3.2) and (3.4) converge.

4. Formal calculation of $\mathcal{M}$ in the multidimensional case. For $J > 1$, the same general procedure is used. For the analogue of (3.2) we introduce vectors

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_J \end{pmatrix}, \quad q = \begin{pmatrix} q_1 \\ \vdots \\ q_J \end{pmatrix}.$$
Then,
\[ v(A) = \sum_{q \in \mathcal{L}} A^q \triangledown q, \quad (4.2) \]

where \( \mathcal{L} \) denotes the lattice
\[ \mathcal{L} = \{ q: \text{each } q_i = \text{integer } \geq 0 \ (i = 1, \ldots, J); \text{at least one } q_i > 0 \}, \quad (4.3) \]

and where \( A^q \) is an abbreviation for \( A_{1}^{q_1} A_{2}^{q_2} \ldots A_{J}^{q_J} \). The analogue of (3.4) is
\[ \hat{A}_j = \sum_{p \in \mathcal{L}} a_j p A^p \quad (j = 1, \ldots, J). \quad (4.4) \]

To find \( \hat{v} \) from (4.2), we must differentiate \( A^q \) with respect to \( A_j \) \( (j = 1, \ldots, J) \) and then use (4.4). We call \( e_j \) the vector whose \( j \)th component is equal to 1 and whose other components are equal to 0. Then,
\[ \frac{\partial}{\partial A_j} A^q = q_j A^q - e_j, \quad (4.6) \]

hence the analogue of (3.5) is
\[ \hat{v} = \sum_{p, q \in \mathcal{L}} \sum_{j=1}^{J} q_j a_j p A^{p+q-e_j} v_q \]
\[ = \sum_{s \in \mathcal{L}} \sum_{j=1}^{J} q_j s_j s^* e_j - q \cdot v_q \quad (s^* e_j - q \in \mathcal{L}) \quad (4.7) \]

We now substitute \( v \) from (4.2) and \( \hat{v} \) from (4.7) into the evolutionary equation (1.1) and equate to zero the net coefficient of \( A^s \) for each \( s \) in \( \mathcal{L} \). The analogue of (3.6) comes from the cases when \( s \) is one of the vectors \( e_1 \); namely,
\[ \sum_{j=1}^{J} a_j e_1 e_j \hat{v}_{e_j} - L v_{e_1} = 0 \quad (1 = 1, \ldots, J). \quad (4.8) \]
The coordinates in $\mathcal{M}$ are $A_1, \ldots, A_J$, where $A_j$ ($j = 1, \ldots, J$) is defined as the coefficient of $v(j)$ in the expansion of $v(A)$ in the complete set of normal-mode eigenfunctions. That is, according to (4.2), $v_j$ is $v(j)$ for each $j = 1, \ldots, J$, while all other $v$ are orthogonal to the first $J$ adjoint functions in the sense that $(v(j), Mv_q) = 0$ for $j = 1, \ldots, J$. It follows from (4.3) and the eigenvalue-eigenfunction equation that

$$a_j \xi_1 = \begin{cases} \lambda_1 & \text{for } j = 1 \\ 0 & \text{for } j \neq 1 \end{cases} \quad (4.9)$$

Hence, to lowest order, for $A \sim 0$, (4.4) gives the correct exponential growth of the $A_j$.

The analogue of (3.7), that is, the equation that results from equating to zero in (1.1) the net coefficient of $A^2$, for any given $s$ not equal to one of the $e_j$, is, using (4.9),

$$(\Sigma \lambda_j M - L) v_s = - \Sigma \Sigma q_{j} a_j s + e_j q_{j} M v_q$$

$$- \Sigma q_{j} R(v_q, v_s) \quad (s, q_{j})$$

$$= \Sigma q_{j} \Sigma q_{j} R(v_q, v_s).$$

We now show that these equations, if taken in the right order for $s \in \mathcal{L}$, determine the unknown functions $v_s$ and unknown coefficients $a_j$ inductively. We call $|q| = q_1 + \ldots + q_J$ for any $q$ in $\mathcal{L}$. We assume the equations (4.10) so ordered that all equations with a given value of $|s|$, say $|s| = \sigma$ (a positive integer) appear earlier than the equations with $|s| = \sigma + 1$, and we assume that when any of the equations is encountered, all functions and coefficients appearing in previous equations have been determined. (The ordering of the equations having a given value of $|s|$ is irrelevant.) We claim that
then all the \( v_q \) appearing on the right of (4.10) are known. In fact, for most of the terms there, \(|q|\) is less than \(|s|\); the only exceptions are terms in which \( q \) is of the form \( s^j - e_j \) for some \( l \neq j \); however, the term with that \( q \) contains the coefficient \( a_j e_1 \), which is 0 by (4.9); hence, all the \( v_q \) that actually appear on the right of (4.10) can be regarded as known. The unknowns in that equation are therefore the function \( v_s \) and the coefficient \( a_j \) \( (j = 1, \ldots, J) \).

To determine the coefficient of \( a_j \) for any \( l = 1, \ldots, J \), we take the inner product with \( v_{\uparrow}(l) \) throughout (4.10). The left member gives zero and all the terms in the first sum on the right give zero except when \( q = e_1 \) and \( j = 1 \); hence,

\[
a_j = \left( v_{\uparrow}(1), \sum_{q \in \mathcal{L}} B(v_q, v_{s-q}) \right). \tag{4.11}
\]

With these coefficients known for \( l = 1, \ldots, J \), equation (4.10) then determines \( v_q \).

In this way, the invariant \( J \)-dimensional manifold \( \mathcal{M} \) is determined in that neighborhood of the origin in \( \mathcal{H} \) in which the series (4.2) and (4.4) converge.

