A Comparison of Quasi-Geostrophic and Primitive Equation Models for Stratospheric Sudden Warming Simulations

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1. **Introduction**

One of the most dramatic and intriguing of atmospheric phenomena is the stratospheric sudden warming, discovered by Scherhag (1952). In a typical sudden warming, an anomalously high amplitude quasi-stationary planetary wave is observed to propagate vertically into the stratosphere. The resulting northward eddy heat flux causes a dramatic polar stratosphere temperature increase followed by a reversal of the zonal wind.

The phenomenon has been extensively studied, both observationally and theoretically (see reviews by Schoeberl (1978), Holton (1980), McIntyre (1982)). Considerable insight into stratospheric sudden warmings has also been obtained by numerical simulation. In fact, Matsuno's (1971) successful simulation, using a simple linearized quasi-geostrophic model, provides the experimental basis upon which rests much of today's theory.

Since Matsuno's (1971) pioneering work, there have been many other such mechanistic simulations of sudden warmings. Most of these simulations have employed stratospheric models forced (usually through the geopotential field) near the tropopause. Some models have employed a form of the quasi-geostrophic equations (Matsuno (1971), Schoeberl and Strobel (1980a)); others have used the primitive equations (Holton (1976), Lordi et al. (1980), Hsu (1981)). Various prescriptions for the dissipation and diabatic heating parameterizations have been used; and both spectral and finite difference discretizations have been tested. Non-linear models used by Lordi et al. (1980) and Hsu (1981) dealt with the role of
wave-wave interactions; but most of the other simulations have only permitted wave-mean flow interactions.

Although there is general agreement amongst the various simulations, there are large differences in detail. Various experiments (in particular, Schoeberl and Strobel, 1980a) suggest that there is considerable sensitivity to the dissipation and diabatic heating parameterizations. Lordi et al. (1980) demonstrate that wave-wave interactions are not unimportant. Bridger and Stevens (1982) have shown that the course of the simulation strongly depends on the initial zonal windfield adopted.

There has been some speculation in the literature that the wavenumber 1 simulation, in particular, may be substantially affected by the choice of a quasi-geostrophic (QG) as opposed to a primitive equation (PE) model. In particular, the normal modes of the linearized PE and QG models differ in the planetary scales. The large-scale Rossby modes of PE and QG are different and QG lacks the large-scale Kelvin waves. In fact, the particular form of the QG equations adopted by Matsuno (1971) and Schoeberl and Strobel (1980a) seems to have been dictated by this concern. However, there have been no careful comparisons between QG and PE models for the simulation of stratospheric phenomena.

In fact, comparisons between QG and PE models are rather rare throughout the range of atmospheric and oceanographic phenomena. Simmons and Hoskins (1976) have compared QG and PE models for the simulation of baroclinic instability. Semtner and Holland (1978) have analyzed QG and PE simulations of the western North Atlantic circulation. Gent and McWilliams (1982a) have compared the phase-space properties of low order
QG, PE and intermediate models. Daley (1982) has compared high resolution barotropic simulations of QG, PE and intermediate models.

One of the goals of the present study is to perform a careful comparison between the primitive equation simulation of Lordi et al. (1981) and a quasi-geostrophic model to be described subsequently. Both models are to be forced in the lower stratosphere and the experimental conditions will be made as similar as possible.

One of the difficulties with mechanistic sudden warming simulations such as those of Matsuno (1971) and Lordi et al. (1980) has been the artificial nature of the forcing. The specification of the geopotential field near the tropopause is not very realistic. Two recent studies have attempted to remove this artificiality by specifying topographic forcing at the ground. In this way, the forcing mechanism becomes more physical and the troposphere enters into the simulation. The experiment of Koermer (1980) used a hybrid sigma-log pressure PE model while the Schoeberl and Strobel (1980b) simulation used a linearized quasi-geostrophic model. Although both simulations successfully produced stratospheric sudden warmings, there were large differences in detail between the two experiments. One difference, though, was particularly striking. The QG simulation of Schoeberl and Strobel (1980b) seemed to require far more topographic forcing to produce a warming than did the PE simulation of Koermer (1980). In the present study, we will examine the response of compatible PE and QG models to topographic forcing in an attempt to resolve this discrepancy.

Section 2 will discuss the formulation of a new quasi-geostrophic model. Section 3 will discuss an intercomparison between the present QG
model and the PE model of Lordi et al. (1980) for geopotential forcing near the tropopause. Section 4 will examine the response of PE and QG models for imposed topographic forcing. Section 5 will present a QG simulation of a stratospheric sudden warming topographically forced from the bottom of the troposphere.
2. Model Description

2.1 Basic Model Equations

The basic equations of the model are written in log-pressure coordinates. The vertical coordinate $Z$ is defined to be $Z = \frac{H \ln P_s}{P}$ where $H$ is the scale height, $P$ is the pressure and $P_s$ is the pressure at the ground (1000 mb). The equations will be developed in spherical coordinates and all Coriolis and metric terms will be handled correctly. The equations in their stratospheric context are similar to those used by Clark (1970) and Cunnold et al. (1975), but differ in detail from the stratospheric quasi-geostrophic equations used by Matsuno (1971) and Schoeberl and Strobel (1980). The model is fully non-linear and includes both wave-wave and wave-zonal flow interactions.

The model equations are the vorticity, thermodynamic, linear balance and continuity equations

\[ \frac{\partial \psi}{\partial t} = - \nabla \cdot (\xi + f) \nabla \psi - \nabla \cdot f \nabla \chi + k \nabla \times F, \quad (1) \]
\[ \frac{\partial \phi}{\partial t} = - \nabla \cdot \nabla \psi \frac{\partial \phi}{\partial Z} - \sigma \frac{\partial \psi}{\partial t} - \nabla \cdot \nabla \psi \frac{\partial \phi}{\partial Z} - \sigma' \frac{\partial \chi}{\partial Z} + \frac{Q}{c_p}, \quad (2) \]
\[ \nabla^2 \psi = \nabla \cdot f \nabla \psi, \quad (3) \]
\[ \nabla^2 \chi + \frac{\partial^2 \psi}{\partial Z^2} - \frac{\chi}{H} = 0, \quad (4) \]

where
\( \psi \) is the streamfunction,
\( \chi \) is the velocity potential,
\( \xi \) is the vorticity \( = \nabla^2 \psi \),
\( H \) is the scale height,
\( V = k \times \nabla \psi \),
\( \nabla, \nabla^2 \) are the horizontal gradient and Laplacian operators,
\( f \) is the (variable) Coriolis parameter,
\( \phi \) is the geopotential,
\( \dot{Z} \) is the vertical motion in \( Z \) coordinates,
\( F \) is the frictional force/unit mass,
\( C_p \) is the specific heat at constant pressure,
\( Q \) is the diabatic heating/unit mass,
\[
\sigma = \frac{R \left[ \frac{RT}{H} \frac{\partial \phi}{\partial Z} + \frac{\partial T}{\partial Z} \right]}{H C_p} \tag{5}
\]
is the static stability, where \( R \) is the gas constant for dry air, \(^{\text{a}}\) indicates a horizontal average, and \(^{\prime}\) indicates a deviation from the horizontal average. Temperature and geopotential are related through the hydrostatic relation
\[
T = \frac{H \frac{\partial \phi}{\partial Z}}{R} \tag{6}
\]
In the notation developed by Gent and McWilliams (1982a and b) equations (1–6) describe a linear balance model (LBE). The important features of the LBE model are that the divergence equation simplifies to the linear balance equation and the complete thermodynamic equation is retained. If the terms inside the dashed rectangle in equation (2) are
omitted, then equations (1 - 6) are known as the global quasi-geostrophic equations (gQG). The gQG model has a static stability ($\sigma$) which is a function only of the vertical coordinate. The prefix lower case $g$ in gQG indicates that this is a global approximation and the full variation of $f$ is retained in equation (3) and not the abbreviated form permissible in an $f$-plane approximation. The gQG version of equations (1 - 6) is formally dynamically equivalent to the model of Cunnold et al. (1975).

In the work that follows we will be mainly referring to the gQG version of equations (1 - 6) but there will be occasional references to simulations performed with the LBE version. In the development of the theory, the gQG model will be stressed. A non-iterative procedure used to integrate the LBE model has been described in Daley (1982) and so will not be discussed further here. In the remainder of the paper it can be assumed that all references are to the gQG model unless otherwise noted.

