Normal Modes of an Atmospheric Prediction Model

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This technical report describes the derivation of the discrete normal modes of a grid point model developed at NCAR by Somerville and Shkoller. The work was completed September 1980. We would like to thank R. Somerville and B. Shkoller for discussions about their model and for assistance in accessing the model.

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. Linear normal mode analysis</td>
<td>2</td>
</tr>
<tr>
<td>3. Examples of normal modes</td>
<td>27</td>
</tr>
<tr>
<td>4. Nonlinear initialization</td>
<td>32</td>
</tr>
<tr>
<td>References</td>
<td>37</td>
</tr>
<tr>
<td>Tables</td>
<td>39–47</td>
</tr>
</tbody>
</table>
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 1</td>
<td>Equivalent depths (m) for 300°K isothermal mean state</td>
<td>39</td>
</tr>
<tr>
<td>Table 2</td>
<td>Frequencies (sec(^{-1})) of Rossby modes for (D = 10) km, (k=1)</td>
<td>40</td>
</tr>
<tr>
<td>Table 3</td>
<td>Frequencies (sec(^{-1})) of Rossby modes for (D = 10) km, (k=4)</td>
<td>41</td>
</tr>
<tr>
<td>Table 4</td>
<td>Frequencies (sec(^{-1})) of Rossby modes for (D = 100) km, (k=1)</td>
<td>42</td>
</tr>
<tr>
<td>Table 5</td>
<td>Frequencies (sec(^{-1})) of Rossby modes for (D = 100) km, (k=4)</td>
<td>43</td>
</tr>
<tr>
<td>Table 6</td>
<td>Frequencies (sec(^{-1})) of gravity modes for (D = 10) km, (k=1)</td>
<td>44</td>
</tr>
<tr>
<td>Table 7</td>
<td>Frequencies (sec(^{-1})) of gravity modes for (D = 10) km, (k=4)</td>
<td>45</td>
</tr>
<tr>
<td>Table 8</td>
<td>Frequencies (sec(^{-1})) of gravity modes for (D = 100) km, (k=1)</td>
<td>46</td>
</tr>
<tr>
<td>Table 9</td>
<td>Frequencies (sec(^{-1})) of gravity modes for (D = 100) km, (k=4)</td>
<td>47</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

The nonlinear normal mode initialization procedure introduced by Machenhauer (1977) for a barotropic spectral model has proven to be a powerful initialization procedure for global models. The procedure has been successfully applied to both baroclinic spectral models (Andersen, 1977; Daley, 1979) and grid point models (Temperton and Williamson, 1979). The basis of the procedure is an expansion of initial data into the linear normal modes of the forecast model. The initialization procedure consists of modifying the modal coefficients of the high frequency modes so that their time tendency is zero initially. This condition gives a nonlinear equation which Machenhauer (1977) proposed be solved iteratively. Once in mode space, the procedure is the same for all models and is well documented in the references cited above. The details of the discrete normal modes and linear expansion procedure, however, do depend on the actual model being used.

In this report, we present these details of the modal derivation for a particular model recently developed at NCAR by Somerville and Shkoller (Somerville, 1980) for forecast studies. We hereafter refer to this model as the S-S model. We describe the normal modes of this model and compare them with the modes of other grid point models. This model has a relatively complicated time stepping procedure which would make the exact model modal initialization procedure expensive with regard to the computing time required. Therefore, in the last section, we offer some suggestions to shorten this procedure for this model.
2. LINEAR NORMAL MODE ANALYSIS

In this section, we determine the linear normal modes of the Somerville-Shkoller model. The numerical approximations of this model are the same as those of the Goddard Institute for Space Studies (GISS) model (Somerville et al., 1974) described in detail by Tsang and Karn (1973). In this report, we follow the approach and modal notation of Williamson and Dickinson (1976) and Temperton and Williamson (1979) which has been used for grid point models. As described in the latter, it is not necessary to keep track of the nonlinear terms of the model formally for the initialization procedure. These can be found in mode space when needed for the nonlinear iteration by subtracting the known linear change from the total change made by the model. Therefore, in the following analysis, we do not carry the nonlinear terms after the equations have been linearized, but rather symbolically add them on in Section 4 where the nonlinear iteration is described. An analysis which retains the nonlinear terms is presented in Temperton and Williamson (1979).

The discrete normal modes are found as separable solutions to the linear discrete equations of the model. We first linearize the model equations and separate the vertical dependence. This gives the vertical component of the modes. We then consider the horizontal dependence. The S-S model is closely based on the TASU (time-alternating space-uncentered) scheme of Arakawa (1972). As used in the S-S model, this consists of a series of pairs of integration procedures, each pair consisting of a two-step Euler backward time difference. The horizontal space differencing employed in a pair can be either centered or
uncentered with centered being used most often. Therefore, we first find the modes corresponding to the centered spatial differences with an arbitrary, unspecified time step. The application of these modes for initialization of the complete model will be discussed in Section 4.

a. Vertical Modes

The S-S model uses a vertical coordinate with the model top at a finite fixed pressure $p_T$

$$\sigma = \frac{P - p_T}{P_S - P_T}$$

where $P_S$ is the variable surface pressure.

The discrete vertical structure of the S-S model consists of $L$ layers each with thickness $\Delta \sigma$, which may vary from layer to layer.

![Vertical finite difference mesh](image)

Fig. 1 Vertical finite difference mesh
The finite difference mesh is shown in Fig. 1; dashed lines correspond to the center of layers and solid lines refer to the interfaces between layers. The layers are indexed from top to bottom, and for each dashed line \( \ell \) there corresponds a solid line above labeled \( \ell^- \) and a solid line below labeled \( \ell^+ \). The surface of the earth corresponds to \( L^+ \). The top and bottom boundary conditions are

\[
\frac{d\sigma}{dt} = 0 \quad \text{at} \quad \sigma = 0, L
\]

In the following we write the difference equations on the discrete \( \sigma \) surfaces. For convenience, we temporarily leave the horizontal and temporal variation in continuous form. We then linearize these equations. We use the same notation as in the description of the GISS model given by Tsang and Karn (1973). The horizontal velocity equations are

\[
\frac{dV_\ell}{dt} + f_k \times V_\ell + \nabla_\sigma \phi_\ell + \sigma \alpha \nabla \pi = 0 \quad \ell = 1, L
\]

(2.1)

where \( V_\ell \) = horizontal velocity vector, \( f = \) Coriolis parameter, \( \nabla_\sigma = \) two-dimensional gradient, \( \sigma = \) vertical coordinate, \( \phi = \) geopotential, \( \alpha = \) specific volume, \( \pi = P_{\text{surface}} - P_{\text{top}}, p = \) pressure. The surface pressure tendency is given by

\[
\frac{\partial \pi}{\partial t} = - \sum_{\ell=1}^{L} (\Delta \sigma_\ell) \nabla_\sigma \cdot (\pi V_\ell)
\]