5. The Taylor problem; cylindrical coordinates. The Navier-Stokes equations for an incompressible fluid of unit density are

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \frac{1}{Re} \nabla^2 u = 0, \tag{5.3}
\]

\[
\nabla \cdot u = 0, \tag{5.4}
\]

where \( u = u(x, t) \) is the fluid's velocity vector field, \( p = p(x, t) \) is its
pressure, and $1/\text{Re}$ is its viscosity. (Then, if suitable units are chosen for
length and time, $\text{Re}$ is the Reynolds number.) The above equations are subject
to no-slip boundary conditions, discussed below.

For the Taylor problem we introduce cylindrical coordinates, $r, \theta, z$.

Then,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \tag{5.3}$$

$$\nabla \cdot \mathbf{V} = \frac{\partial}{\partial r} u + \frac{\partial}{\partial \theta} v + \frac{\partial}{\partial z} w, \tag{5.4}$$

where $u, v, w$ are velocity components in the directions of increasing
$r, \theta, z$, so that if $\mathbf{k}_r, \mathbf{k}_\theta, \mathbf{k}_z$ are unit vectors in those directions,

$$\mathbf{u} = u \mathbf{k}_r + v \mathbf{k}_\theta + w \mathbf{k}_z. \tag{5.5}$$

When (5.3) and (5.4) are applied to (5.5), it must be kept in mind that the
unit vectors $\mathbf{k}_r$ and $\mathbf{k}_\theta$ are not constants, but depend on $\theta$:

$$\mathbf{k}_r = \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix}, \quad \mathbf{k}_\theta = \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{bmatrix};$$

hence,

$$\frac{\partial}{\partial \theta} \mathbf{k}_r = \mathbf{k}_\theta, \quad \frac{\partial}{\partial \theta} \mathbf{k}_\theta = -\mathbf{k}_r.$$

Hence,

$$\nabla^2 u = \mathbf{k}_r \left( \nabla^2 u - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) + \mathbf{k}_\theta \left( \nabla^2 v - \frac{1}{r} \frac{\partial}{\partial \theta} \right) + \mathbf{k}_z \nabla^2 w \tag{5.6}$$

and

$$\nabla \cdot \mathbf{V} u = \mathbf{k}_r \left[ \left( \frac{\partial}{\partial r} u + \frac{1}{r} \frac{\partial}{\partial \theta} v + \frac{\partial^2}{\partial \theta^2} u - \frac{v^2}{r} \right) \right]$$

$$+ \mathbf{k}_\theta \left[ \left( \frac{\partial}{\partial r} u + \frac{1}{r} \frac{\partial}{\partial \theta} v + \frac{\partial^2}{\partial \theta^2} v + \frac{u v}{r} \right) \right]$$

$$+ \mathbf{k}_z \left( \frac{\partial}{\partial r} u + \frac{1}{r} \frac{\partial}{\partial \theta} v + \frac{\partial^2}{\partial \theta^2} w \right). \tag{5.7}$$
The basic Couette flow is the exact solution of the Navier-Stokes equation given by \( \{u, v, w, p\} = \{0, V(r), 0, P(r)\} \), where

\[
V(r) = C_1 r + C_2/r, \quad P = \int V(r)^2 \, dr + \text{const.} \tag{5.8}
\]

(The formula for \( P(r) \) will not be used.) Henceforth, \( \dot{u} \) and \( p \) will denote the departure from the basic flow; therefore, we replace \( u \) and \( p \) in the Navier-Stokes equations by

\[
\begin{pmatrix}
0 \\
V(r) \\
0
\end{pmatrix} + \dot{u} \quad \text{and} \quad P(r) + \ddot{p}.
\]

The result in cylindrical coordinates is

\[
\frac{\partial u}{\partial t} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{2v}{r} \dot{u} + \frac{\partial p}{\partial r} - \frac{1}{Re} \left( \frac{1}{r} \frac{\partial^2 v}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial v}{\partial \theta} - \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \right)
\]

\[
- \frac{u^2}{r} - \frac{v}{r} \frac{\partial v}{\partial \theta} - \frac{w}{r} \frac{\partial w}{\partial z} + \left( \frac{v}{r} \frac{\partial}{\partial \theta} + \frac{w}{r} \frac{\partial}{\partial z} \right) u - \frac{v^2}{r} = 0 \tag{5.9}
\]

\[
\frac{\partial v}{\partial t} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{v'}{r} u + \frac{v}{r} u + \frac{1}{r} \frac{\partial p}{\partial \theta} - \frac{1}{Re} \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2} + \frac{1}{r^2} \frac{\partial u}{\partial \theta} \right)
\]

\[
+ \frac{u}{r} \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{v}{r} \frac{\partial v}{\partial z} + \frac{uv}{r} = 0 \tag{5.10}
\]

\[
\frac{\partial w}{\partial t} + \frac{v}{r} \frac{\partial w}{\partial \theta} + \frac{\partial p}{\partial z} - \frac{1}{Re} \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 w}{\partial z^2} \right)
\]

\[
+ \frac{u}{r} \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \theta} + \frac{w}{r} \frac{\partial w}{\partial z} = 0 \tag{5.11}
\]

\[
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{5.12}
\]

In the first of these, the divergence equation (5.12) has been used to eliminate \( \partial u/\partial r \) and \( \partial^2 u/\partial r^2 \).