In spherical polar coordinates the model equations (1 - 6) can be expressed in the following form:

\begin{align*}
\frac{\partial}{\partial t} \psi & = R(A, B) - L(\chi), \\
\frac{\partial}{\partial t} \frac{\partial \phi}{\partial z} & = R(C, D) - \sigma(z)^2 + Q/C_p + S, \\
\nabla^2 \psi & = L(\psi), \\
\nabla^2 \chi & = -e^{Z/H} \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} e^{-Z/H} \right),
\end{align*}

where
\[ R(A,B) = - \frac{1}{a \cos^2 \phi} \left[ \frac{\partial A}{\partial \lambda} + \cos \phi \frac{\partial B}{\partial \phi} \right], \]

\[ L(\psi) = 2 \Omega \sin \phi \psi^2 + \frac{2\Omega}{a^2} \cos \phi \frac{\partial \psi}{\partial \phi}, \]

\[ A = u\psi \cos \phi [\xi + 2\Omega \sin \phi] - F \cos \phi, \]

\[ B = v\psi \cos \phi [\xi + 2\Omega \sin \phi] + F \lambda \cos \phi, \]

\[ C = u\psi \cos \phi \frac{\partial \phi}{\partial z}, \]

\[ D = V\psi \cos \phi \frac{\partial \phi}{\partial z}, \]

\[ a = \text{earth's radius}, \]

\[ \Omega = \text{the earth's rotation rate}, \]

\[ \lambda, \phi = \text{longitude and latitude}, \]

\[ u\psi = - \frac{1}{a} \frac{\partial \psi}{\partial \phi} \text{ is the east-west rotational wind component}, \]

\[ v\psi = \frac{1}{a \cos \phi} \frac{\partial \psi}{\partial \lambda} \text{ is the north-south rotational wind component}. \]

S indicates the non-linear terms in the dashed rectangle in equation (2). If S = 0, the equations (7 - 10) refer to the gQG model, if S ≠ 0 to the LBE model.
2.2 Horizontal Discretization

The model is horizontally discretized using spherical harmonic expansions in the manner of Eliasen et al. (1970) and Bourke (1971). Thus any variable $G(\lambda, \phi, Z, t)$ is expanded in a finite sum of spherical harmonic functions.

$$G(\lambda, \phi, Z, t) = \sum_{m=-J}^{J} \sum_{j=0}^{L} G^m_j(Z,t) Y^m_j(\lambda,\phi)$$

where

$$Y^m_j(\lambda, \phi) = P^m_j(\sin \phi) e^{im\lambda}.$$ 

$P^m_j(\sin \phi)$ is the associated Legendre polynomial,

$m$ is the zonal wavenumber,

$\lambda$ is the meridional wavenumber,

$J$ and $L$, the zonal and meridional wavenumber truncation limits,

define a parallelogramic spherical harmonic truncation.

The $Y^m_j$ are orthogonal and normalized satisfying

$$[Y^m_j Y^m_k] = \delta^m_j \delta^m_k$$

where $\delta^m_j$ is the Kronecker Delta, (*) indicates complex conjugation and

$$[\ ] = \frac{1}{2\pi} \int_0^{\pi/2} \int_0^{2\pi} [\ ] \cos \phi \, d\phi \, d\lambda$$

is the horizontal average as in Section (2.1).

Several spherical harmonic identities are helpful in the derivation of the spectral (discretized) form of the equations.
Expanding the variables $\psi$, $\chi$, $\phi$, $Z$, $Q$ in terms of spherical harmonic series (equation 11), the model equations (7) - (10) can be written in the following discretized form

\[
\begin{align*}
\frac{\partial \chi^m_\ell}{\partial \lambda} &= i m \chi^m_\ell, \\
\cos \phi \frac{\partial \psi^m_\ell}{\partial \phi} &= -\ell \psi^m_{\ell+1} + (\ell+1) \psi^m_{\ell-1}, \\
\sin \phi \psi^m_\ell &= \psi^m_{\ell+1} + \psi^m_{\ell-1}, \\
\n\frac{\partial^2 \chi^m_\ell}{\partial \phi^2} &= \frac{\ell(\ell+1)}{a^2} \chi^m_\ell, \\
\varepsilon^m_\ell &= (\ell^2 - m^2)/(4\ell^2 - 1)^{1/2}, \\
\n\end{align*}
\]

Expanding the variables $\psi$, $\chi$, $\phi$, $Z$, $Q$ in terms of spherical harmonic series (equation 11), the model equations (7) - (10) can be written in the following discretized form

\[
\begin{align*}
-\ell(\ell+1) \psi^m_\ell &= a^2 R^m_\ell(A,B) + 2i[m \chi^m_\ell + \chi^m_{\ell+1}], \\
\frac{\partial \psi^m_\ell}{\partial Z} &= R^m_\ell(C,D) - \sigma(Z) \frac{\partial^2 \chi^m_\ell}{\partial \phi^2} + \chi^m_\ell + Q^m_\ell + S^m_\ell, \\
\ell(\ell+1) \phi^m_\ell &= 2i[m \chi^m_\ell + \chi^m_{\ell+1}], \\
\ell(\ell+1) \chi^m_\ell &= a^2 e^{+Z/H} \frac{\partial^2 \chi^m_\ell}{\partial Z^2} \left( \frac{\partial \chi^m_\ell}{\partial Z} e^{-Z/H} \right), \\
\varepsilon^m_\ell &= (\ell+1)(\ell-1) \psi^m_\ell, \\
R^m_\ell(A,B) &= -\frac{\pi/2}{-\pi/2} \int \cos \phi [i m A^m_\ell \chi^m_\ell - B^m_\ell \cos \phi \frac{\partial \chi^m_\ell}{\partial \phi}] d\phi, \\
\end{align*}
\]
\[ A(\lambda, \phi, Z, t) = a \sum_{m=-J}^{J} A_m(\phi, Z, t) e^{im\lambda} \]

\[ B(\lambda, \phi, Z, t) = a \sum_{m=-J}^{J} B_m(\phi, Z, t) e^{im\lambda} \]

and \( \psi^m, \chi^m, \phi^m, Z^m \) and \( Q^m \) are the spherical harmonic expansion coefficients of \( \psi, \chi, \phi, Z, \) and \( Q \) respectively.

The nonlinear terms \( A, B, C, \) and \( D \) are calculated on a spectral transform grid (equally spaced in the east-west direction and Gaussian latitudes) following Eliasen et al. (1970) and Bourke (1971).

2.3 Vertical Discretization

The vertical discretization is obtained by using second order finite differencing. Figure 1 indicates the vertical grid used.

The streamfunction \( \psi \), geopotential \( \phi \), wind components \( u, v \), and the velocity potential \( \chi \) are defined at the integer levels, while the vertical velocity \( \dot{Z} \) is defined at the half integer levels. The top boundary condition is \( \dot{Z} = 0 \) while the bottom boundary condition assumes \( \frac{d\phi}{dt} \) at the bottom is externally specified. The vertical level index is defined by \( 0 \leq n \leq N \).

We define

\[ E_{n+1} = e^{n+1/H} \quad 0 \leq n \leq N \]
\[ A Z_{n+i} = Z_{n+.5+i} - Z_{n-.5+i} \quad i = 0, .5 \]
\[ \Delta Z_{N+1/2} = Z_{N+1/2} - Z_N, \quad \Delta Z_0 = Z_{1/2} - Z_{1/2} \]

The vertically discretized form of the model equations (14) - (21) can then be written

\[ -\mathcal{E}(\ell+1) v^m,^\ell_n = a^2 R^{m,^\ell}_n(A, B) + 2\Omega(a^m_{\ell} \chi_n^{m,^\ell-1} + a^m_{\ell+1} \chi_n^{m,^\ell+1}) \]
\[ 0 \leq n \leq N \quad (23) \]
\[ \frac{\Delta Z_{n-.5}}{\Delta Z_{n-.5}} = R^{m,^\ell}_{n-1}(C, D) - \sigma_{n-.5} Z_{n-.5} + \frac{\alpha^{m,^\ell}_{n-.5}}{\sigma} + S_{n-.5} \quad 1 \leq n \leq N \quad (24) \]
\[ \mathcal{E}(\ell+1) \phi^m,^\ell_n = 2\Omega(a^m_{\ell} \psi_n^{m,^\ell-1} + a^m_{\ell+1} \psi_n^{m,^\ell+1}) \quad 0 < n < N \quad (25) \]
\[ \mathcal{E}(\ell+1) \chi^m,^\ell_n = \frac{a^2 E_n}{\Delta Z_n} \frac{Z_n^{+5}}{Z_n^{+5}} - \frac{Z_n^{-5}}{Z_n^{-5}} \quad 1 < n < N-1 \quad (26) \]

2.4 Vertical Boundary Conditions

The upper boundary condition is

\[ Z_{N+.5} = 0 \]

The lower boundary condition is

\[ \frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \nabla \cdot V^\phi + \frac{RT}{H} \cdot Z_0 + \frac{V^\phi \cdot V^\phi}{\chi} + \frac{RT^1}{H} \cdot Z_0 = gW_s \]

where \( W_s \) is externally specified. The terms in the dashed rectangle are included in the LBE model, but not the gQG model.
These boundary conditions result in modifications to the continuity equation (26) at the top. Thus, at \( n = N \)

\[
\lambda(n+1) \chi_N^m,\lambda = - \frac{a^2 E_1}{\Delta Z_N E_{N-.5}} \frac{Z}{Z_{N-.5}}
\]  

(27)

At \( n = 0 \) the boundary condition becomes

\[
\Phi_0 = R_0^{m,\lambda} (C_0, D_0) - \frac{R}{H} Z_0^{m,\lambda} + g W_s^{m,\lambda} + S_o^{m,\lambda}
\]  

(28)

where

\[
C_0 = u^3 \cos \phi \Phi_0
\]

\[
D_0 = v^3 \cos \phi \Phi_0
\]

\( S_o^{m,\lambda} \) are the terms in the dashed rectangle.