(2.2)

and the vertical motion by

\[
\frac{\cdot \ell}{\ell} = - \frac{1}{\pi} \left[ \sum_{i=1}^{L} (\Delta \sigma_i) \nabla_\sigma \cdot (\pi V_i) \right] - \frac{\sigma_\ell}{\pi} \left[ \frac{\partial \pi}{\partial t} \right]
\]

(2.3)
The hydrostatic equation is

\[ \phi_\ell - \phi_{\ell+1} = C_p \left( T_{\ell+1} - T_\ell \right) + C_p \pi \frac{p^K}{p_{\ell+1}} \left( \frac{\Delta \sigma_{\ell} T_\ell}{p_{\ell+1}^\kappa - p_{\ell}^\kappa} - \frac{\Delta \sigma_{\ell+1} T_{\ell+1}}{p_{\ell+1}^\kappa - p_{\ell}^\kappa} \right) \]

(2.4)

where \( C_p \) = specific heat at constant pressure, \( T \) = temperature, and \( \kappa = \frac{R}{C_p} \). For the lowest half layer at \( \ell = L \), the hydrostatic equation becomes

\[ \phi_L = \phi_s + C_p \left\{ \frac{\pi \sigma_{\ell} (\kappa+1)}{p^\kappa_{\ell+1} - p^\kappa_{\ell}} \right\} T_L \]

(2.5)

The thermodynamic equation is

\[ \frac{\partial (\pi T_{\ell})}{\partial t} + \nabla \cdot (\pi \nabla T_{\ell}) + \left( \frac{p_{\ell}}{P_o} \right) \kappa \pi \left[ \frac{\theta_{\ell+1} - \theta_{\ell}}{\Delta \sigma_{\ell}} - \frac{\theta_{\ell+1} - \theta_{\ell}}{\Delta \sigma_{\ell+1}} \right] \]

\[ = \frac{\sigma_{\ell} \pi}{C_p} \left[ \frac{\partial T}{\partial t} + \nabla \cdot \nabla \nabla \pi \right] \]

(2.6)

where \( \theta = \) potential temperature.

Here, \( P_o \), a reference pressure, is taken as 1000 mb and \( \theta_{\ell+1} \) is defined from

\[ \theta_{\ell+1} = \beta \theta_{\ell} + (1 - \beta) \theta_{\ell+1} \]

where \( \beta = \frac{(\Delta \sigma)_{\ell+1}}{(\Delta \sigma)_{\ell} + (\Delta \sigma)_{\ell+1}} \)
We linearize equations (2.1) - (2.6) around a mean with state \( \bar{u} = 0, \bar{v} = 0, \bar{a} = 0, \bar{T} = T(\sigma), \bar{\pi} = \text{constant} \). An overbar denotes the mean state, and those quantities without overbars are the perturbations.

The perturbation equations are

\[
\begin{align*}
\frac{\partial u}{\partial t} - f v + \left( \frac{1}{a \cos \theta} \frac{\partial \phi}{\partial \lambda} \right) \sigma + \frac{\sigma \bar{a}}{a \cos \theta} \frac{\partial \pi}{\partial \theta} &= 0 \\
\ell &= 1, L \\
\frac{\partial v}{\partial t} + f u + \left( \frac{1}{a \cos \phi} \frac{\partial \phi}{\partial \lambda} \right) \sigma + \frac{\sigma \bar{a}}{a \cos \phi} \frac{\partial \pi}{\partial \phi} &= 0 \\
\ell &= 1, L
\end{align*}
\]

where \( \vec{V} = (u,v) \), \( a = \text{radius of earth} \), and \( \theta \) and \( \lambda \) are latitude and longitude.

\[
\begin{align*}
\frac{\partial}{\partial t} (T_L) + \bar{T}_L \nabla_\sigma \cdot (V_L) + \left\{ \frac{\bar{P}_L}{P_0} \right\}^\kappa \left[ \bar{\phi}_L \frac{\partial \bar{\phi}_L}{\partial \sigma} - \frac{\sigma_L}{\Delta \sigma_L} \right] \\
= \left( \frac{\sigma_L \bar{\pi}}{P_L} - 1 \right) \frac{T_L}{\pi} \frac{\partial \pi}{\partial t} \\
\ell &= 1, L
\end{align*}
\]

\[
\frac{\partial \pi}{\partial t} = -\bar{\pi} \sum_{\ell=1}^{L} (\Delta \sigma)_L \left[ \nabla_\sigma \cdot (V_L) \right]
\]

(2.10)
\[ \dot{\sigma}_{\xi} = - \frac{1}{i=1} (\Delta\sigma)_{i} \nabla_{\sigma} \cdot (V_{i}) = - \frac{\sigma_{\xi}}{T} \]  
\( \xi = 1, L \)  

\[ \phi_{\xi} - \phi_{\xi+1} = C_{p} \left[ 1 - \frac{(\bar{P}_{\xi}^{+})^{k}(\kappa+1)\Delta\sigma_{\xi+1}}{\Delta\bar{P}_{\xi+1}} \right] T_{\xi+1} \]

\[ - C_{p} \left[ 1 - \frac{(\bar{P}_{\xi}^{+})^{k}(\kappa+1)\Delta\sigma_{\xi}}{\Delta\bar{P}_{\xi}^{+}} \right] T_{\xi} \]

\[ + C_{p} \left[ (\bar{P}_{\xi}^{+})^{k-1}(\pi+1) \right] \left\{ \left[ \frac{(\bar{P}_{\xi}^{+})(\pi\eta_{\xi+1}^{+}-1) - \kappa\pi\sigma_{\xi+1}^{+}}{\Delta\bar{P}_{\xi+1}} \right] \Delta\sigma_{\xi+1} T_{\xi+1} \right\} \]

\[ \xi = 1, L - 1 \quad (2.12) \]

\[ \phi_{L} = \sigma_{s} + C_{p} \left[ (\kappa+1)(\bar{P}_{s}^{+})^{k-1} \Delta\sigma_{L} \right] T_{L} \]

\[ - C_{p} \left[ (\kappa+1)(\bar{P}_{s}^{+})^{k-1} \left[ \frac{\pi[\bar{P}_{s}^{+}(\eta_{s}^{+}-1) - \kappa\pi\sigma_{s}^{+}]}{\Delta\bar{P}_{s}^{+}^{k+1}} \right] \Delta\sigma_{L} T_{L} \right] \]

\( \xi = L \quad (2.13) \)
where

$$\Delta P_{\kappa+1} = \frac{P_{\kappa+1}^+ - P_{\kappa+1}^-}{(P_{\kappa+1}^-)^{\kappa+1} - (P_{\kappa+1}^+)^{\kappa+1}}$$

$$\eta_{\kappa} = \frac{\kappa+1}{(P_{\kappa+1}^-)^{\kappa+1} - (P_{\kappa+1}^+)^{\kappa+1}}$$