In the Taylor problem, the fluid occupies the region
\[ r_1 \leq r \leq r_2, \quad 0 \leq \Theta \leq 2 \pi, \quad \text{all } z. \]

On the surfaces of the cylinders, which have angular velocities \( \Omega_1 \) and \( \Omega_2 \), we have the no-slip boundary conditions

\[ V(r_1) = r_1 \Omega_1, \quad V(r_2) = r_2 \Omega_2 \quad (5.13) \]

\[ u = 0 \text{ at } r = r_1 \text{ and at } r = r_2 \quad (5.14) \]

6. The Hilbert Space; Eagles' vector notation. Following Eagles 1971, we introduce a six-component vector \( \underline{U} = \underline{U}(r, \Theta, z, t) \) whose components are \( p, \frac{\partial v}{\partial r}, \frac{\partial w}{\partial r}, u, v, \) and \( w \). Then equations (5.9) - (5.12) can be written in matrix-vector form as

\[ \frac{\partial}{\partial r} \underline{U} - \underline{A} \underline{U} - \underline{M} \frac{\partial}{\partial t} \underline{U} - \underline{K}(\underline{U}) \underline{U} = 0, \quad (6.1) \]

where \( \underline{A}, \underline{M}, \) and \( \underline{K}(\underline{U}) \) are 6 by 6 operator-valued matrices, namely

\[ A = \begin{bmatrix}
0 & -1 & 0 & B & \frac{2V}{r} & \frac{1}{r^2} \frac{\partial}{\partial r} \\
\frac{\text{Re} \frac{\partial}{\partial r}}{r} & -1 & 0 & \text{Re}(V' + \frac{V}{r}) & -2 & -\text{Re}B + \frac{1}{r^2} & 0 \\
\frac{\text{Re} \frac{\partial}{\partial z}}{r} & 0 & -1 & 0 & 0 & 0 & \text{Re}B \\
0 & 0 & 0 & -1 & -\frac{1}{r} & -\frac{1}{r} \frac{\partial}{\partial r} & -\frac{\partial}{\partial z} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix} \quad (6.2) \]

where

\[ B = \frac{1}{\text{Re}} \left( \frac{1}{r^2} \frac{\partial^2}{\partial \Theta^2} + \frac{\partial^2}{\partial z^2} \right) - \frac{V}{r} \frac{\partial}{\partial r}, \quad (6.3) \]
All elements of $M$ and $K(U)$ are zero except in the upper right quadrants of the matrices. To identify (6.1) with (1.1), we must identify $v$ with $U$, and we must set

$$L = \frac{\partial}{\partial r} - A, \quad B(v, v) = K(U) U. \quad (6.6)$$

The eigenvalue–eigenfunction problem obtained from (6.1) and corresponding to (1.2) of the abstract formulation is

$$\frac{\partial}{\partial r} U - A U - \lambda M U = 0 \quad (6.7)$$

$$U_4 = U_5 = U_6 = 0 \text{ at } r = r_1 \text{ and } r = r_2. \quad (6.8)$$

To simplify the formulation, we restrict consideration to fluid motions that are periodic in the $z$ direction, as observed in experiments. We call $\mathcal{R}$ the region of space

$$\mathcal{R} : r_1 < r < r_2, 0 < \theta < 2\pi, 0 < z < 2\pi/\alpha.$$
where $\alpha$ is the wave number of the periodicity in the $z$ direction; it is a free parameter in the calculation. We take $\mathcal{H}$ as the Hilbert space $L^2(\mathbb{R})^6$ with the inner product

$$(\underline{u}^{(1)}, \underline{u}^{(2)}) = \frac{\alpha}{4\pi^2} \int \int \underline{u}^{(1)} \cdot \underline{u}^{(2)} \, dr \, d\theta \, dz.$$  

(6.9)

Then the adjoint eigenvalue-eigenfunction problem is

$$-\frac{\partial}{\partial r} \underline{u} - A^* \underline{u} - \lambda M^* \underline{u} = 0, \quad (6.10)$$

$$U_1 = U_2 = U_3 = 0 \text{ at } r = r_1 \text{ and at } r = r_2. \quad (6.11)$$

In (6.10), $A^*$ and $M^*$ are the transposes of the matrices $A$ and $M$ with $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial z}$ replaced by $-\frac{\partial}{\partial \theta}$ and $-\frac{\partial}{\partial z}$.

It is generally believed, as has been assumed in the abstract discussion, that

1. For a given value of $\lambda$, (6.7) and (6.8) have a nontrivial solution (eigenfunction) if and only if (6.10) and (6.11) have.

2. Each of these eigenvalue problems has a complete set of eigenfunctions, and for this purpose it is not necessary to include generalized eigenfunctions of higher order, e.g. solutions of $(\frac{\partial}{\partial r} - A - \lambda M)^2 \underline{u} = 0$, etc.

3. The eigenfunctions $\underline{u}^{(j)}$ of the direct problem and those $\underline{u}^{\dagger(j)}$ of the adjoint problem are biorthogonal sets in the sense that

$$(\underline{u}^{\dagger(j)}, M \, \underline{u}^{(l)}) = \delta_{j,l}. \quad (6.12)$$

In connection with (6.12), we note that if $\lambda_j \neq \lambda_l$, it follows from (6.7), (6.8), (6.10), and (6.11) that the left member of (6.12) is $0$; hence, it
only remains to establish that the functions can be suitably orthonormalized when \( \lambda j = \lambda 1 \).

The variables can be separated in the eigenvalue problems, and eigenfunctions of the form \( U = Y(r) e^{i\alpha z + im\theta} \) are obtained, where \( \alpha \) and \( m \) are integers. For the range of Reynolds numbers considered, \( \alpha \) is always \( \pm 1 \); other values of \( \alpha \) correspond to eigenvalues in the left half-plane. Hence we can denote the relevant eigenvalues as \( \lambda (m) \). As the Reynolds number is increased, the eigenvalues in the right half-plane are first just the real eigenvalue \( \lambda (0) \), next the complex conjugate pair \( \lambda (1) \) and \( \lambda (-1) \), then the pair \( \lambda (2) \) and \( \lambda (-2) \), and so on. Each is of multiplicity two, the corresponding eigenfunctions being complex conjugate functions, for example

\[
V(r)e^{iaz} \quad \text{and} \quad \overline{V(r)}e^{-iaz}
\]
or

\[
V(r)e^{i\alpha z + im\theta} \quad \text{and} \quad \overline{V(r)}e^{-i\alpha z - im\theta}
\]

The adjoint function \( U^+ \) corresponding to an eigenfunction \( U \) has the same \( z \) and \( \theta \) dependence as \( U \), for if that were not so, then \( (U^+, MU) \) would be \( = 0 \), contradicting (6.12).