2.5 Energy Conservation

The analytic form of the (gQG) model equations (7) to (10) conserves the sum of globally integrated kinetic and potential energy in the case where \( W_s = 0 \). Thus, if the globally integrated kinetic and potential energies are defined as

\[
KE = - \int_0^{Z_T} \left[ \Phi \frac{\partial^2 \Phi}{\partial z^2} \right] e^{-Z/H} \, dz,
\]  

(29)

\[
PE = \int_0^{Z_T} \frac{1}{\sigma(z)} \left[ \frac{\partial \Phi}{\partial z} \frac{\partial \Phi}{\partial z} \right] e^{-Z/H} \, dz + \frac{H}{R} \left[ \Phi \cdot \Phi \right]_o
\]

(30)

then \( KE + PE = \text{constant} \). Here \( \left[ \right] \) is defined by equation (13), and \( Z_T \) is the top of the atmosphere.
The appropriate discretized forms of (29) and (30) also have this conservative property. Thus

\[ KE = \sum_{m, \lambda} \sum_{n=0}^{N} \lambda(\lambda+1) \psi_n^m, \lambda \psi_n^m, \lambda E_n^{-1} \Delta Z_n, \] (31)

\[ PE = \sum_{m, \lambda} \sum_{n=1}^{N} \left( \phi_n^m, \lambda - \phi_{n-1}^m, \lambda \right) \left( \phi_n^m, \lambda - \phi_{n-1}^m, \lambda \right) + \frac{H}{RT_0} \sum_{m, \lambda} \phi_o^m, \lambda \phi_o^m, \lambda. \] (32)

Again, \( KE + PE \) is a constant.

As shown by Lorenz (1960), the LBE equations also conserve energy when \( W_s = 0 \). Burger and Riphagen (1979) have discussed energy conservation for PE, LBE and gQG models in the case \( W_s \neq 0 \) (although for a simpler lower boundary condition than equation 28). Their results show that although energy is conserved for the PE case, there is a discrepancy in energy conservation of \( O(R_o) \) in the LBE case and \( O(1) \) in the gQG case.

Here \( R_o \) is the Rossby number.

2.6 Dissipation and Forcing

The frictional force terms \( F_\phi \) and \( F_\lambda \) are composed of a surface drag term and a Rayleigh friction term. Thus

\[ F_\phi = - \frac{gH}{\rho_n} \frac{\partial \phi}{\partial z} - F_r(z)[u - u_e], \] (33)

\[ F_\lambda = - \frac{gH}{\rho_n} \frac{\partial \lambda}{\partial z} - F_r(z)[v - v_e], \] (34)
where

\[ \tau_\phi = \tau_\lambda = 0 \quad \text{for} \quad n \neq 0 \]

\[ \tau_\phi = -\rho_s c_D |\nabla| u_1 \quad n = 0 \]

\[ \tau_\lambda = -\rho_s c_D |\nabla| v_1 \quad n = 0 \]

\( \rho \) is the density, \( \rho_s \) density at the ground, and \( c_D \) is the drag coefficient = .0025.

The Rayleigh friction coefficient \( F_r(z) \) is given by Holton (1976)

\[
F_r(z) = \begin{cases} 
10^{-7} \text{ s}^{-1}, & Z \leq 50 \text{ km} \\
10^{-7} \text{ s}^{-1} + 5 \times 10^{-6} \times [1 - \exp\left(\frac{50-Z}{40}\right)] \text{ s}^{-1}, & Z > 50 \text{ km}.
\end{cases}
\]

The equilibrium velocity fields are given by \( u_e \) and \( v_e \) and are assumed to be specified.

The diabatic heating rate is given by Holton (1976) and is expressed as follows:

\[
Q/C_p = \beta(z)[T_e - T]
\]

(36)

where

\[ \beta(z) = [1.5 + \tanh \left(\frac{Z-35}{7}\right)] \times 10^{-6} \text{ s}^{-1} \]

Here \( T \) is related to \( \phi \) by the hydrostatic equation (6) and \( T_e \) is the (specified) equilibrium temperature distribution.

There is also a subgridscale parameterization similar to that used by Lordi et al. (1980). A Fickian type diffusion is added to the vorticity (1) and thermodynamic (2) equations. Thus,
\[
\begin{align*}
\frac{\partial \xi}{\partial t} &= \ldots + k_H \left[ \nabla^2 + 2a^{-2} \right] \left[ \xi - \xi_e \right], \\
\frac{\partial^2 \phi}{\partial t \partial z} &= \ldots + k_H \nabla^2 \left[ \frac{\partial \phi}{\partial z} - \frac{\partial \phi_e}{\partial z} \right]
\end{align*}
\] (37) (38)

where \( \xi_e \) and \( \phi_e \) are the equilibrium vorticity and geopotential distributions related to \( u_e, v_e \) and \( T_e \). The dots in equations (37) and (38) indicate the omitted right-hand side terms in equations (1) and (2).

2.7 Method of Solution

The model equations (23) to (26) together with the boundary conditions (27) and (28) and the forcing and dissipation of Section 2.6 were integrated by centered time differencing (leapfrog method). To control the weak computational time mode excited by this scheme, a weak time filter (Asselin, 1972) was used.

The non-linear terms \( R_{n}^{m, \lambda} (A, B) \), and \( R_{n-0.5}^{m, \lambda} (C, D) \) and \( S_{n-0.5}^{m, \lambda} \) were calculated using the spectral transform technique of Eliasen et al. (1970). That is, the spherical harmonic coefficients were transformed into gridpoint values by Legendre and Fourier transforms. The non-linear multiplications were performed on the transform grid and the results were reverse transformed (Fourier followed by Legendre) into spherical harmonic coefficients, using the operators defined after equation (22). As in Eliasen et al. (1970), the transform grid was equally spaced in longitude and used the Gaussian latitudes. The vectorized fast Fourier transform
technique of Temperton (1977) was used for the forward and reverse Fourier transforms.

For the gQG model, the most difficult remaining problem is the simultaneous solution of equations (23) - (24) subject to the constraints (25) - (26) given that all the non-linear and forcing terms have been determined. \( z^m_k \), \( x^m_n \), and \( \psi^m_n \) can be eliminated to produce a set of equations in which the only unknowns are the \( z^m_n \). This set of equations is analogous to the discrete form of the quasi-geostrophic omega equation. The \( z^m_n \) are given implicitly and a large matrix must therefore be inverted to obtain the \( z^m_n \). The form of the matrix is block tridiagonal in the meridional (\( \lambda \)) and vertical (\( n \)) directions for each zonal wavenumber (\( m \)). The problem in the zonal direction is trivial, but the problem in the meridional and vertical directions is more difficult.

In general, we expect \( N \gg L \) (i.e. considerably more vertical than meridional resolution). The procedure adopted for inverting the matrices was the eigenvector decomposition technique used by Staniforth and Daley (1979). For each zonal wavenumber (\( m \)), the eigenstructure in the meridional direction (mainly given by the \( \lambda(\lambda+1) \) and \( \alpha^m_\lambda \) operators) was found using standard algebraic eigenvalue techniques. Using the inverse of the matrix of eigenvectors and the eigenvalues, each original block tridiagonal matrix could be reduced to a set of ordinary tri-diagonal matrices in the vertical coordinate (\( n \)) for each meridional eigenvector. These systems were solved by standard direct solvers. The results were then back transformed using the transpose of the eigenvector matrix (see Staniforth and Daley (1979) for details).
The only disadvantage of this scheme is that it requires storage of the meridional eigenvector matrix, its inverse and the eigenvalues for each zonal wavenumber \( (m) \). These matrices are of course independent of the vertical index \( n \). However, since the order of these matrices (defined by \( L \)) is not expected to be too large, this problem is not serious. It might be noted that the meridional eigenvalue problems break up into separate symmetric and anti-symmetric problems which simplifies the calculations. There is a simple degeneracy in the case \( m = 0 \), but this causes no difficulty.