The derivation of the vertical modes is most easily done by rewriting the equations in vector form. Let

$$\phi = [\phi_1, \phi_2, \phi_3, \ldots, \phi_L]^T$$

$$T = [T_1, T_2, \ldots, T_L]^T$$

and let the L x L matrix A have components

$$A = [a_{i,j}] = \begin{cases} 
\frac{C_p \pi \Delta \sigma_j (P_{\kappa}^+ - P_{\kappa}^-)(\kappa+1)}{\Delta P_j^{\kappa+1}} & i < j < L \\
\frac{C_p \pi \Delta \sigma_j (P_{\kappa}^+ - P_{\kappa}^-)(\kappa+1)}{\Delta P_j^{\kappa+1}} & i = j < L \\
\frac{C_p \pi \Delta \sigma_j (P_{\kappa}^+ - P_{\kappa}^-)(\kappa+1)}{\Delta P_j^{\kappa+1}} & i > j \\
\frac{C_p \pi \Delta \sigma_j (P_{\kappa}^+ - P_{\kappa}^-)(\kappa+1)}{\Delta P_j^{\kappa+1}} & i = j = L \\
0 & i > j \end{cases}$$
Let $\mathbf{b}$ be the vector of length $L$ with components

$$
\begin{align*}
\mathbf{b}_L &= - C \frac{(\kappa+1)(\tilde{P}_s^\kappa - P_{L+1}^\kappa)}{\Delta \tilde{P}_L^{\kappa+1}} \mathbf{b}_L + \frac{\Delta \sigma_L}{\Delta \tilde{P}_L^{\kappa+1}} \frac{\tilde{T}_L}{\Delta \tilde{P}_L^{\kappa+1}} \\
\mathbf{b}_L &= \left( \sum_{i=0}^{L-1} \mathbf{b}_{L-i} \right) + C \frac{\tilde{P}_L^{\kappa+1}}{\tilde{P}_L^{\kappa+1}}(\kappa+1) \left\{ \mathbf{P}_T \left( \chi_1 - \chi_2 \right) \right\}
\end{align*}
$$

where

$$
\begin{align*}
\chi_1 &= \left\{ \kappa - \frac{(\kappa+1)(\tilde{P}_L^\kappa - \tilde{P}_{L+1}^\kappa)}{\Delta \tilde{P}_L^{\kappa+1}} \right\} \frac{\Delta \sigma_L}{\Delta \tilde{P}_L^{\kappa+1}} \frac{\tilde{T}_L}{\Delta \tilde{P}_L^{\kappa+1}} \\
\chi_2 &= \left\{ \kappa - \frac{(\kappa+1)(\tilde{P}_L^\kappa - \tilde{P}_{L-1}^\kappa)}{\Delta \tilde{P}_L^{\kappa+1}} \right\} \frac{\Delta \sigma_L}{\Delta \tilde{P}_L^{\kappa+1}} \frac{\tilde{T}_L}{\Delta \tilde{P}_L^{\kappa+1}}
\end{align*}
$$

With these definitions, we may rewrite the hydrostatic equations (2.12) and (2.13) in vector form

$$
\phi = \phi_s \mathbf{e} + \mathbf{A} \mathbf{T} + \mathbf{\pi} \mathbf{b}
$$

(2.14)
where $\mathbf{e}$ is the vector with unity elements

$$\mathbf{e} = (1, 1, 1, \ldots)$$

Define the vectors

$$\mathbf{d} = (\Delta \sigma_1, \Delta \sigma_2, \ldots, \Delta \sigma_L)^T$$

and

$$\mathbf{\delta} = (\nabla \sigma \cdot \mathbf{V}_1, \nabla \sigma \cdot \mathbf{V}_2, \ldots, \nabla \sigma \cdot \mathbf{V}_L)$$

Equation (2.10) then becomes

$$\frac{\partial \mathbf{\pi}}{\partial t} = -\mathbf{\pi} \mathbf{d}^T \mathbf{\delta}$$ \hspace{1cm} (2.15)$$

Similarly, let $\mathbf{T}$ be the $L \times L$ matrix with components

$$\mathbf{T} = [t_{ij}] = \begin{pmatrix}
-T_1 & 0 \\
-T_2 & -T_3 \\
0 & -T_L
\end{pmatrix}$$

and the vector

$$\mathbf{Y} = \begin{bmatrix}
\left(\frac{1}{p_1} - 1\right) T_1, \\
\left(\frac{1}{p_2} - 1\right) T_2, \\
\ldots, \\
\left(\frac{1}{p_L} - 1\right) T_L
\end{bmatrix}^T$$
and \( D \) the \( L \times L \) matrix with components

\[
D = [d_{ij}] = \begin{cases} 
-\frac{\Delta \sigma_i}{\sigma_i} & i = j \\
\frac{\Delta \bar{\sigma}_{i+1}}{\bar{\sigma}_{i+1}} & i = j - 1 \\
\frac{\Delta \bar{\sigma}_{i-1}}{\bar{\sigma}_{i-1}} & i = j + 1 \\
0 & \text{otherwise}
\end{cases}
\]

The thermodynamic equation (2.9) can then be written as

\[
\frac{\partial T}{\partial t} = -\bar{T} \Delta \bar{\sigma} - Y \bar{d}^T \bar{\sigma} + D \dot{\bar{\sigma}}
\]

where

\[
\dot{\bar{\sigma}} = [\dot{\bar{\sigma}}_1, \dot{\bar{\sigma}}_2, \dot{\bar{\sigma}}_3, \ldots, \dot{\bar{\sigma}}_{L-1}, 0]^T
\]

Finally, if we let \( B \) be the \( L \times L \) matrix with components

\[
B = [b_{ij}] = \begin{cases} 
\Delta \sigma_j & j \leq i \leq L \\
0 & \text{otherwise}
\end{cases}
\]

equation (2.11) becomes
\[ \dot{z} = B(e^T - I)\dot{\xi} \tag{2.17} \]

where \( I \) is the identity matrix and \( e \) was defined following (2.14).

Now define a variable \( h_4 \) by

\[ gh_4 = \phi h + \sigma \alpha \pi \tag{2.18} \]

where \( g = \text{gravity} \), and let

\[ h = [h_1, h_2, \ldots, h_L]^T \]

\[ f = [\frac{\sigma_1}{p_1} RT_1, \frac{\sigma_2}{p_2} RT_2, \ldots, \frac{\sigma_L}{p_L} RT_L]^T \]

The derivative of \( h_4 \) can be rewritten in vector form

\[ \frac{3h}{3t} = \frac{\partial h}{\partial t} + f \frac{\partial f}{\partial t} \tag{2.19} \]

Combining (2.14) - (2.19) gives

\[ \frac{3h}{3t} = C \dot{\xi} \tag{2.20} \]

where

\[ C = A \left[ T - \frac{Y}{T} d^T + D B(e^T - I) \right] - \pi [b + f] d^T \]

Let \(-gD_m\) be an eigenvalue of \( C \) and let \( \phi_m \) be the corresponding eigenvector. These eigenvectors are the vertical modes of this model.