7. Consequences of the symmetries. For the Taylor problem, the calculation outlined in the abstract formulation is very considerably simplified by the symmetries of the problem, and would probably not be feasible without the resulting simplifications. We shall see that the variables \( r, \theta, \) and \( z \) separate not only in the eigenfunctions but also in the other functions \( \tilde{\varphi}_q \), which correspond to the \( \varphi_q \) of the abstract theory, so that only one function of \( r \) needs to be computed and stored for each lattice point. Furthermore,
there are relations among those functions of \( r \) so that only about one fourth of them need to be computed and stored. Thirdly, many of the coefficients \( a_q \) are \( 0 \) and need not be computed, and that decreases the amount of time spent in evaluating the quite time consuming sum (4.11).

The symmetries of the evolutionary equation (6.1) are:

1. Invariance under \( z \rightarrow z + \text{constant} \)
2. Invariance under \( \Theta \rightarrow \Theta + \text{constant} \)
3. Invariance under \( z \rightarrow -z, \ w \rightarrow -w \)
4. Invariance under \( U \rightarrow \overline{U} \).

The first two symmetries have already been taken into account in the separation of variables in the eigenfunction equations. The fourth symmetry was taken into account in asserting that the complex conjugates of the eigenfunctions are also eigenfunctions.

In connection with the third symmetry above, we introduce the notation \( \hat{\underline{\Psi}} \) for the vector whose components are \( U_1, \ U_2, \ -U_3, \ U_4, \ U_5, \ -U_6 \) in terms of the components \( U_1, \ldots, \ U_6 \) of \( \overline{\Psi} \) (recall that \( U_3 = \partial \ w/\partial \ r \) and \( U_6 = \ w \)). Then, only one function \( \underline{\Psi}(r) \) needs to be computed for each eigenvalue pair. The first six eigenfunctions are of the following forms, and are listed together with their eigenvalues:

\[
\begin{align*}
\underline{\Psi}^{(1)} &= \underline{\Psi}^{(1)}(r) \ e^{i\alpha z} & \lambda (0) \ (\text{real}) \\
\underline{\Psi}^{(2)} &= \underline{\Psi}^{(1)}(r) \ e^{-i\alpha z} & \lambda (0) \\
\underline{\Psi}^{(3)} &= \underline{\Psi}^{(3)}(r) \ e^{i\alpha z + i\theta} & \lambda (1) \ (\text{complex})
\end{align*}
\]
The adjoint eigenfunctions have the same forms and the same eigenvalues and are distinguished by superscript daggers attached to the vectors $\hat{U}$ and $\hat{V}$.

From here on, the eigenfunctions appear in groups of four, corresponding to $\hat{U}^{(3)}$ to $\hat{U}^{(6)}$, but with $m = 2, 3, \ldots$ instead of $m = 1$. For example, the seventh eigenfunction is

$$\hat{U}^{(7)} = \hat{V}^{(7)}(r) e^{i\alpha z} + 2i \lambda (2) \text{ (complex)} \quad (7.1b)$$

There is one group of four for each eigenvalue pair $\lambda_{\pm 1}$, $\lambda_{\pm 2}$, $\ldots$ in the right half plane, so that the number of dimensions is 2, 6, 10, 14, $\ldots$.

We also introduce integer-valued functions $p(q)$ and $m(q)$ defined on the lattice $\Lambda$ by

$$p(q) = q_1 - q_2 + \sum (m) (q_{4m-1} - q_{4m} - q_{4m+1} + q_{4m+2}) \quad (7.2)$$

$$m(q) = \sum (m) m(q_{4m-1} - q_{4m} + q_{4m+1} - q_{4m+2}) \quad (7.3)$$

We recall that, for each $l = 1, \ldots, J$, $e_1$ denotes the lattice point in $\Lambda$ (a $J$-component vector) whose $l$th component is $= 1$ and whose other components are $= 0$.

**Lemma 1**: For general $l$ and $s$,

$$a_{l \ s} = 0 \text{ unless } p(s) = p(e_1) \text{ and } m(s) = m(e_1); \quad (7.4)$$

furthermore, $\hat{U}_s$ is of the form
\[ U \sim = V_\sim(r) e^{i p(\sim) az + i m(\sim) \theta}. \] (7.5)

Proof: The proof is by induction on the norm \(|s| = s_1 + \ldots + s_J\). First, we show that (7.4) and (7.5) hold for \(|s| = 1\), in which case \(s = e_1\) for some \(1 = 1, \ldots, J\). According to the abstract formulation, \(U_\sim\) is then equal to the \(1\)th eigenfunction \(U^{(1)}\), and (7.5) follows from (7.1a), (7.1b), (7.2), and (7.3), while (7.4) follows from equation (4.9) of the abstract formulation.

Now suppose that (7.4) and (7.5) have been established for all \(s\) such that \(|s_\sim| \leq \sigma\) (a positive integer), and consider a point \(s\) such that \(|s_\sim| = \sigma + 1\). Each term in the sum in (4.11) has \(|q_\sim| \leq \sigma\) and \(|s_\sim - q_\sim| \leq \sigma\); furthermore,

\[ p(q_\sim) + p(s_\sim - q_\sim) = p(s_\sim) \] (7.6)

\[ m(q_\sim) + m(s_\sim - q_\sim) = m(s_\sim); \] (7.7)

therefore, each term of that sum depends on \(\Theta\) and \(z\) through the factor

\[ e^{i p(\sim) az + i m(\sim) \theta}, \] (7.8)

hence the sum contains that factor, and the inner product in (4.11) vanishes unless \(v^{(1)}_\sim\) also contains that factor. But we have seen that the dependence of \(v^{(1)}_\sim = U^{(1)}_\sim\) on \(\Theta\) and \(z\) is the same as that of \(U^{(1)}\); hence, (7.4) follows.