We note that if \( f \) were constant in those equations, the meridional matrices would be diagonal and the block tri-diagonal problem would simplify into an ordinary tri-diagonal problem for each \( m \) and \( \ell \). The variability of \( f \) causes the difficulties.

After solving for \( \psi_{n}^{m,\ell} \), back substitution into the vorticity equation will produce \( \psi_{n}^{m,\ell} \) from which \( \psi, \phi \) at the new timestep can be evaluated.

The LBE model (which includes the \( S_{n-\frac{1}{2}}^{m,\ell} \) term) was integrated using the non-iterative procedure of Daley (1982) with an iterative restart procedure applied every 24 hours.

The model integration has been made very efficient by the use of vectorized fast Fourier and Legendre transforms. The matrix multiplications involved in the solution of the block tri-diagonal problems are also vectorized.

A timing comparison between the present model and the semi-implicit global non-linear spectral primitive equation stratospheric model of
Koermer (1980) is instructive. A timing comparison was performed in which the models were made as similar as possible. Thus, the number of degrees of freedom (N = 31, J = 4, L = 24), the timestep (30 minutes), the physical parameterizations and the computer used (the NCAR CRAY-1) were the same. (Both models were run hemispherically so there were really only 12 meridional degrees of freedom.) Both models were core contained.

A 24-hour integration with Koermer's (1980) PE model required 9 minutes CPU. The same integration with the gQG version of the present model took 18 seconds, while the LBE version took 30 seconds. Wave-mean flow simulations performed with the present gQG or LBE models can be integrated considerably more rapidly. In all fairness, it should be mentioned that the PE model was not fully optimized and the processing of the PE model results was done as the integration proceeded.
3. Stratospheric sudden warmings forced from the lower stratosphere

The gQG and LBE models described in Section 2 were first tested in the simulation of sudden warmings forced from the lower stratosphere. The results were compared directly with those of Lordi et al. (1980) - hereinafter referred to as LKK. Indirect comparisons were also made with the simulations of Matsuno (1971), Holton (1976), and Schoeberl and Strobel (1980a) - hereinafter referred to as SSa.

In order to simulate forcing from the lower stratosphere, the model described in Section 2 had to be slightly modified. The lower boundary of the model was defined to be 12 km as in LKK. The lowest model working level (integer level) was at 13.5 km with 26 levels to 88.5 km as in LKK. The vertical placing of variables in the present model is slightly different than in LKK. Thus, in the present model, the geopotential, streamfunction and velocity potential are defined at the integer levels (13.5 km, 16.5 km ....) while the vertical motion Z and the temperature T are defined at the half levels (15 km, 18 km ....). In the LKK simulation, the vertical specification of temperature and geopotential are reversed.

The geopotential is specified at the lower boundary (12 km) using the time dependent formulation of equation 36 of LKK (see also Equation 40 of present paper). Because of the different staggering of geopotential and temperature used in the present model, an extra working half integer level is added between the bottom boundary (12 km) and the first integer working level (13.5 km). At this half integer level (12.75 km), the vertical motion Z and temperature T are defined.
The physical parameterizations used in the present simulation are identical to those of LKK. Thus, the Rayleigh friction coefficient and diabatic heating rate are given in equations (35) and (36). The sub-grid scale dissipation is given by equations (37) and (38) and uses the same value of $K_H$ as given in equation (38) of LKK. The magnitude of the lower boundary forcing is 350 m and the time constant is $2.5 \times 10^5$ s. $M = 4$, $L = 24$, $\Delta t = 1800$ s and the model is run hemispherically as in LKK.

The horizontally averaged temperature $T$ was set equal to 244°K, which gives a scale height of 7.14 km and a constant mean static stability as in LKK. The initial state wind field was zonal and is depicted in Figure 2. For this simulation, the domain of integration goes from Line B (12 km) to the top of the model (90 km). The initial state temperature field was obtained using equations (3) and (6) from the initial wind field. These initial state fields were also used as the equilibrium fields for the Rayleigh friction and diabatic heating parameterizations. These initial fields were very close to those used in LKK, but not identical.

We can summarize the differences between the present model and LKK as follows: (1) PE versus gQG or LBE, (2) slight differences in the vertical discretization, (3) slight differences in the initial conditions. Otherwise the two models appeared to be virtually identical.

3.1 Inviscid simulation

The first test of the model was to reproduce the inviscid wave-mean flow forced warmings of Matsuno (1971) and SSa. In these simulations the
wave-wave interactions are suppressed and the Rayleigh friction, diabatic cooling and sub grid-scale parameterizations are turned off.

These comparisons were not intended to be exact as there are many differences in detail between the present model and those of Matsuno (1971) and SSa. In particular, the domain, vertical and meridional discretization, lower boundary forcing and the initial state are all different. In addition, both Matsuno (1970) and SSa use a slightly modified quasi-geostrophic set of equations which are designed to be more appropriate for the planetary waves using the scaling arguments of Burger (1958).

We show in Figures 3 and 4 height-time cross-sections of the zonal wind $\bar{u}$ and zonal temperatures $\bar{T}$ at 60°N and 86.6°N respectively for the gQG model when forced at 12 km and run inviscidly with wave-wave interactions suppressed. Figure 3 is for zonal wavenumber 1 forcing and Figure 4 is for zonal wavenumber 2 forcing. Figure 3 can be compared with case $C_1$ of Matsuno (1971) and Figure 2 of SSa. Figure 4 can be compared with case $C_2$ of Matsuno (1971) and Figure 4 of SSa. Figure 3 shows the strong double warming between 15 - 20 days as in Matsuno (1971), which is unrealistic and is caused by spurious reflection off the top of the model. In the wavenumber 2 case (Figure 4), the maximum warming is in excess of 60°, similar to the results of Matsuno (1971) and SSa.

3.2 Damped warmings - wavenumber 1 case

We will now proceed to a more exact comparison of damped stratospheric sudden warming simulations between the present model and that of LKK. All comparisons with Lordi's simulation were with the results
published in LKK, there was no attempt made to rerun the Lordi models. We will also compare our results in a more indirect fashion with those of Holton (1976) and SSa.

We will concentrate first on the damped wavenumber 1 case forced at 12 km. As in LKK, we will perform both wave mean-flow simulations and complete non-linear simulations. There are thus four cases as in LKK. In this section, we will concentrate on case L1 and N1, which are the wave mean-flow and the fully non-linear wavenumber 1 forcing simulations. Both gQG and LBE simulations were performed. We will concentrate first on the gQG simulations.

We present in Figure 5 latitude-height cross-sections of the mean zonal wind $\bar{u}$ for the two cases (N1 and L1) in the same format as Figure 3 of LKK. The stippled areas indicate easterlies and the contour interval is 10 ms$^{-1}$. The time, in days, is indicated in the upper right hand corners. We have presented these snapshots at slightly different times than in LKK because the actual warming event may be slightly advanced or retarded in the present gQG simulation.

Concentrating first on the case L1, we see considerable similarity between the present simulation and that of LKK. In particular, the development of the easterly regime in the lower stratosphere is very similar. Both the present gQG-L1 simulation and that of LKK are markedly different from those of Holton (1976) or SSa. In particular, the primitive equation simulation of Holton (1976) develops a warming much sooner which is probably caused by a lack of dissipation in the zonal flow. SSa, on the other hand, never really achieves an L1 warming at all, probably due to weaker forcing and apparently stronger dissipation. Possibly, the most noticeable difference between the L1 warmings in the
present case and in LKK is the much weaker easterlies in the mid-latitudes in the pre-warming stage (day 22) in the present case.

Turning now to the N1 warming, we again see great similarities between the present gQG case and the LKK simulation. In particular, both the present simulation and LKK show a relative weakening of the westerly stratospheric jet with respect to the L1 case in the pre-warming stage. Also, when the warming does occur, the polar easterlies stretch throughout the stratosphere in both the present and LKK N1 simulations. The strength of the polar easterlies is under-predicted in the present case, however.

In figures 6 and 7 we show time-height cross-sections of the zonal mean wind $\bar{u}$ and temperature $\bar{T}$ at 60°N and 86.6°N respectively for the N1 and L1 cases for the gQG model. N1 is in the top panel and L1 is in the bottom panel. These figures can be compared directly with Figures 3 and 4 of Section 3.1 and with Figures 5 and 7 of LKK.

In Figure 6, the comparative trapping of the polar easterly regime in the lower stratosphere in the L1 case as contrasted with the N1 case shows up clearly as in LKK. LKK noted the similarity between the LKK N1 case and the L1 case of Matsuno (1971) and Holton (1976) and speculated that the wave-wave interaction terms in the N1 simulation may tend to counteract the Rayleigh friction terms in the upper stratosphere. The present results seem to suggest the same conclusion.