Let \( \phi \) be the matrix whose columns are the eigenvectors of \( C \)

\[ \phi = (\phi_1, \phi_2, \ldots, \phi_L) \]
and let $D$ be the diagonal matrix of eigenvalues with diagonal elements $D_m$. The eigenvalues and eigenvectors of $C$ can be written

$$C \phi = -g D \phi$$

If we define the vectors of vertical grid point values $u$ and $v$, the vertical transformation

$$\bar{u} = \phi^{-1} u,$$
$$\bar{v} = \phi^{-1} v,$$
$$\bar{h} = \phi^{-1} h,$$

of (2.7), (2.8) and (2.20) gives

$$\bar{u} = \phi^{-1} u$$
$$\bar{v} = \phi^{-1} v$$
$$\bar{h} = \phi^{-1} h$$

of (2.7), (2.8) and (2.20) gives

$$\frac{\partial \bar{u}}{\partial t} - f\bar{v} + \frac{g}{a \cos \theta} \frac{\partial \bar{h}}{\partial \lambda} = 0$$
$$\frac{\partial \bar{v}}{\partial t} + f\bar{u} + \frac{g}{a} \frac{\partial \bar{h}}{\partial \theta} = 0$$
$$\frac{\partial \bar{h}}{\partial t} + D \frac{\partial \bar{h}}{\partial \theta} = 0$$

The components of the transformed vectors $\bar{u}$, $\bar{v}$ and $\bar{h}$ represent the vertical mode coefficients and, because $D$ is diagonal, satisfy the shallow water equations with the corresponding mean depth $D_m$. 
\[ \begin{align*}
\frac{\partial u_m}{\partial t} - f \frac{v_m}{a \cos \theta} + g \frac{\partial h_m}{\partial \lambda} &= 0 \\
\frac{\partial v_m}{\partial t} + f \frac{u_m}{a \cos \theta} + \frac{g}{a} \frac{\partial h_m}{\partial \theta} &= 0 \\
\frac{\partial h_m}{\partial t} + D_m \frac{\delta h_m}{\delta m} &= 0
\end{align*} \]

b. Horizontal Modes

We now consider the horizontal difference approximations of the model. The GISS model uses a staggered grid in which the net of points on which thermodynamic variables (and thus \( h \)) are defined interlaces with the net on which \( u \) and \( v \) are defined. For convenience, two different points are represented by the same index as illustrated in Fig. 2.

![Horizontal finite difference mesh](image)
The u, v grid includes the equator (j = 0) and the h grid includes the pole (j = J). The cos \( \theta \) is defined at h grid points. Let \( \delta_t \) represent the discrete time difference which is left general for now. We first consider centered space differences as applied in the model. These represent a fundamental subset of a complete cycle of the GISS model. We will discuss the more complicated complete cycle of the model later.

\[
\delta_t u_{ij} = v_{ij} - \frac{g}{\Delta \lambda} \left[ \frac{1}{(\cos \theta_j + \cos \theta_{j-1})} \right] h_{i+1,j}^{\tau} - h_{i,j}^{\tau} + h_{i+1,j-1}^{\tau} - h_{i,j-1}^{\tau}.
\]

(2.24)

\[
\delta_t v_{ij} = -u_{ij} - \frac{g}{2a} \left[ \frac{h_{i,j}^{\tau} - h_{i,j-1}^{\tau} + h_{i+1,j}^{\tau} - h_{i+1,j-1}^{\tau}}{\Delta \theta} \right].
\]

(2.25)

\[
\delta_t h_{ij} = \frac{-D_m}{2a \cos \theta_j} \left[ \frac{u_{i+1,j}^{\tau} + u_{i,j+1}^{\tau} - u_{i-1,j}^{\tau} - u_{i,j+1}^{\tau}}{\Delta \lambda} \right] + \left( \frac{v_{i-1,j+1}^{\tau} + v_{i,j+1}^{\tau}}{2} \right) \left( \frac{\cos \theta_j + \cos \theta_{j+1}}{\Delta \theta} \right) - \left( \frac{v_{i-1,j}^{\tau} + v_{i,j}^{\tau}}{2} \right) \left( \frac{\cos \theta_{j-1} + \cos \theta_j}{\Delta \theta} \right).
\]

(2.26)

At the pole for h and next to the pole for u and v, equations (2.24), (2.25), and (2.26) take a special form, namely for \( j = J \)

\[
\delta_t u_{ij} = \frac{v_{ij}^{\tau}}{f_{ij}} - \frac{g}{\Delta \lambda} \left[ \frac{1}{(\cos \theta_j + \cos \theta_{j-1})} \right] h_{i+1,j}^{\tau} - h_{i,j}^{\tau} + h_{i+1,j-1}^{\tau} - h_{i,j-1}^{\tau}.
\]

(2.27)
\[ \delta v^\tau_{ij} = - u^\tau_{ij} f_j - \frac{g}{2a} \left[ \frac{h^\tau_{ij} - h^\tau_{i,j-1} + h^\tau_{i+1,j} - h^\tau_{i+1,j-1}}{\Delta \theta} \right] \]

\[ \delta h^\tau_{ij} = \frac{-4D_m}{(a \cos \theta) \Delta \theta} \left[ \frac{1}{I} \sum_{i=1}^{I} \left[ v_{i-1,J} + v_{i,J} \right] \cos \theta \left( \frac{j-1}{2} \right) \right] \]

where \( I \) = the number of points in longitudinal direction.

We diagonalize the longitudinal variation by Fourier transforms.

\[
\begin{pmatrix}
  -u^\tau \\
  -v \\
  -h
\end{pmatrix}
= \sum_{k=0}^{K} 
\begin{pmatrix}
  -u^t \\
  -v \\
  -h
\end{pmatrix}
\begin{pmatrix}
  e^{ik(j-1)\Delta \lambda} \\
  e^{ik(j-1)\Delta \lambda} \\
  e^{ik(j-1)\Delta \lambda} e^{-ik \frac{\Delta \lambda}{2}}
\end{pmatrix}
\]

where \( k \) is the longitudinal wavenumber and the extra term \( e^{-ik \Delta \lambda / 2} \) in our representation for \( h \) is a phase shift introduced because we are dealing with a staggered grid on which the longitude of the \( j \)th \( h \) point is not the same as that of the \( j \)th \( u, v \) point. Substituting (2.30) into equations (2.24) – (2.26) yields
\[ \delta t u_j = v_j f_j - \frac{g}{a (\cos \theta_j + \cos \theta_{j-1})} (2 \sin \frac{k \Delta \lambda}{2} (h_j + h_{j-1})) \] (2.31)

\[ \delta t v_j = -u_j f_j - \frac{g}{2a} \left( \frac{2 \cos \frac{k \Delta \lambda}{2} (h_j - h_{j-1})}{\Delta \theta} \right) \] (2.32)

\[ \delta t^\sim h_j = \frac{-D_m}{2a \cos \theta_j} \left\{ \left( v_{j+1} + \frac{\cos \theta_j + \cos \theta_{j+1}}{2} \right) \right\} - v_j \left\{ \frac{\cos \theta_{j-1} + \cos \theta_j}{2} \right\} \] (2.33)

Equation (2.31) is also valid next to the pole \( j = J \) except the \( h_j \) term is not present as seen by substituting (2.30) into (2.27).