We can now see that the first term on the right of (4.10) also depends on \(\Theta\) and \(z\) through the factor (7.8). We have already seen from (4.9) that whenever \(|q_\sim| = |s_\sim|\), the coefficient \(a_j \delta_\sim e_j - q_\sim\) is \(0\), because then \(s_\sim + e_j - q_\sim\) has norm \(= 1\) and is therefore one of the \(e_1\), but not \(e_j\), because the case \(q_\sim = s_\sim\) was excluded. Therefore, the only nonvanishing terms in the first sum on the right of (4.10) have \(|q_\sim| \leq \sigma\), and the inductive hypotheses (7.4) and (7.5) can be used, according to which a term of the sum is \(= 0\) unless
\[ p(s + e_j - q) = p(e_j) \text{ and } m(s + e_j - q) = m(e_j), \]
i.e., unless
\[ p(s) = p(q) \text{ and } m(s) = m(q); \]
hence each term in the sum contains the factor (7.8), and the solution \( V_s = V_s \)
of the equation (4.10) contains that factor; so (7.5) holds, as was claimed.

Owing to (7.4), the first subscript on \( a_{\sim} \) is unnecessary, because for
given \( s \) there is at most one value of \( l \) such that \( a_{\sim} \) is \( \neq 0 \). We define
another integer-valued function \( j(s) \) on the lattice \( \mathcal{L} \) by saying that if
there is an \( e_j \) such that \( p(s) = p(e_j) \) and \( m(s) = m(e_j) \), then \( j(s) = j \), other-
wise \( j(s) = 0 \). We define

\[
a_s = \begin{cases} 
a_j(s), & \text{if } j(s) \neq 0 \\
0, & \text{if } j(s) = 0 
\end{cases} \quad \text{(7.9)}
\]

Lemma 2: If \( s \) and \( s' \) are lattice points related according to
\[ s'_1 = s_2, \ s'_2 = s_1, \]
\[ s'_{4m-1} = s_{4m}, \ s'_m = s_{4m-1}, \]
\[ s'_{4m+1} = s_{4m+2}, \ s'_{4m+2} = s_{4m+1}, \quad \text{(7.10)}\]
for all \( m \), in which case \( p(s') = -p(s) \) and \( m(s') = -m(s) \), then
\[ V_{s'}(r) = V_s(r) \text{ and } a_{s'} = a_s. \quad \text{(7.11)}\]
Proof: This lemma is also proved by induction on the norm $|s|$. For $|s| = 1$ (hence $|s'| = 1$), the assertions (7.11) simply represent known properties of the eigenfunctions and eigenvalues (7.1). For larger values of $|s|$, the assertions then follow inductively from (4.10) and (4.11), after replacing $s$, $q$, and $e_j$ throughout by $s'$, $q'$, and $e'_j$ according to the scheme (7.10), whereupon all quantities are replaced by their complex conjugates.

Lemma 3: If $s$ and $s'$ are lattice points related according to

\begin{align*}
  s'_1 &= s_2, \quad s'_2 = s_1, \\
  s'_{4m-1} &= s_{4m+1}, \quad s'_{4m+1} = s_{4m-1}, \\
  s'_{4m} &= s_{4m+2}, \quad s'_{4m+2} = s_{4m} 
\end{align*}

for all $m$, in which case $p(s') = -p(s)$ and $m(s') = m(s)$, then

\begin{align*}
  \hat{V}_{s'}(r) &= \hat{V}_s(r) \quad \text{and} \quad a_{s'} = a_{s} \quad \text{(7.13)}
\end{align*}

The proof of this lemma is similar to that of Lemma 2; it involves symmetry (3) described at the beginning of this section, rather than symmetry (4).

Lemma 4: If $s$ and $s'$ are lattice points related according to

\begin{align*}
  s'_1 &= s_1, \quad s'_2 = s_2, \\
  s'_{4m-1} &= s_{4m+2}, \quad s'_{4m+2} = s_{4m-1}, \\
  s'_{4m} &= s_{4m+1}, \quad s'_{4m+1} = s_{4m}, 
\end{align*}

for all $m$, in which case $p(s') = p(s)$ and $m(s') = -m(s)$, then

\begin{align*}
  \hat{V}_{s'} &= \hat{V}_s \quad \text{and} \quad a_{s'} = a_{s} 
\end{align*}

\quad (7.15)
This is simply a combination of Lemmas 2 and 3. As a further corollary of Lemmas 2 and 3, we note that if \( \mathbf{s} \) is any lattice vector such that \( s_{4m-1} = s_{4m} = s_{4m+1} = s_{4m+2} \) for all \( m \), then components 1, 2, 4, and 5 of \( \mathbf{V} \) are real, and components 3 and 6 are imaginary.