Figure 7 again shows considerable similarity with Figure 7 of LKK. In particular the L1 warming takes longer to develop than the N1 warming and does not penetrate above 40 km. Also, the L1 warming, when it does occur, is stronger than the N1 warming.
Figure 8 shows a latitude-height cross-section of zonal wavenumber 1 amplitude and phase for the N1 case at day 17. The amplitude is density weighted as in LKK and the results are directly comparable with Figure 8a of LKK. The westward phase tilt is clearly indicated as in LKK. The results are reasonably similar except that the present case has slightly less amplitude.

Figure 9 is a plot of geopotential wavenumber 1 amplitude at 60°N as a function of height and time. The geopotential amplitude in meters is not density weighted. Figure 9a is for the gQG-L1 case, Fig. 9b is for the gQG-N1 case, and Fig. 9c is for the LBE-N1 case. Concentrating on Figs. 9a and 9b, we see that in the N1 case there is considerably more amplitude in wavenumber 1 than in the L1 cases and the maximum amplitude reaches higher into the stratosphere. This result is consistent with Figs. 6 and 7 and also with LKK. Thus, for the wavenumber 1 case, the non-linear terms appear to counteract the Rayleigh friction, allowing wavenumber 1 to penetrate higher.

Not all aspects of the present gQG simulation are consistent with LKK. The forcing of wavenumber 1 is applied (as in LKK) at 12 km with a ridge at 60°N, 90°W and a trough at 60°N, 90°E. We show in Figure 10 geographical plots of the geopotential deviation at 28.5 km and 40.5 km at day 25 and day 34 for the L1 case for the gQG model. These results can be compared with Figures 15c, 15d, 16c and 16d of LKK. The heights are plotted in meters and the geographical position is indicated by the dotted continental outlines. At day 25 (in the pre-warming stage) at 28.5 km the present results are very similar to those of LKK. Thus the polar low is displaced southward and a high pressure cell develops at
about 170° east. At 40.5 km, at day 25, the westward tilt of the disturbance is quite evident. The results at day 34 (after the warming) can be compared with Figures 15d and 16d of LKK. There is a large difference in phase shift, suggesting that the wave configuration after the warming is very sensitive.

As mentioned earlier, the N1 and L1 cases were also run with the LBE model. The LBE L1 results were not substantially different than the gQG results, except that the wave amplitudes were slightly larger. The N1 results, however, differed substantially. We show in Fig. 11 - top panel a height-time cross section of $\bar{u}$ at 60°N in a format strictly comparable with Fig. 6 for the N1-LBE case. In Fig. 11 - bottom panel is a height-time cross section of $\bar{T}$ at 86.6° for the N1-LBE case in a format strictly comparable with Fig. 7. Comparing Fig. 11 - top with Fig. 6 - top we see that the region of easterlies is more intense and penetrates lower in the stratosphere in the LBE case. Comparison of Fig. 11 - bottom with Fig. 7 - top indicates a more intense warming in the LBE case. Fig. 9-C which shows the height-time cross section of wavenumber 1 amplitude for the N1-LBE case indicates substantially more wave amplitude high in the stratosphere than for the N1-gQG case.

There are two general conclusions to be drawn from this section. Firstly, for the wavenumber 1 case, primitive equation and quasi-geostrophic simulation of stratospheric sudden warmings are not very different. Secondly, consistent with the findings of LKK, the addition of non-linear terms seems to counteract Rayleigh friction, thus allowing more vertical wave propagation and hence a more intense warming. This
result is even more striking in the LBE model. The reason for the enhanced vertical propagation in the LBE model is not clear and will be investigated in subsequent work.

3.3 Damped wavenumber 2 case

We will now turn to the damped cases with wavenumber 2 forced at 12 km. As in LKK these cases will be referred to as case L2 for the wave-mean flow interaction case and case N2 for the fully non-linear case. Both gQG and LBE cases were run, but in both the L2 and N2 case the gQG and LBE results were very similar. Thus in this subsection we show only gQG results.

We present in Figure 12, latitude-height cross-sections of the mean zonal wind $\bar{u}$ for the two cases N2 and L2 in the same format as Figure 4 of LKK. Concentrating first on the case L2, we see considerable resemblance between the present simulation and that of LKK. Thus, the westerlies are decelerated and a vertical critical line moves slowly northward in the lower stratosphere. By day 22, there is a strong easterly flow at mid-latitudes and the polar night jet has been split. The warming occurs at day 27 with strong easterlies through the polar region except in the lower stratosphere. The principal differences between the L2 simulations of LKK and the present appear to be that the mid-latitude easterly maximum is stronger and farther north in the LKK simulation. Comparisons with the L2 cases of Holton (1976) and SSa also indicate considerable similarity, although there were differences in timing, probably caused by the same differences in dissipation discussed in Section 3.2.

LKK found that the N2 and L2 simulations were quite similar, much more so than the N1 and L1 simulations. One difference noted by LKK,
however, was that the L2 warming appeared to be more intense and proceeded more rapidly than the N2 warming. LKK attributed this difference to wave-wave interaction which removed energy from wavenumber 2 and transferred it into wavenumber 4.

The results in the present case are somewhat different from those of LKK. The N2 simulation is quite different from the L2 simulation. The mid-latitude easterly flow in the lower stratosphere is much weaker and further south in the N2 case and the warming is delayed until day 33. When the polar wind reversal does occur, it is weaker than in the case L2 and is mostly confined to the lower stratosphere.

Figures 13 and 14 show time-height cross-sections of the zonal wind $\bar{u}$ at 60°N and temperature $\bar{T}$ at 86.6°N respectively for the N2 and L2 cases. N2 is in the top panel and L2 in the bottom panel. These figures can be compared directly with Figures 3 and 4 of Section 3.1 and with Figures 5 and 7 of LKK.

The L2 case in both $\bar{u}$ and $\bar{T}$ is similar to the corresponding inviscid case discussed in Section 3.1. That is, the wind reversal seems to occur throughout the stratosphere in the present model. Figure 5d of LKK, by contrast, would seem to indicate that the wind reversal is confined to the lower stratosphere. Figure 5d of LKK is slightly misleading in this respect, as can be seen by examining Fig. 4 of LKK which indicates that the easterly flow region of day 29 tilts northward with height, but does extend throughout the stratosphere. In the L2 simulation of Holton (1976), the wind reversal extends throughout the stratosphere, but in SSa it is confined to the lower stratosphere. The temperature $\bar{T}$ evolution
for the L1 simulation shown in Figure 13 indicates the same 40° warming in the lower stratosphere and cooling at 60 km as in the LKK L1 simulation.

LKK speculated that the N2 warming appeared to be weaker than the L2 warming because some of the zonal wavenumber 2 energy was being leaked to wavenumber 4 in the N2 case. In the spirit of this conjecture, we speculate that the N2 warming is so weak and delayed in the present case because even more energy is being leaked to wavenumber 4 in the gQG model. To give credence to this speculation, we have plotted in Figure 15 time-height cross-sections of geopotential amplitude (not density weighted) at 60°N for zonal wave 2 for the L2 and N2 cases. We see that indeed case N2 propagates less energy vertically in wavenumber 2 than in the case L2. Plots of geopotential amplitude for zonal wavenumber 4 (not shown) indicate that a substantial amount of energy has been transferred from wavenumber 2 to wavenumber 4.

The lower boundary forcing for the L2 and N2 cases has ridges at 90°W and 90°E and troughs at 0° and 180°, as in LKK. The geographical response of the geopotential for the case N2 at 28.5 km and 40.5 km is shown in Figure 16, which can be compared with Figure 17 of LKK. The results, as in Figure 10, are very similar to those of LKK in the pre-warming stage (day 22). The results after the warming are also fairly similar, although there is more evidence of wavenumber 4 in the subtropics at 40.5 km in the LKK N2 simulation (Figure 17d of LKK).

The results from this section suggest that for the wave-zonal flow case the gQG and LBE models and PE model give similar results for wavenumber 2 simulations forced from the lower stratosphere. The results
with a fully non-linear model, however, suggest some differences between the PE and gQG formulations. In both the PE and gQG simulations the wave-wave interactions tend to remove energy from wavenumber 2 and thus inhibit the warmings. This effect, however, seems to be somewhat stronger in the gQG models.
4. Topographic forcing

One of the shortcomings of mechanistic sudden warming simulations such as those of Matsuno (1971), SSa and LKK has been the artificial nature of the forcing. Clearly, specifying the geopotential field at 12 km is not very realistic.

Two recent studies have attempted to remove this artificiality by specifying topographic forcing at the bottom of troposphere. In this way, the forcing mechanism becomes more physical and the troposphere enters into the simulation. The principal objection to this type of experiment is the requirement to "turn on" the forcing slowly enough so as to minimize the excitation of transients. The resulting time-dependent topography field is clearly non-physical. In fact, one suspects that if the topography does play a role in sudden warmings, then it is presumably the basic state flow which varies in time while the topography remains fixed, rather than the reverse.