Equation (2.28) becomes the same as (2.32), and (2.29) becomes, for \( j = J \)

\[ \delta t^\sim h_j = \frac{-4D_m}{a (\cos \theta_{J+1}) \Delta \theta} \left[ v_J \left( \frac{\cos \theta_J + \cos \theta_{J-1}}{2} \right) \right] \] (2.34)

We note that \( h_j \) is non-zero at the pole \( j = J \) for \( k = 0 \) only.

If we make the following substitutions:

\[ u_j' = i u_j \]

\[ v_j' = v_j \]

\[ h_j' = i \left( \frac{g}{D_m} \right) h_j \]
\[ C_m = (g D_m)^{1/2} \]
\[ k' = 2 \left( \sin \frac{k \Delta \lambda}{2} \right) / \Delta \lambda \]
\[ t = \cos \left( \frac{k \Delta \lambda}{2} \right) \]

Equations (2.31) to (2.33) become

\[ -i \delta_{t} \tilde{u}_j = \tilde{v}_j f_j - \frac{C_m}{a} \left[ \frac{1}{(\cos \theta_j + \cos \theta_{j-1})} \right] [k' \tilde{h}^j_j + \tilde{h}^j_{j-1}] \]  \hspace{1cm} (2.35)

\[ -i \delta_{t} \tilde{v}_j = \tilde{u}_j f_j + \frac{C_m}{a} \left[ \frac{t \tilde{h}^j_j - \tilde{h}^j_{j-1}}{\Delta \theta} \right] \]  \hspace{1cm} (2.36)

\[ -i \delta_{t} \tilde{h}^j_j = \frac{-C_m}{a \cos \theta_j} \left[ \frac{k' (\tilde{u}^j_j + \tilde{u}^j_{j+1})}{2} \right] \]
\[ \quad + \frac{\tilde{v}^j_{j+1} (\cos \theta_j + \cos \theta_{j+1})}{2 \Delta \theta} - \tilde{v}^j_j (\cos \theta_{j-1} + \cos \theta_j) \]  \hspace{1cm} (2.37)

And next to the pole, at \( j = J \), the equations become, for \( k \neq 0 \)

\[ -i \delta_{t} \tilde{u}_j = \tilde{v}_j f_j - \frac{C_m}{a} \left[ \frac{1}{(\cos \theta_j + \cos \theta_{j-1})} \right] [k' \tilde{h}^j_{j-1}] \]  \hspace{1cm} (2.38)

\[ -i \delta_{t} \tilde{v}_j = \tilde{u}_j f_j + \frac{C_m}{a} \left[ \frac{t \tilde{h}^j_{j-1}}{\Delta \theta} \right] \]  \hspace{1cm} (2.39)
and for $k = 0$ (remembering $k' = 0$ and $t = 1$)

\[-i \delta_t \tilde{u}'_j = \tilde{v}'_j f_j \tag{2.41}\]

\[-i \delta_t \tilde{v}'_j = u'_j f_j + \frac{C_m}{a} \left[ \frac{\tilde{h}_j - \tilde{h}_{j-1}}{\Delta \theta} \right] \tag{2.42}\]

\[-i \delta_t \tilde{h}'_j = \left( \frac{4C_m}{a(\cos \theta_j^{-1})\Delta \theta} \right) \tilde{v}' \left[ \frac{\cos \theta_j + \cos \theta_{j-1}}{2} \right] \tag{2.43}\]

Equations (2.35) to (2.43) represent a coupled set of equations from south pole to north pole. Although we have not presented the details of the difference approximations, the equations at the south pole can be determined in a similar manner. This coupled set of equations has two classes of solution. The symmetric solutions, which have $u_j = u_{-j}'$, $v_j = -v_{-j}'$, $h_j = h_{-1-j}'$, and the antisymmetric solutions, with $u_j = -u_{-j}'$, $v_j = v_{-j}'$, $h_j = -h_{-1-j}'$. Both the symmetric and the antisymmetric problems can be written as a coupled set of equations from the equator $j = 0$ to the pole $j = J$ for longitudinal wavenumber $k = 0$. For other longitudinal wavenumbers, $h_J = 0$, so that $h_{J-1}$ is the last unknown for height, while $u_J$ and $v_J$ are the last unknowns for the winds. These coupled equations can be written in the matrix form
\[ i Q \delta_t \bar{T} + LT = 0 \]  

(2.44)

where

\[
\bar{T} = \begin{bmatrix}
T_0 \\
\vdots \\
T_{J-1} \\
T_J
\end{bmatrix}
\quad \text{and} \quad
\bar{T}_j = \begin{bmatrix}
\bar{u}_j \\
\bar{v}_j \\
\bar{h}_j
\end{bmatrix}
\quad j = 1, 2, \ldots, J - 1
\]

\( \bar{T}_J \) is specified later for the two cases, and \( L \) and \( Q \) contain the terms in the equations:

\[
L = \begin{bmatrix}
A_0 & B_0 & 0 \\
C_1 & A_1 & B_1 \\
0 & C_2 & A_2 & B_2 \\
& & & & C_{J-1} & A_{J-1} & B_{J-1} \\
& & & & & C_J & A_J
\end{bmatrix}
\]

\[
Q = \begin{bmatrix}
Q_0 \\
\vdots \\
\vdots \\
Q_J
\end{bmatrix}
\]
\[ A_j = \begin{bmatrix} f_jq_j - \frac{Ck'}{2a} & 0 & -Ck' m \\ f_jq_j - 1 & 0 & \frac{Cm}{a\Delta\theta} q_j - 1 \\ 0 & \frac{Cm}{a\Delta\theta} q_j - 1 & 0 \end{bmatrix} \quad j = 1, \ldots, J-1 \]

\[ C_j = \begin{bmatrix} 0 & 0 & -Ck' m \\ 0 & 0 & \frac{Cm}{a\Delta\theta} q_j - 1 \\ 0 & 0 & 0 \end{bmatrix} \quad j = 2, \ldots, J-1 \]

\[ B_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad j = 1, \ldots, J-2 \]

\[ Q_j = \begin{bmatrix} q_j - 1 & 0 & 0 \\ 0 & q_j - 1 & 0 \\ 0 & 0 & q_j \end{bmatrix} \quad j = 1, \ldots, J-1 \]

where
\[ q_{j-1}^+ = \frac{\cos \theta_j + \cos \theta_{j-1}}{2} \quad \text{and} \quad q_j^0 = \cos \theta_j. \]

We have four cases to consider: \( k = 0 \) vs. \( k \neq 0 \) and symmetric vs. antisymmetric.