8. **Plan of the calculation: the long lists.** The lattice \( \mathcal{L} \) is enumerated by an index \( I = 1, 2, \ldots \) such that the norm \( |\mathbf{q}| \) of \( \mathbf{q} \) in \( \mathcal{L} \) is a nondecreasing function of \( I \). Various quantities are tabulated against \( I \) in long lists prepared at the beginning of the calculation. The quantities are

<table>
<thead>
<tr>
<th>old notation</th>
<th>new notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{q} )</td>
<td>( \mathbf{q}(I) )</td>
</tr>
<tr>
<td>( j(\mathbf{q}) )</td>
<td>( j(I) )</td>
</tr>
<tr>
<td>( p(\mathbf{q}) )</td>
<td>( p(I) )</td>
</tr>
<tr>
<td>( m(\mathbf{q}) )</td>
<td>( m(I) )</td>
</tr>
<tr>
<td>( \text{id}(\mathbf{q}) )</td>
<td>( \text{id}(I) )</td>
</tr>
<tr>
<td>(</td>
<td>\mathbf{q}</td>
</tr>
</tbody>
</table>

The quantity \( \text{id}(\mathbf{q}) \) is an identification number defined as

\[
\text{id}(\mathbf{q}) = \sum_{j=1}^{J} 10^{J-j} q_j ;
\]

it is used to simplify searching through the long lists for entries that correspond to a given lattice point \( \mathbf{q} \). (An alternative would be to use an enumerating polynomial to compute the index \( I \) from \( \mathbf{q} \) and avoid the searches altogether, but it is my impression that the computation of such a polynomial would take as much machine time as a search through a few hundred lattice...
Two more long lists are computed during the main calculation. They are
(see next section) \( \text{loc}(I) \)

\[ a_q^a = a_j(q), \quad q_a(I) \]

9. **Plan of the calculation: tables of the radial functions.** As we go through the lattice in the main calculation, it is only necessary, according to Lemmas 2, 3, and 4 of Section 7, to calculate the function \( V_s(r) \) when \( p(q) \geq 0 \) and \( m(q) \geq 0 \). That function can then be used for certain other lattice points after being suitably modified according to the Lemmas. Each time such a function is calculated, it is stored in a special table at a location indicated by an index "loc", which is then recorded in a long list as \( \text{loc}(I) \) for those values of \( I \) that correspond to the relevant lattice points; "loc" is then increased by 1 before the next function \( V_s(r) \) is computed and stored. The special tables are in an array where

\[ VV(\text{loc}, k + 1, 1) = 1^{\text{th}} \text{ component of } V_s(r_1 + k \Delta r) \quad (\text{complex}) \]

\[ (k = 0, 1, 2, \ldots, k_{\text{max}}, 1 = 1, \ldots, 6) \]

Whenever \( V_q(r) \) is needed later in the calculation for a given lattice point \( q \), it is obtained by a subroutine "FINDV", which obtains \( V(r) \) from the appropriate table in the above array and then modifies it in accordance with Lemmas 2, 3, and 4 if \( p(q) < 0 \) or \( m(q) < 0 \) or both.

10. **Runge-Kutta integration of the ordinary differential equations.**

Three kinds of ordinary differential equations are to be solved for functions \( V(r) \): the direct eigenfunction problem (6.7) with the boundary conditions
(6.8), the adjoint problem (6.9) and (6.10), and the inhomogeneous problem (4.10) of the abstract formulation, which here takes the form

\[ \frac{\partial}{\partial r} \mathbf{V} - \mathbf{A}(p, m) \mathbf{V} = \lambda \mathbf{M} \mathbf{V} = \mathbf{R}, \quad (10.1) \]

where \( \mathbf{R} = R(r) \) is given by minus the right member of (4.10) in terms of previously computed functions. \( \mathbf{A}(p, m) \) denotes the matrix \( \mathbf{A} \) with \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial \theta} \) replaced by \( ip \alpha \) and \( im \) respectively. The boundary condition for (10.1) is that \( V_4, V_5, \) and \( V_6 = 0 \) for \( r = r_1 \) and \( r = r_2 \). The shooting method is used, with occasional reorthonormalization, to reduce loss of accuracy by cancellations - see Keller 1968.

For the direct eigenfunction-eigenvalue problem, three integrations are performed for each trial \( \lambda \), from \( r = r_1 \) to \( r = r_2 \), all satisfying the left boundary condition \( V_4 = V_5 = V_6 = 0 \) but having different initial values of \( V_1, V_2, \) and \( V_3 \). The requirement that a linear combination of the three functions should have components 4, 5, and 6 = 0 at \( r = r_2 \) is a secular equation for \( \lambda \) in the form of the vanishing of a 3 x 3 determinant. The value of \( \lambda \) for which the determinant is zero is found by a simple iterative scheme, starting with a presumably rather accurate initial \( \lambda \) obtained from previous authors or earlier calculations.

The adjoint eigenfunction-eigenvalue problem is similar.

In the inhomogeneous problem (10.1), \( \lambda \) is not adjustable, but is fixed in advance. Four functions are computed, all satisfying the left boundary condition, three satisfying the homogeneous equation corresponding to (10.1) and the fourth satisfying (10.1) itself. We must then find a linear combination of the first three functions, which, when the fourth function is added to it,
satisfies the right boundary condition. The coefficients of the linear combination are therefore given by a system of three inhomogeneous linear equations.

Fourth order Runge-Kutta is used. The interval \((r_1, r_2)\) is divided into \(k_{\text{max}}\) subintervals, where \(k_{\text{max}}\) is even. The Runge-Kutta algorithm carries the solution from one even value of \(k\) to the next even one; the intermediate values of \(V(r)\) for odd \(k\) are then obtained by \(4^{\text{th}}\) order interpolation and stored for later use. For the Runge-Kutta algorithm, a subroutine DERIV computes \(\frac{\partial}{\partial r} V\) from (10.1) for any \(k\), when the other quantities in the equation are given.

11. An orthogonality correction. In consequence of various tests of accuracy, which were designed principally for debugging, it was found worthwhile to perform an orthogonalization operation on some of the radial functions, as follows. For \(|q| > 1\), each \(U_q\) is supposed to be orthogonal to the adjoint eigenfunctions \(\tilde{U}^+(j)\) \((j = 1, \ldots, J)\). In most cases that is automatic because of the dependence on \(\theta\) and \(z\), but when \(j(q) \neq 0\), the orthogonality of \(U_q\) to \(\tilde{U}_q\) \((j(q))\) involves the dependence on \(r\), hence is slightly disturbed by the residual errors of the Runge-Kutta integration. To prevent the accumulation of errors, we modify each such \(U_q\), immediately after it has been computed, by adding a small multiple of \(\tilde{U}_q(j(q))\) so as to achieve exact orthogonality (to machine accuracy). This correction is normally of the order of a few parts in \(10^5\). It turns out to be quite important when the Taylor number is close to the first critical value, for then the corresponding inhomogeneous ordinary differential equations are nearly singular, as noted at the end of Section 3, so that a small error in the right members can have the
effect of causing a magnified multiple of the corresponding eigenfunction to
be added to the solution.