Schoeberl and Strobel (1980b) - hereinafter referred to as SSb - used the basic quasi-geostrophic model described in SSa with a troposphere and time-dependent topographic forcing. As in SSa, only wave-mean flow interactions were permitted. The topographic forcing was of the type $W_m = m \bar{\omega} h_m$, where $m$ indicates the zonal wavenumber, $h_m$ is the topographic height and $\bar{\omega}$ is the angular velocity of the zonal flow. Real topography was used and $\bar{\omega}$ was assumed to be the 500 mb flow. The mountains were gradually switched on during the first 15 days of the integration.
Koermer (1981) - subsequently referred to as K - used a primitive equation model with a hybrid vertical coordinate system. Basically, the model consisted of the stratospheric log pressure model used in LKK joined together at 12 km with the tropospheric sigma coordinate model of Bourke (1974). The topographic forcing was applied in a straightforward manner using the exact lower boundary condition of the sigma coordinate tropospheric model. The topography field had a maximum height of 600 m at 45°N over Asia and was switched on slowly during the first 7 days of the integration.

Stratospheric sudden warmings were produced in both simulations. There were many differences in detail because the experimental conditions were very different in the two experiments. One difference, though, seemed very striking. The simulation of SSb seemed to require far more topographic forcing to produce a warming than does the simulation of K. This difference could be due to the use of a QG rather than a PE model, or it could be due to other factors.

SSa discussed the forcing in the lower stratosphere required to produce a sudden warming. Their results showed that there must be at least 200-300 m amplitude in the geopotential field in the lower stratosphere (≈ 10 km) for a warming to occur. It is quite clear that in both the simulations of SSb and K, the geopotential amplitude in the lower stratosphere exceeds the threshold value, even though the topographic forcing is apparently quite different in the two cases.

To examine this question, an experimental comparison was performed between the present QG model and a primitive equation model. The idea
was to run both models with equivalent topographic forcing to determine the amplitude of the geopotential response in the lower stratosphere. The domain of both models was the troposphere and lower stratosphere and there was no attempt made to produce a sudden warming.

The primitive equation model used was the fully non-linear sigma coordinate spectral global model described in Otto-Bliesner et al. (1982), but with the physical parameterizations modified to be consistent with the gQG model as noted below. The dynamical framework of the model was identical to that of Bourke (1974), except that a triangular, rather than a rhomboidal spherical harmonic truncation was used. The model had 14 vertical levels \( \sigma = P/P_s = .95, .85, .75, .65, .55, .45, .35, .25, .15, .125, .1, .075, .050, .025 \). The upper and lower boundary conditions were \( \sigma = 0 \). The horizontal resolution was triangular 15. The surface drag formulation was as in equation (34), and there was no Rayleigh friction. The horizontal dissipation parameter \( K_H \) (see equations 37 and 38) was \( 2.5 \times 10^4 \text{ m}^2 \text{ s}^{-1} \) and there was no convective adjustment. The Newtonian cooling coefficient \( \beta(Z) \) was the following:

\[
\beta(Z) = [1.5 + \tanh \left( \frac{Z-18}{3} \right)] \times 10^{-6} \text{ s}^{-1}. \tag{39}
\]

This formulation has the effect of supplying extra damping above 18 km in order to suppress spurious reflections off the upper boundary of the model. The model was run hemispherically and dry. The earth's topography was analyzed to triangular 15 and turned on slowly using the same time dependency as LKK, viz.

\[
\phi_s(t) = \phi_s [1 - \exp(-t/to)], \text{ to} = 2.5 \times 10^5 \text{ s}, \tag{40}
\]
where $\phi_s$ is the earth's topography. The topography ($\phi_s$) can be seen in Figure 17.

The present non-linear gQG model was modified slightly to duplicate the PE model as closely as possible. Thus, the horizontal diffusion, surface drag, Rayleigh friction, Newtonian cooling and topography were the same. The horizontal resolution was triangular 15 and the vertical levels were the same except that the level at $\sigma = .95$ was replaced by one at $P = 1000$ mb or $Z = 0$. Unlike the sigma surface model, the log-pressure model has a formal top at $P = 10$ mb or $Z = 36$ km. The topographic forcing was applied by setting

$$g W_s = \frac{V_s}{\phi_s} \cdot \nabla \phi_s,$$

(41)

where $V_s$ is the velocity field at the bottom of the model.

The initial zonal wind field $\hat{u}$ for both integrations are shown in Figure 2. In this case, the domain of integration goes from the ground to line A (35 km).

Both models were integrated for 40 days. We were concerned both with the transient response and the stationary (or at least the 40 day time-averaged) response to the topographic forcing. We first examined the transient response by comparing the integrations at 6 days, which is the time when the topography essentially stops rising.

4.1 Transient response

The output from the sigma surface primitive equation model was interpolated back to pressure coordinates for comparison with the gQG model. We show in Figure 18, latitude-longitude plots of the geopotential
fields (m) at day 6 at 200 mb (= 13 km) in the two models. The primitive
equation sigma coordinate model is Figure 18a and the gQG model is Figure
18b. The level 200 mb has been chosen because it is approximately the
level where the lower boundary geopotential forcing is applied in the
strictly stratospheric models of Section 3. Also, this level is below
the level where the heavy damping is applied (equation 39). As noted
earlier, an amplitude of 200-300 m at this level seems to be necessary if
a warming is to occur. Figure 18a and 18b show quite clearly that the
topographic forcing in the gQG model at 6 days is much weaker than in the
PE models.

Now, in addition to the PE-gQG differences between the models, there
is also the difference between pressure or log-pressure coordinates in
the gQG model and terrain following $\sigma$ coordinates in the PE model. In
order to investigate this difference we modified the sigma coordinate PE
model slightly to simulate a pressure coordinate PE model. This was done
as follows. Normally, in a sigma coordinate model, $\sigma = 0$ at the terrain
level and the topography $\Phi_s$ enters as the lower boundary condition for
the integration of the hydrostatic equation. To simulate a pressure co-
ordinate PE model, $\Phi_s$ was set equal to zero in the hydrostatic equa-
tion and $\sigma$ at the ground was modified as follows:

$$\dot{\sigma} = \frac{d}{dt} \left( \frac{P}{P} \right) = \frac{\omega}{P} - \frac{P}{s} \frac{P}{p^2},$$

Thus,
\[
\dot{\sigma}_s = \frac{\omega_s}{P_s} - \frac{\sigma_s \dot{P}_s}{P_s}, \tag{42}
\]

where \( \dot{\sigma}_s \) is the vertical velocity in sigma coordinates at the ground.

Assuming

\[
\omega = -\frac{P g W}{R T}, \]

we have

\[
\dot{\sigma}_s = -\frac{g W_s}{R T_s} - \frac{\dot{P}_s}{P_s}, \tag{43}
\]

where \( W_s \) is to be specified as in equation (41). Then assuming the balance between \( \dot{P}_s/P_s \) and the non-mountain portion of \( g W_s/RT_s \) is not affected by the mountain, we have

\[
\dot{\sigma}_s = -\frac{V_s \cdot \nabla \phi_s}{R T_s}. \tag{44}
\]

This modification to the sigma coordinate equations will affect both the vertical advection terms and the three-dimensional divergence.

We show in Figure 18c, the 6-day 200 mb geopotential field simulation from this "pressure coordinate" PE model in the same format as Figures 18a and 18b. The resemblance between Figures 18b and 18c is obvious, but the discrepancy with sigma coordinate PE model (Figure 18a) remains.
The mystery is solved when we recall that the topography field is rising with time. The rising topography in the sigma coordinate PE model will generate upward vertical motions at the ground, independently of the model initial conditions. This effect can be simulated by the gQG model and the "pressure coordinate" PE model simply by modifying equation (41). Thus, including the local time derivative of the topography in equation (41) gives

\[ g \frac{\partial \phi_s}{\partial t} + \nabla_s \cdot \nabla \phi_s = g W_s, \tag{45} \]

A similar modification is made in equation (44). The term \( \frac{\partial \phi_s}{\partial t} \) can be calculated analytically from equation (40).

With these modifications the QG model and the "pressure coordinate" PE model are again integrated to produce day 6 200 mb geopotential fields in the format of Figure 18. The results are shown in Figure 19a - the QG model, and Figure 19b - the "pressure coordinate" PE model. The correspondence with Figure 18a is obvious.

Clearly, then, in order to get the same transient response between the sigma surface PE model and the log pressure coordinate gQG model in the presence of time-dependent topography, we must include the \( \dot{\phi}_s \) term in the gQG model. The remaining experiments in this section included this term.