For \( k \neq 0 \)
\[
\begin{pmatrix}
  u_j' \\
  v_j'
\end{pmatrix}
\]
\[
T_j =
\]

Since \( h_j = 0 \), then
\[
A_j = \begin{pmatrix}
0 & f_j q_{j-1}^+ \\
\end{pmatrix}
\]
\[
B_{j-1} = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
\frac{-C k'}{2a} & \frac{-C t}{m (a \Delta \theta)} q_{j-1}^+
\end{pmatrix}
\]
\[
C_j = \begin{pmatrix}
0 & 0 & \frac{-C k'}{2a} \\
0 & 0 & \frac{-C t}{m (a \Delta \theta)} q_{j-1}^+
\end{pmatrix}
\]
\[
Q_j = \begin{pmatrix}
q_{j-1}^+ & 0 \\
0 & q_{j-1}^+
\end{pmatrix}
\]
For $k = 0,$

$$T_j = \begin{pmatrix}
\tilde{u}_j' \\
\tilde{v}_j' \\
\tilde{h}_j'
\end{pmatrix}$$

$$A_j = \begin{bmatrix}
0 & f_j q_{J-1}^+ & 0 \\
f_j q_{J-1}^+ & 0 & \frac{C_m q_{J-1}^+}{a\Delta \theta} \\
0 & \frac{C_m q_{J-1}^+}{a\Delta \theta} & 0
\end{bmatrix}$$

$$B_{J-1} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -\frac{C_m q_{J-1}^+}{a\Delta \theta} & 0
\end{bmatrix}$$

$$C_j = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -\frac{C_m}{a\Delta \theta} q_{j-1}^+ \\
0 & 0 & 0
\end{bmatrix}$$

$$Q_j = \begin{bmatrix}
q_{J-1}^+ & 0 & 0 \\
0 & q_{J-1}^+ & 0 \\
0 & 0 & 1/4 q_{J-1}^0
\end{bmatrix}$$
The symmetric vs. antisymmetric nature of the formulation enters in at the equator. For the symmetric problem, \( v_j = -v_{-j} \) implies \( v = 0 \) at the equator, so

\[
T_0 = \begin{bmatrix}
  u_0' \\
  0 \\
  h_0'
\end{bmatrix}
\]

\[
A_0 = \begin{bmatrix}
  0 & -c\frac{m}{k'} \\
  0 & \frac{m}{2a} \\
  -c\frac{m}{k'} \\
  \frac{m}{2a} & 0
\end{bmatrix}
\]

\[
C_1 = \begin{bmatrix}
  0 & -c\frac{m}{k'} \\
  0 & \frac{m}{a\Delta\theta} q^{-1} \\
  0 & 0
\end{bmatrix}
\]

\[
B_0 = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & -c\frac{m}{k'} & -c\frac{m}{a\Delta\theta} q^{-1} \\
  0 & \frac{m}{a\Delta\theta} q^{-1} & 0
\end{bmatrix}
\]

\[
Q_0 = \begin{bmatrix}
  1/2 q^{+1} & 0 \\
  0 & q_0
\end{bmatrix}
\]

For the antisymmetric case,

\[
T_0 = \begin{bmatrix}
  v_0' \\
  h_0'
\end{bmatrix}
\]
and $C_1$, $B_0$ and $Q_0$ remain the same.

Equation (2.44) can be transformed to simpler form by writing

$$\hat{L} = \hat{Q}^{-1/2} L \hat{Q}^{-1/2}$$

$$\hat{T} = \hat{Q}^{1/2} T$$

then

$$-i \delta_t \hat{T} + \hat{L} \hat{T} = 0 \quad (2.45)$$

This system of equations is diagonalized by using the eigenvalues and eigenvectors of $\hat{L}$. Let $\hat{Y}$ be the matrix whose columns are the eigenvectors $\hat{Y}_k^\ell$ of $\hat{L}$ and let $\hat{\Lambda}$ be the diagonal matrix of corresponding eigenvalues $\nu_k^\ell$.

$$\hat{Y}^{-1} \hat{L} \hat{Y} = \hat{\Lambda}$$

Equation (2.45) can then be written

$$\delta_t \hat{Y}^{-1} \hat{T} = -i \hat{\Lambda} \hat{Y}^{-1} \hat{T}$$

Since $\hat{\Lambda}$ is diagonal, this represents independent equations for the components of the vector $\hat{Y}^{-1} \hat{T}$. Denote these components by $c(k, \ell, m)$
\[ \hat{Y}^{-1} \hat{T} = \begin{pmatrix} c(k,1,m) \\ \vdots \\ c(k,L,m) \end{pmatrix} \]

Since \( \hat{L} \) is symmetric, its eigenvectors are orthogonal so that
\[ \hat{Y}^{-1} = \hat{Y}^T. \] Thus, if we define \( \hat{Y} = Q^{-1/2} \hat{Y}, \)

\[ \hat{c}(k, m) = Y^T Q \hat{Y}. \]

The linear equation for each coefficient is

\[ \delta_t \hat{c}(k, \ell, m) = -i \nu' \hat{c}(k, \ell, m) \quad (2.46) \]
3. **EXAMPLES OF NORMAL MODES**

a. **Vertical**

The vertical modes are the eigenvectors of the matrix $C$ in (2.20). Each vertical mode has a corresponding eigenvalue or equivalent depth $D$ which corresponds to the mean depth in the shallow-water equations used to determine the horizontal modes associated with that vertical mode. Table 1 lists the equivalent depths for the vertical modes of the 9-layer S-S model with a 300° isothermal atmosphere and pressure at the top $P_T$ of 1 and 10 mb. The mean thickness $\bar{\tau}$ is 1000 mb and the $\sigma$ levels are spaced equally. These equivalent depths can be compared with those for a 300° isothermal atmosphere in the ECMWF model (Temperton and Williamson, 1979). The general structure of the vertical modes of the GISS model is similar to that of other models shown in Hoskins and Simmons (1975), Williamson and Dickinson (1976), Daley (1979), or Temperton and Williamson (1979), and is not shown here.

b. **Horizontal Modes**

The horizontal normal modes are divided into two classes – *Rossby* modes and eastward and westward propagating *gravitational* modes. The discrete modes are identified according to their correspondence with the modes of the continuous equations. For the continuous case, the classification is made on the basis of the behavior of the frequencies and is discussed in Longuet-Higgins (1968), Dickinson and Williamson (1972), Kasahara (1976), and Temperton (1977).
In our discrete system, the Rossby and gravity wave modes are distinguishable in terms of their frequency and can be ordered by frequency. The symmetric system includes the odd-indexed Rossby modes and even-indexed gravity modes, while the antisymmetric system gives the even-indexed Rossby modes and odd-indexed gravity modes. The index increases with increasing frequency for the gravity modes and decreasing frequency for the Rossby modes. The number of zero crossings does not always correspond exactly with the index but in general the latitudinal structure becomes more complex as the index increases.