12. Axial flow in the helical vortex modes. By integrating $w$, the $z$-
component of the velocity, with respect to area over the annular region
between the cylinders, one finds that there is a net axial flow of fluid in
the helical vortex modes. Right- and left-handed helixes have opposite axial
flows, while the Taylor vortices and wavy vortices have none.

Clearly end effects due to enclosing caps (or an endcap at the bottom and
a free surface at the top, if the cylinders are arranged vertically) will
prevent the axial flow and hence inhibit or at least alter the helical modes,
even if the cylinders are long and the ends are far away from the region con-
sidered, because the fluid is incomprehensible. Furthermore, it seems unclear
whether those end effects can alter the stability of the Taylor and wavy vor-
tices, hence it was thought appropriate to undertake a modification of the
mathematical model in which the axial flow is prevented.

If the cylinders are long, the flow far from the ends is expected to be
periodic in the $z$ direction, but it does not follow that the pressure is
periodic, since only its gradient appears. Rather, the pressure should be the
sum of a linear function of $z$ (independent of $r$ and $\theta$) and a periodic func-
tion. Hence we should write the total pressure as

$$zf(t) + p,$$  \hspace{1cm} (12.1)

where $p$ is the periodic part. The overall pressure gradient $f(t)$ represents
whatever force the endcaps must apply to prevent the axial flow. (That force
is of course balanced by a net $z$ component of the viscous drag on the cylinder
walls.) The effect of that is to replace the right member of (5.11) by $-f(t)$ and the right member of (6.1) by $f(t)Mv_o$, where

$$v_o = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(12.2)

Hence the right member of the evolution equation (1.1) becomes $-f(t)Mv_o$. The extra freedom thus introduced is compensated by the auxiliary condition of zero axial flow, which takes the form

$$(\tilde{v}_o, v(t)) = 0,$$

(12.3)

with

$$\tilde{v}_o = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ r \end{pmatrix},$$

(12.4)

where $r$ is the cylindrical radial coordinate; it appears because the inner product (6.9) contains $\int \cdots dr$ rather than $\int \cdots rdr$, which is required for an integration with respect to area. The auxiliary conditions can be written also in the form

$$(M \tilde{v}_o, M v(t)) = 0;$$

(12.5)

either form says that $w$, the sixth component of $v(t) = U(t)$, integrates to zero. (See (6.4) for $M$.)

We now show that the eigenfunctions of interest, the ones that span
are unchanged. If we take the inner product of the new equation of evolution ((1.1) with \(-f(t) M v_o\) on the right) with \(M \tilde{v}_o\) and use (12.5), we see that \(f(t)\) is given by

\[
(M \tilde{v}_o, L v - B(v, v)) = \text{const.} \ f(t),
\]

where the constant is \((M \tilde{v}_o, v_o)\). Hence the equation of evolution now has the form

\[
M \dot{v} - L v + B(v, v) = \text{const.} \ (M \tilde{v}_o, L v - B(v, v)) M v_o.
\]

By linearizing this equation for small \(v\), we see that the eigenfunction-eigenvalue equation is

\[
M v - L v = \text{const.} \ (M \tilde{v}_o, L v) M v_o
\]

In the Taylor problem we seek solutions \(v\) of the form \(e^{ipz + im\theta}\) times a function of \(r\), as before. For any such function, however, the inner product above is zero, owing to the dependence of \(v\) on \(\Theta\) and \(z\); hence the eigenfunctions of interest satisfy the same equation (1.2) as before.

To solve the evolution equation in the form

\[
M \dot{v} - L v + B(v, v) = -f(t) M v_o
\]

with the auxiliary condition (12.3) or (12.5), we write

\[
f(t) = \sum_{s \in L} b_s A_s^s, \quad |s| > 1.
\]

There is then an additional term \(-b_s M v_o\) on the right of (4.10), where \(b_s\) is not yet known but has to be determined so as to satisfy the condition of zero axial flow. Hence, equation (4.10) has to be solved twice, once with the
old right hand side and once with $M v_o$ as right hand side, and then $v_s$ has to
be taken as the first solution, say $v_s^{(1)}$, minus $b_s$ times the second solution,
say $v_s^{(2)}$, and $b_s$ is then determined by the condition

$$0 = (\tilde{v}_o, v_s) = (\tilde{v}_o, v_s^{(1)}) - b_s (\tilde{v}_o, v_s^{(2)}).$$

The first inner product in the third member of this equation is zero, unless
$p(s) = m(s) = 0$, hence it is only in this case that the second solution of
(4.10) must be computed. We note that for this same reason the final step in
the proof of Lemma 1 remains valid, because the solution of (4.10) is modified
only when it is independent of $\Theta$ and $z$, and then only by a correction that is
also independent of $\Theta$ and $z$.

Lastly, we note that equation (4.11) continues to hold, because the extra
term on the right of (4.10), from which (4.11) is derived, is orthogonal to
all the $v^{+(1)}$, owing to their dependence on $\Theta$ and $z$.

It was found that this modification of the program has a small destabil-
izing effect on the helical vortex mode at low Reynolds numbers, but almost no
effect on the other modes.