4.2 Stationary response

We were also interested in the stationary (or at least time-averaged) response of the PE and gQG models. We show in Figure 20a, a latitude-longitude plot of the 40-day average of the 200 mb geopotential
field from the sigma coordinate PE model. Comparisons were also performed with the gQG model. One rather arbitrary aspect of the topographic forcing in the gQG model is $V_s$ in equation (41). It is not quite clear at what level $V_s$ should be applied. SSb, for example, take $V_s$ to be the windfield at 500 mb. SSb justify this choice because the average height of the Himalayan Plateau is 5 - 6 km or about 500 mb. On the other hand, applying $V_s$ at $P = 1000$ mb or $Z = 0$ is not appropriate because much of this level is below ground.

We performed two 40-day integrations with the gQG model using two different values of $V_s$. In Figure 20b is shown the 40-day time-averaged geopotential field in the same format as Figure 20a for the case where $V_s$ is assumed to be the wind at 550 mb. In Figure 20c, we show the same result for the case where $V_s$ is assumed to be the wind at 1000 mb. We note considerable differences in detail between Figure 20a, b and c, but the major features are apparent in all three. By and large, $V_s = 1000$ mb wind underestimates the amplitude of the stationary component, and $V_s = 550$ mb wind overestimates it. Consequently, $V_s = 700$ mb wind seems a reasonable compromise. This level, 700 mb or 3 km, represents approximately the maximum height of the Himalayas in this model (see Figure (17)).

According to the criteria of SSa, the approximately 120 m stationary component in the sigma coordinate PE model at 200 mb (Figure 20a) would be inadequate to initiate a stratospheric sudden warming. We note, however, that there are also transients present and the total amplitude would probably be sufficient.
There remains some discrepancy with the simulations of K. The topography field in K was defined by

$$\phi_s(\lambda, \phi) = g[A_0 + A_1 \sin \lambda + A_2 \sin (2\lambda - \frac{\pi}{2})] \sin^2 \phi,$$

(46)

where $A_0 = A_2 = 240$ m and $A_1 = 150$ m. This results in a maximum height of the mountain field of 600 m over the Tibetan plateau (as opposed to more than 3000 m using the real topography in Figure 17). This topography (equation 46) was tested in both our sigma coordinate PE model and our gQG model. The transient and stationary (time-averaged) response in both of the models was considerably smaller than when using real topography. The correspondences between the two models, as noted earlier, were maintained. The present results suggest that the topographic forcing used by K would be too weak to initiate a warming. K clearly did simulate a sudden warming, however, and we are unable to explain the discrepancy.

4.3 Baroclinic instability

One rather large discrepancy between the sigma coordinate PE model and the gQG model was noticed. That is, the strength of the tropospheric jet (see initial $u$ field in Figure 2) tended to be maintained in the PE model, but was much weakened in the gQG simulation.

In both the PE and gQG simulations, the flow becomes baroclinically unstable after 8 - 12 days of integration and there is a rapid growth of small-scale disturbances after this time. Simmons and Hoskins (1976) in their study of baroclinic instability in PE and gQG models noted that the lack of vertical eddy heat flux terms in the gQG model affected the time
rate of change of the zonally averaged winds and temperatures. In the gQG model there was more warming poleward of the disturbance than in the PE model, resulting in greater deceleration of the tropospheric westerly flow. Lorenz (1960) has also pointed out that the neglect of the vertical eddy heat flux terms can result in excessive baroclinic conversion of potential to kinetic energy.

A simulation was performed using the LBE model which includes the vertical eddy heat flux terms. The results (not shown) indicated that the tropospheric westerly jet was much better maintained in the LBE model than the gQG model.
5. A simulation of a topographically forced stratospheric sudden warming

In this section we will briefly describe a simulation, using the present gQG model, of a topographically forced stratospheric sudden warming. In the design of the experiment, we will attempt to make use of the knowledge that has been gained in the previous sections.

The model was basically as described in Section 2 and included both wave-mean flow and wave-wave interactions. There were 30 levels, with a vertical spacing of 1.5 km below 12 km and a spacing of 3.5 km from 12 km to 85.5 km. The model was run hemispherically with $M = 4$, $L = 20$, $\Delta t = 1800$ seconds.

Real topography was used, analyzed spectrally to the resolution of the model. $v_s$ in equation (41) was prescribed to be the wind at 3 km. The horizontally-averaged temperature $T$ was set equal to 244° above 12 km, and standard atmosphere values were used below.

The physical parameterization used included the Rayleigh friction given in equation (35), the Newtonian cooling given in equation (36) and the surface drag given in equations (33 and 34). The Fickian diffusion formulation in equations (37) and (38) with $K_H = 2.5 \times 10^4$ m$^2$ s$^{-1}$ was included.

The initial state was defined by the zonal windfield $\overline{u}$ shown in Figure 2. The initial state temperature was obtained using equations (3) and (6) from the initial windfields. These initial fields were also used as the equilibrium fields for the Rayleigh friction and diabatic heating parameterizations.
The diabatic heating formulation of equation (36), with a time constant of \( \approx 20 \) days, is appropriate for the stratosphere where it is primarily radiation effects which are being parameterized. In the troposphere, however, convective and planetary boundary layer effects must also be considered. Consequently, we have chosen the following formulation for \( \beta(Z) \) below 12 km:

\[
\beta(Z) = [1.5 + \tanh \left( \frac{10-Z}{2} \right)] \times 10^{-6} \text{ s}^{-1}. \tag{47}
\]

This gives a diabatic time constant in the troposphere of about 5 days. Considering also the results of Section 4.3, this extra diabatic parameterization in the troposphere has the effect of maintaining the tropospheric jet against the spurious high latitude warming (in the gQG model) caused by baroclinicity.

Two 60-day integrations were performed. The first case T1 included the \( \dot{\psi}_g \) term in equation (45), while the second case T2 did not include the \( \dot{\psi}_g \) term.

We first show in Figure 21, longitude-height cross-sections of geopotential (not density weighted) at 45°N at day 5 to illustrate the different transient behavior in the two cases. The top panel is case T1, while the bottom panel is case T2. The difference is striking.

Figure 22 shows height-time plots of the zonal wind \( \vec{u} \) at 75°N in the two cases. The top panel is case T1, while the bottom panel is case T2. We see that there is a wind reversal at about 40 days in both cases, with an apparent minor event at about 30 days. The wind reversal is stronger...
in the case T1, but by and large the inclusion of the \( \hat{s} \) term does not greatly affect the simulation.

Figure 23 shows a height-time cross-section of the zonal temperature \( \bar{T} \) at 90° in the two cases. As in Figure 22, the case T1 is in the top panel. Case T1, in particular, is very reminiscent of the N1 warming shown in Figure 7-top. The warming is relatively weak.

Figure 24 is a height-latitude cross-section of \( \bar{u} \) for the T1 case at 45 days, just after the warming has taken place. Again, this topographically-induced warming is very reminiscent of the N1 case shown in Figure 5.

We show, in Figure 25, height-time cross-sections of wavenumber 1 geopotential amplitude at 60°N (not density-weighted) for the two cases T1 and T2. (T1 is in the top panel.) Figure 26 shows the same plot except for zonal wavenumber 2. The units are meters and the contour interval is 200 m. The transient activity in the T1 case shows up clearly in the top panel of Figure 25. There is some evidence that the amplitude peaks of wavenumbers 1 and 2 are out of phase (particularly in the case T2). The amplitude of wavenumber 1 is much greater than that of wavenumber 2, confirming that this is basically an N1 warming.

The topographically-induced stratospheric sudden warmings of K and SSb were both basically wavenumber 2 events, while the present simulation was dominated by wavenumber 1. There are so many differences in detail between the three experiments as to make comparison virtually impossible. It might be noted, however, that the results of Section 3.3 suggest that it might be difficult to get a wavenumber 2 warming in a quasi-geostrophic model which includes wave-wave interactions.
6. Summary and Conclusions

A fully non-linear model in log pressure coordinates has been constructed for the purpose of simulating stratospheric sudden warmings. Two versions of the model were built - the linear balance equation (LBE) version included the complete thermodynamic equation, the global quasi-geostrophic (gQG) model used a horizontally-averaged static stability. Most of the integrations were performed with the simpler (gQG) model.

The model was global and had a vertical domain which stretched from the ground to the lower mesosphere. The model used a spherical harmonic representation in the horizontal and a finite difference discretization in the vertical. All non-linear terms were calculated using transform techniques and the block tri-diagonal matrices that appeared in the model \( \omega \) equation were solved by an eigenvalue decomposition technique. Simple parameterizations of dissipation, diabatic heating, planetary boundary layer and topographic forcing were included. The model was extremely efficient.