For \( k \neq 0 \), there are \( 3J + 1 \) modes which divide in \( J \) Rossby modes, \( J \) westward propagating gravity modes and \( J + 1 \) eastward propagating gravity modes. If we let \( R_k \) denote the \( k \)th Rossby mode, \( G^W_k \) and \( G^E_k \) the \( k \)th westward and eastward propagating gravity modes respectively, and order the modes by decreasing frequency, we find they are ordered in the following way:

\[
G^W_{2J-2}, \ldots, G^W_2, G^W_0, R_1, R_3, \ldots, R_{2J-1}, G^E_0, G^E_2, \ldots, G^E_{2J}
\]

For low wavenumbers, one of the eastward propagating gravity modes is a special type of computational polar inertial mode with a frequency which is determined by the grid interval. For convenience, this mode is isolated and labeled \( G^E_{2J} \). The details of this mode will be discussed later. Similarly, the antisymmetric modes are ordered:

\[
G^W_{2J-1}, \ldots, G^W_3, G^W_1, R_0, R_2, \ldots, R_{2J-1}, G^E_1, G^E_3, \ldots, G^E_{2J+1}
\]
The special computational mode mentioned above can also occur in the antisymmetric problem and is again isolated and labeled $G_{2J+1}^E$.

With this ordering or identification, we can now consider the frequencies of the modes. Tables 2–5 list the frequencies of the Rossby modes of $D = 10$ km, $k = 1$ and 4 and $D = 100$ m, $k = 1$ and 4 respectively, for $2\ 1/2^\circ$ and $5^\circ$ grids ($J = 36$ and 18 respectively). The exact frequencies provided by A. Kasahara are also listed for comparison. They were computed using the method described in Kasahara (1976). These frequencies are all for a continuous time derivative rather than being modulated by a response function of a particular temporal finite difference approximation. The frequency of the Rossby waves for centered differences with time step as large as 360 sec would be the same. Therefore, these frequencies can be compared with those of Dickinson and Williamson (1972) and Temperton (1977) which take into account the particular time differencing. The most notable difference in the general characteristics of the frequencies as the grid interval decreases is that with the S-S finite differences the frequencies of the Rossby modes decrease as the grid interval decreases because the discrete modes have a frequency higher than the continuous modes. With the unstaggered mesh of Dickinson and Williamson (1972), the frequencies increase as the grid interval decreases. Their discrete Rossby modes have lower frequencies than the continuous modes. The S-S staggered grid does not show the phenomenon of computational Rossby modes which propagate in the wrong direction. Temperton (1977) shows that these computational modes are a consequence of aliasing in the finite-differencing scheme on a nonstaggered grid.
Tables 6 - 9 present the frequencies of the gravity modes for $D = 10$ km, $k = 1$ and 4 and $D = 100$ m, $k = 1$ and 4 respectively. For $k = 1$, the low order modes of the $5^\circ$ S-S staggered grid described here are closer to those of the $2 1/2^\circ$ unstaggered grid of Dickinson and Williamson (1972). This agrees with Temperton's (1977) results for a different staggered grid and finite difference scheme. The correspondence is not as clear with higher wavenumber. For $k = 4$, the frequencies of the $5^\circ$ staggered grid modes are between those of the $5^\circ$ and $2 1/2^\circ$ unstaggered grid. Unlike the Rossby modes, the frequencies of the gravity modes increase with increased resolution for both the staggered and unstaggered grids.

Similar comparisons were found with other wavenumbers and equivalent depths. We do not show any graphs of the modes as they essentially resemble those of Dickinson and Williamson (1972) and Temperton (1977).

The particular S-S staggered grid does result in one particular computational mode as described earlier. This mode has $\tilde{u}$ and $\tilde{v}$ large at the point next to the pole and several orders of magnitude smaller elsewhere while $\tilde{h}$ is several orders of magnitude smaller everywhere. Since $\tilde{h}$ is several orders of magnitude smaller everywhere, (2.31) and (2.32) just give inertial motion at the point next to the pole.

$$\frac{\delta}{\delta t} \tilde{u}_j = \tilde{v}_j f_j$$

$$\frac{\delta}{\delta t} \tilde{v}_j = -\tilde{u}_j f_j$$
The oscillatory solution of these equations has frequency \( u = \pm f_j \) and \( u_j = \pm V_j \). For \( k = 1 \), the divergence in (2.33) due to these terms cancels to several orders of magnitude making \( \delta \chi_j \) several orders of magnitude smaller. Thus, this polar computational inertial wave has frequency \( f_j \) which is \(-1.45 \times 10^{-4}\) for both the 5° and 2 1/2° grids. Only the negative frequency appears as this computational mode. As the wavenumber increases above 1, the mode has additional structure in the polar region since the divergence term in (2.33) no longer cancels to the same extent. However, this computational mode remains identifiable.
4. NONLINEAR INITIALIZATION

In Section 2, we found the linear normal modes for centered horizontal differences which, as pointed out there, represent just a subset of a complete cycle of the S-S model. As described in Tsang and Karn (1973), a complete cycle consists of six pairs of procedures, each pair involving a two-step Euler backward time differencing, given by

\[ Q^{\tau+n} = Q^{\tau+(n-1)} + \Delta t \, D_c [Q^{\tau+(n-1)}] \]  \hspace{1cm} (4.1a)

\[ Q^{\tau+n} = Q^{\tau+(n-1)} + \Delta t \, D_n [Q^{\tau+n}] \]  \hspace{1cm} (4.1b)

where the \( D_n \) operators represent the horizontal differences which are not always the same and each time step corresponds to some value \( n \) of the cycle, \( n = 1 \) to 6. The forward time step of the pair (4.1) always uses the centered horizontal differences \( D_c \) described earlier. For the corrector or backward step (4.1b), different differencing is used depending on the cycle \( n \). When \( n = 1 \), \( D_1 \) is evaluated using up-right uncentered space differencing, when \( n = 2 \), \( D_2 \) is evaluated using down-left uncentered space differencing, and when \( n = 3 \) to 6, \( D_n \) is evaluated using the centered space differencing \( D_c \) described in Section 2.

In principle, one could find the normal modes of the complete six-cycle procedure. In practice, this becomes an extremely large eigenvalue problem and the modes have undesirable, inconvenient properties. The uncentered space differences in cycles 1 and 2 produce nonsymmetric matrices and thus the orthogonality property of the modes is no longer
guaranteed. As pointed out in Temperton and Williamson (1979), orthogonality is a very desirable property for practical application.

The model itself is used in the iterative nonlinear initialization procedure. To perform a complete six-cycle integration procedure, each iteration would require substantially more computer time than a procedure based on only one of the cycles. Each cycle itself should not generate large gravity waves once the initial data are balanced with respect to a different cycle. Therefore, nonlinear iteration could be based on one cycle using centered space differences in both steps (4.1, 4.2). By the same argument, one could also initialize using just the forward time step (4.1). This is essentially what has been done with other models (Machenhauer, 1977; Andersen, 1977; Daley, 1979; Temperton and Williamson, 1979) which use centered differences except for a forward first time step. However, one of the advantages of the nonlinear modal initialization is that it can be applied without knowledge of the details of the model code itself (Temperton and Williamson, 1979; Daley, 1980). If the fields are not available in output form after the forward step, but rather are available after the second step of any cycle, it is more convenient to perform the initialization based on one complete Euler backward pair.

The equation for the coefficients (2.46) including the nonlinear contribution can be written for a general time step operator $\delta_t$

$$\delta_t c(k,\xi,m) = -i \nu'(k,\xi,m) c(k,\xi,m) + r(k,\xi,m)$$  \hspace{1em} (4.2)
where the eigenvalues are from the centered space differences. Applying the Euler-backward differences of one cycle (4.1, 4.2) gives

\[ c^{\tau+1} = c^\tau + \Delta t(-i v' c^\tau + r) \]
\[ c^{\tau+1} = c^\tau + \Delta t(-i v' c^{\tau+1} + r) \]

We have assumed the nonlinear term does not vary in the two steps. Combining these two equations gives

\[ \frac{c^{\tau+1} - c^\tau}{\Delta t} = (1 - i v' \Delta t) (-i v' c^\tau + r) \] (4.3)

which has solution

\[ c(t) = \frac{r}{i v'} + \left[ c(0) - \frac{r}{i v'} \right] e^{-v_it} e^{i v_r t} \] (4.4)

where

\[ v_r \Delta t = \arctan \left[ \frac{-v' \Delta t}{1 - (v' \Delta t)^2} \right] \]
\[ v_i \Delta t = -1/2 \ln \left[ 1 - (v' \Delta t)^2 + (v' \Delta t)^4 \right] \]

For stable time step, \(|v' \Delta t| \leq 1\), the \( e^{-v_i t} \) factor represents the well-known damping of the Euler-backward scheme.

The normal mode initialization procedure modifies the coefficients of the undesirable modes to eliminate their time variation

\[ c(0) = \frac{r}{i v'} . \] (4.5)

This is the same condition as with forward differences. However, the equation to determine \( r \) from the model forecast is different in the case
of the Euler backward time step than in the case of the forward time step. As mentioned before, the nonlinear terms can be obtained without actually keeping track of them separately in the model by subtracting the linear part from the total change given by the model. Solving (4.3) for \( r \) gives

\[
r(k,\ell,m) = i v' c^T + \frac{c^{T+1} - c^T}{\Delta t (1 - i v' \Delta t)}
\]

(4.6)

for the complete Euler-backward procedure.

Once the coefficients are modified, grid point values of \( u, v, \) and \( h \) are obtained by summing over all the modes. The only slight complication arises in separating \( \pi_\ell \) and \( \phi_\ell \) from \( h_\ell \). We do this following Temperton and Williamson (1979) by assuming that the change made by the initialization removes linear gravity waves. Elimination of the divergence from (2.20) and (2.15) relates the \( \Delta \pi \) and \( \Delta h \) where \( \Delta \) denotes the change made by the initialization

\[
\frac{\partial \Delta \pi}{\partial t} = - g \pi \mathbf{d}^T \mathbf{C}^{-1} \frac{\partial \Delta h}{\partial t}
\]

(4.7)

or for the change in each gravity wave

\[
i v' \Delta \pi = - i v' g \pi \mathbf{d}^T \mathbf{C}^{-1} \Delta h
\]

Since the frequency \( v' \) is nonzero for gravity waves

\[
\Delta \pi = - g \pi \mathbf{d}^T \mathbf{C}^{-1} \Delta h
\]

(4.8)
which also holds for the sum of the modes being modified. Thus (4.8) can be used to relate the grid point changes. Once $\Delta \pi$ is known, the new $\pi^*$ is computed by

$$\pi^* = \pi + \Delta \pi$$

and the new geopotential from inverting (2.18)

$$\phi^* = g h^* - \sigma \alpha \pi^*$$

The new temperature is obtained by inverting (2.14)

$$T = A^{-1}[\phi^* - \phi_s e - \pi^* b]$$

Since these last two steps are linear, one could also deal with the changes rather than the total field.
REFERENCES


REFERENCES

Page 2


### TABLE 1

Equivalent depths (m) for 300°K isothermal mean state

<table>
<thead>
<tr>
<th>Vertical mode index</th>
<th>$P_T = 1 \text{ mb}$</th>
<th>$P_T = 10 \text{ mb}$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>11,700.0</td>
<td>11,480.0</td>
</tr>
<tr>
<td>2</td>
<td>2,588.0</td>
<td>2,259.0</td>
</tr>
<tr>
<td>3</td>
<td>603.3</td>
<td>546.8</td>
</tr>
<tr>
<td>4</td>
<td>180.2</td>
<td>167.1</td>
</tr>
<tr>
<td>5</td>
<td>69.51</td>
<td>65.58</td>
</tr>
<tr>
<td>6</td>
<td>27.42</td>
<td>26.06</td>
</tr>
<tr>
<td>7</td>
<td>10.61</td>
<td>10.15</td>
</tr>
<tr>
<td>8</td>
<td>3.025</td>
<td>2.90</td>
</tr>
<tr>
<td>9</td>
<td>.2485</td>
<td>.2396</td>
</tr>
</tbody>
</table>
TABLE 2

Frequencies (sec\(^{-1}\)) of Rossby modes for D = 10 km, k = 1

<table>
<thead>
<tr>
<th>(\ell)</th>
<th>5°</th>
<th>2 1/2°</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7.12 x 10(^{-5})</td>
<td>6.13 x 10(^{-5})</td>
<td>6.14 x 10(^{-5})</td>
</tr>
<tr>
<td>1</td>
<td>1.45 x 10(^{-5})</td>
<td>1.45 x 10(^{-5})</td>
<td>1.45 x 10(^{-5})</td>
</tr>
<tr>
<td>2</td>
<td>8.81 x 10(^{-6})</td>
<td>8.77 x 10(^{-6})</td>
<td>8.77 x 10(^{-6})</td>
</tr>
<tr>
<td>3</td>
<td>5.96 x 10(^{-6})</td>
<td>5.92 x 10(^{-6})</td>
<td>5.91 x 10(^{-6})</td>
</tr>
<tr>
<td>4</td>
<td>4.27 x 10(^{-6})</td>
<td>4.22 x 10(^{-6})</td>
<td>4.21 x 10(^{-6})</td>
</tr>
<tr>
<td>5</td>
<td>3.20 x 10(^{-6})</td>
<td>3.14 x 10(^{-6})</td>
<td>3.13 x 10(^{-6})</td>
</tr>
<tr>
<td>6</td>
<td>2.48 x 10(^{-6})</td>
<td>2.42 x 10(^{-6})</td>
<td>2.41 x 10(^{-6})</td>
</tr>
<tr>
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TABLE 3

Frequencies (sec⁻¹) of Rossby modes of D = 10 km, k = 4

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TABLE 5

Frequencies (sec$^{-1}$) of Rossby modes for $D = 100$ m, $k = 4$

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Frequencies (sec\(^{-1}\)) of gravity modes for \( D = 10 \) km, \( k = 1 \)
Table 7

Frequencies (sec\(^{-1}\)) of gravity modes for D = 10 km, k = 4

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### TABLE 8

Frequencies (sec⁻¹) of gravity modes for D = 100 m, k = 1

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Frequencies (sec$^{-1}$) of gravity modes for $D = 100$ m, $k = 4$