13. The modes of motion. The only modes that were observed in the cal-
culations (by modes we mean the fixed points and closed orbits of the dynamical
system in the unstable manifold) were of the kinds already described by
Davey, DiPrima, and Stuart, namely,

1. The Taylor vortex mode: axisymmetric steady ring vortices uniformly
   spaced in the $z$ direction.

2. Wavy vortices: These are similar to the Taylor vortices, but the vortex
core is displaced alternately up and down in the z direction, as $\theta$ varies; we call $m'$ the number of complete waves as $\theta$ goes from 0 to $2\pi$. The pattern rotates with approximately the mean angular velocity of the Couette flow. Necessary for such a mode to exist is that the eigenvalue $\lambda(m)$ with $m = m'$ be in the right half plane. However, that is not sufficient; a wavy vortex mode (with $m' = 1$ for a narrow gap between cylinders and $m' > 1$ for a wider gap) appears generally at a Reynolds number considerably higher than that for the first appearance of $\lambda(m')$ in the right half plane; it bifurcates from the Taylor vortex mode, not from the Couette flow.

3. Helical vortices: These are similar to the Taylor vortices, except that if the core of one is followed round the cylinders by letting $\theta$ increase from 0 to $2\pi$, it is joined not onto itself to form a ring but onto the second, or fourth, ..., vortex above it (or below it) in the z direction; we write correspondingly $m' = 1, 2, ...$. (It cannot be joined onto the first, or third, ..., vortex above it because those have the opposite direction of circulation of the fluid about the core.) The helical vortices bifurcate not from the Taylor vortices but from the Couette flow after that flow (which continues to be a solution of the Navier-Stokes equation) has become unstable; they are also unstable when they first appear, but may become stable at higher Reynolds numbers, thus providing an instance of a stable mode that cannot be reached by an adiabatic sequence of stable modes, as the Reynolds number is increased. The helical vortex with $m' = 1$ appears as soon as $\lambda(m')$ is in the right half plane. All these things are in agreement with the findings of Davey, DiPrima, and Stuart.
4. In-out wavy vortices: Similar to the ordinary wavy vortices, but the vortex core is displaced alternately in and out in the radial direction, as $\theta$ varies. They are apparently never stable.

5. Nonaxiasymmetric simple mode: This also bifurcates from the laminar Couette flow after that flow has become unstable. It may be regarded in lowest approximation as the superposition of two helical vortex modes, one right handed and one left handed. It is apparently never stable.

14. A few typical results. An early series of calculations was for a narrow gap, namely $r_2 - r_1 = .05$, with the mean radius $1/2 (r_1 + r_2)$ taken as unity, with the outer cylinder at rest, and with the expansions (4.2) and (4.4) carried to fifth degree terms, in order to compare with the results of Eagles.

A ten-dimensional manifold was studied, based on the first five of the eigenvalues of the linearized problem that cross into the right half of the complex plane; each has multiplicity two. The corresponding eigenfunctions depend on $\Omega$ as $e^{i m \theta}$, where $r, \theta, z$ are cylindrical coordinates, and where $m = 0, \pm 1, \pm 2$. The periodicity in the $z$ direction was such that the axial extent of a single vortex was 1.0047 times the gap. Axial flow was not suppressed in this series. The calculations were carried to a Taylor number $T$ (see, for example, Eagles 1968 for the definition) up to 3000. The power series expansions were carried to the 5th order. The results can be summarized as follows:

- Taylor ring vortices appear at $T = 1753.1$
- They are stable for $T < 1971.5$
- Wavy vortices $m = 1$ appear at $T = 1971.5$
- They are stable for $T < 1985$
Wavy vortices $m = 2$ appear at $T = 1970$
They are always stable

Helical vortices $m = 1$ appear at $T = 1768.4$
They are stable for $T > 1770.3$

Helical vortices $m = 2$ appear at $T = 1784$
They are stable for $T > 1815$

The values of $T$ given may not be very precise because of the small number of intervals used (24) in the solution of the radial ordinary differential equations. It is perhaps worth noting that for an interval of $T$ from about 1970 to about 1985 both wavy vortex modes, $m = 1$ and $m = 2$, appear to be stable.

This series was discontinued, because the number of dimensions of the unstable manifold increases rather rapidly so that for $T$ above about 1820, the ten-dimensional manifold, although invariant, is only a part of the unstable manifold, hence perturbations can take the solution out of that part.

That difficulty is somewhat less severe for the wide gap cases, hence we made a second series of calculations with gap $= 0.5$ times the mean radius. Eigenvalue calculations show that for that gap, the eigenvalue with $m = 4$ remains in the left half plane to the highest values of $T$ considered, hence a 14-dimensional manifold based on $m = 0, \pm 1, \pm 2, \pm 3$ was studied. (However, for large $T$, further eigenvalues appear in the right half plane for given $m$, but with different radial functions, so that there is here the same difficulty as in the previous series, but it now appears only at higher values of $T$.)

For large values of $T$, the number of steps in the radial integration was increased to 240, but in this series the power series expansions were carried only to the 3rd order. Axial flow was suppressed. It was found that the Tay-
lor vortices give way to wavy vortices with $m = 5$ somewhere between $T = 320,000$ and $640,000$, and the wavy vortices are stable from there to $T \approx 1,148,000$. For $T \geq 1,150,000$, no stable modes were found, either by mode analysis of the kind made by the previous authors or by orbit calculations; all orbits escape from the neighborhood of the origin in the configuration space (where the Taylor and wavy vortices had been) to larger distances, where the power series expansions used in computing the invariant manifold are inaccurate.

We must apparently assume that the bifurcation at $T = 1,148,000$ is subcritical. It may correspond to an unstable mode at $T < 1,148,000$ consisting of pulsating wavy vortices. For $T > 1,148,000$ there may be an aperiodic behavior (strange attractor), but it was felt that the present method is inadequate for further study without increasing the number of dimensions and the order of the power series expansions far beyond the capabilities of the present computer program.
References