Three sets of experiments were conducted with the model. The first set attempted to compare gQG and LBE with primitive equation (PE) mechanistic simulations of stratospheric sudden warmings which were forced from the lower stratosphere. The second set of experiments attempted to compare the response of gQG and PE models to topographic forcing at the ground. The third set of experiments examined the simulation of a stratospheric sudden warming produced by a gQG model topographically forced at the ground. We shall discuss the results of each of these three experiments in turn.
In Section 3 we described an intercomparison between the present gQG models and the PE model of Lordi et al. (1980) for the simulation of stratospheric sudden warmings forced at 12 km. The experimental conditions (horizontal and vertical discretization, resolution, initial conditions, boundary conditions, physical parameterization, forcing) were made as identical as possible, the most serious discrepancy being in the vertical discretization. There were four cases run: wavenumbers 1 and 2 fully non-linear (cases N1 and N2) and wave-mean flow (cases L1 and L2).

The PE and gQG results were found to be very similar for the L1, L2 and N1 cases; somewhat different for the N2 case.

The similarity of the L2 case was not unexpected, but the closeness of the PE and gQG L1 simulations was surprising because of previous speculation in the literature that the gQG simulation of the L1 case might be deficient.

In both the PE and gQG N1 simulations, the wind reversal stretched throughout the stratosphere rather than being trapped in the lower stratosphere as in the L1 case. Lordi et al. (1980) speculated that the wave-wave interactions in the N1 case tended to counteract the Rayleigh friction terms in the upper stratosphere, a conclusion which seemed to be supported by the present results. The results of the LBE-N1 simulation were even closer to the PE-N1 simulation of Lordi et al. (1980).

The gQG N2 simulations were both late and weak compared to the PE N2 simulation or either of the L2 simulations. Lordi et al. (1980) noticed that the PE N2 warming was a little weaker than the PE L2 warming and speculated that this effect might be due to the leakage of wavenumber 2.
energy to wavenumber 4 by wave-wave interaction. The present results tend to support this conclusion.

In Section 4 we described an intercomparison between PE and the gQG models for the purpose of determining the response to topographic forcing at the ground. The two models were essentially tropospheric and, as in Section 3, were made as compatible as possible.

Two recent studies of stratospheric sudden warmings have specified topographic forcing at the ground rather than geopotential forcing at the tropopause. Schoeberl and Strobel (1980b) used a quasi-linear quasi-geostrophic model while Koermer (1980) used a hybrid sigma-log pressure coordinate non-linear PE model. Both experiments produced sudden warmings, but there seemed to be a large discrepancy in the strength of the topographic forcing required to produce the warming. Schoeberl and Strobel (1980a) demonstrated that there must be a geopotential amplitude of 200–300 m in the lower stratosphere (≈12 km) for a warming to take place. Thus the experiment in Section 4 attempted to determine the response at 12 km of compatible PE and QG non-linear models topographically forced at the ground. In both models the topography was raised slowly in time from zero to its full height. Both the transient and the stationary (or time-averaged) response were examined.

It was found that the log pressure coordinate gQG model and the sigma coordinate PE model had the same transient response to topographic forcing, provided that the gQG model properly took account of the time-dependence of the topography. The stationary responses of the gQG and PE models were similar in magnitude, but differed in phase. The
stationary response over the Himalayas was similar in the two models, but differed somewhat over Greenland and the Rockies. Even with the use of a realistic topography field in the PE and gQG models, it was thought that the response at 12 km in both models was barely adequate to initiate a sudden warming according to the criteria of Schoeberl and Strobel (1980a). In this respect the present experiment seems to be much more in accord with Schoeberl and Strobel (1980b) than with Koermer (1980), where a warming was initiated by apparently much weaker topographic forcing.

In Section 5 an integration was performed with the complete tropospheric-hemispheric version of the gQG model, topographically forced at the ground. A stratospheric sudden warming occurred at day 45 of the integration and appeared to be primarily of wavenumber 1 (N1) nature. This result was somewhat at variance with both the simulations of Koermer (1980) and Schoeberl and Strobel (1980b) which produced warmings of primarily wavenumber 2 character. The present results are, however, consistent with Section 3 which suggest that it might be difficult to get an N2 warming in the non-linear gQG model.
The authors would like to acknowledge helpful discussions with A. Bridger, R. Dickinson, A. Kasahara, J. Koermer, and D. Stevens. The manuscript was typed by Mary Niemczewski and the graphics prepared by Stephanie Honaski and the NCAR Graphics Department. Partial support for this research has been provided through the National Oceanic and Atmospheric Administration under P.O. No. NA82AAG00927.
Figure 1  Vertical grid of the quasi-geostrophic model.
Figure 2 The initial state zonal wind fields $\bar{u}$ (ms$^{-1}$) as a function of height and latitude. Line A indicates the top of the tropospheric experiment (Section 4) while line B indicates the bottom of the stratospheric experiment (Section 3).
Figure 3  Height-time cross-sections of zonal wind field $\bar{u}$ (ms$^{-1}$) at 60°N (left panel) and temperature $\bar{T}$ (°K) at 86.6°N (right panel) for the inviscid wavenumber 1 case.
Figure 4 as Figure 3 except for the inviscid wavenumber 2 case.
Figure 5  Latitude-height cross-sections of the zonal wind $\bar{u}$ (ms$^{-1}$) for the case N1 and L1 (gQG model). The time in days is indicated in the upper right-hand corner. The shaded areas indicate easterlies.
Figure 6  Height-time cross-sections of $\bar{u}$ (ms$^{-1}$) at 60°N in the same format as Figure 3 for the N1 (top panel) and L1 (bottom panel) cases (gQG model).
Figure 7  Height-time cross-sections of temperature $\overline{T}$ at 86.6° in the same format as Figure 3 for the N1 (top panel) and L1 (bottom panel) cases.
Figure 8  Latitude-height cross-sections of zonal wavenumber 1 amplitude (solid lines) and phase (dashed lines) for the case N1 at day 17. The amplitudes are density weighted.
Figure 9  Height-time cross-sections of geopotential amplitude (not density weighted) in meters at 60°N for zonal wave 1.  (a) L1-gQG, (b) N1-gQG, (c) N1-LBE.
Figure 10 Geopotential deviation (m) at 28.5 km and 40.5 km for the case LI at days 25 and 32.
Figure 11 Height-time cross-sections of zonal wind $\bar{u}$ (ms$^{-1}$) at 60°N (upper panel) and temperature $\bar{T}$ (°k) at 86.6°N (lower panel) for the case N1 with the LBE model.
Figure 12 Latitude-height cross-sections of $\bar{u}$ for the cases N2 and L2 in the same format as Fig. 5.
Figure 13 Height-time cross-sections of $\bar{u}$ at 60°N for the case N2 (top panel) and L2 (bottom panel) in the same format as Fig. 6.
Figure 14 Height-time cross-sections of $\tilde{T}$ at 86.6°N for the case N2 and L2 in the same format as Fig. 7.
Figure 15 Height-time cross-sections of geopotential (not density weighted) in meters at 60°N for zonal waves 2 in the N2 and L2 cases.
Figure 16  Geopotential deviation (m) at 28.5 km and 40.5 km for the case N2 at days 22 and 33.
Figure 17 The earth's topography (m) truncated to triangular 15 used for the topographic forcing experiments of Section 4 (zero contour removed).
Figure 18  Latitude-longitude plots of the geopotential (m) at Day 6 at 200 mb for (a) PE sigma coordinate model, (b) gQG model, (c) PE pressure coordinate model.
Figure 19 Latitude-longitude plots of 200 mb geopotential at Day 6 in the same format as Fig. 16 for (a) gQG model with $\Phi_S$ term, (b) PE pressure coordinate model with $\Phi_S$ term.
Figure 20 Latitude-longitude plots of 40 day time average geopotential (m) at 200 mb for (a) PE model, (b) gQG model with $\vec{V}_s$ applied at 550 mb, (c) QG model with $\vec{V}_s$ applied at 1000 mb.
Figure 21 Longitude-height cross-sections of geopotential (m) at 45°N at day 5. Case T1 - top panel, Case T2 - bottom panel.
Figure 22 Height-time plot of $\bar{u}$ (ms$^{-1}$) at 75°N in cases T1 (top panel) and T2 (bottom panel) in the same format as Fig. 6.
Figure 23 Height-time plot of $T\, (\text{°K})$ at 90°N in cases T1 (top panel) and T2 (bottom panel) in the same format as Fig. 7.
Figure 24 Height-latitude plot of $\bar{u}$ (ms$^{-1}$) for the case T1 at day 45 in the same format as Fig. 2.
Figure 25  Height-time cross-section of zonal wavenumber 1 geopotential (m) amplitude (not density weighted) at 60°N. Case T1 - top panel, case T2 - bottom panel.
Figure 26 Same as Fig. 25 except for zonal wavenumber 2.
REFERENCES


