The First Moments of the Variance- and Cross-Spectra of Standard and Interferometric Clear-Air-Doppler-Radar Signals

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BOULDER, COLORADO
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Preface

This Technical Note was prepared during the author's year-long research stay at NCAR's Atmospheric Technology Division (ATD). It is a translated, revised and slightly extended version of the author's habilitation thesis "Die ersten Momente der Varianz- und Kreuzspektren der Signale von konventionellen und interferometrischen Clear-air-Doppler-Radars" (in german), which had been also accomplished during the stay at NCAR.

This work was motivated by NCAR's and the author's interest to understand in some more detail effects of inhomogeneity and anisotropy of refractive-index irregularities on the signals measured with NCAR's MAPR radar system and other standard and interferometric atmospheric Doppler radars.
Acknowledgements

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The MAPR radar system was initially designed jointly by NCAR and the Radian Corporation. Funding from both NCAR and grants to NCAR from the Department of Energy's ARM (Atmospheric Radiation Measurement) Program have been used in the subsequent testing and development.

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1 Introduction

1.1 Richardson’s dream

To be able to predict the weather is an old dream of mankind. This is known from countless “farmer rules”, which often considerably differ from each other with respect to descent and reliability.

About two decades prior to the dawn of the era of electronic computing-machines, RICHARDSON (1922) had worked out the physical and numerical foundations of numerical weather forecasting. He wrote (RICHARDSON 1922, S. xi):

"The scheme is complicated because the atmosphere is complicated. But it has been reduced to a set of computing forms. These are ready to assist anyone who wishes to make partial experimental forecasts such incomplete observational data as are now available. In such a way it is thought that our knowledge of meteorology might be tested and widened, and concurrently the set of forms might be revised and simplified. Perhaps some day in the dim future it will be possible to advance the computations faster than the weather advances and at a cost less than the saving to mankind due to the information gained. But that is a dream."

Nowadays, three quarters of a century later, Richardson’s dream is reality. In the big weather forecasting centers, one relies on numerically generated weather predictions. ROEBBER & BOSART (1996) assess the task of a present-day forecaster as follows: “The skill advantage of human forecasters over numerical guidance continues to diminish and now largely reflects the human ability to recognize occasional departures from the linear relationship between forecast information and future observations.” RICHARDSON (1992, S. xii) had considered his work as a paradigm contrary to the theory of fronts (Frontenlehre) of the Norwegian School led by V. Bjerknes (BERGERON 1928). Now, it appears as if things have turned out in favour of Richardson’s.

The more the development of powerful computers advances, the finer becomes the spatial and temporal resolution of the numerical models, and correspondingly, the requirements with respect to the spatial and temporal resolution of the data sets increase, with which the numerical models have to be initialized (e.g., STEPFELER 1997, FOKEN et al. 1997). In the foreseeable future, the network of radiosonde stations, which still provides the major portion of the routine data for weather-forecasting, will be insufficient to fulfill those requirements. There is hope, however, that a network of remote sensing facilities will first complement and later completely replace the radiosonde network.
1.2 Doppler radars for remote sensing of the optically clear atmosphere — overview and state of the art

1.2.1 UHF/VHF radars

During the last two decades, there has been an enormous distribution of phase-coherently pulsed Doppler radars, and they have proved to be versatile instruments to probe the atmosphere (see, e.g., RADIO SCIENCE 1985; METEOROL. RDSCH. 1990; RADIO SCIENCE 1990; ATLAS 1990; EDWARDS 1994; RADIO SCIENCE 1995; EDWARDS 1996; WILCZAK et al. 1996; RADIO SCIENCE 1997). Theoretical and technical foundations of the Doppler radar technique are described in the standard text by DOVIAK & ZRNIĆ (1993). In the United States, an operational network of “wind-profilers” has already been realized (STRAUCH et al. 1984), the data of which have been operationally used as input for numerical weather forecasting. In the efforts COST 74 (METEOROL. RDSCH. 1990) and COST 76 (RICHNER et al. 1997), the operational utilization of wind-profiler data has been prepared also in Europe.

At the AL (Aeronomy Laboratory) in Boulder, Colorado, a 915-MHz profiler has been developed, and meanwhile, more than 50 systems of this kind have been brought into operation (CARTER et al. 1995). They have been used in many measurement campaigns (see, e.g., GAGE et al. 1994), among other things for the measurement of turbulent fluxes in the atmospheric boundary layer (e.g., ANGEVINE et al. 1994). 915-MHz profilers are also components of the ISSs (“Integrated Sounding System”), which have been developed by AL und NCAR (National Center for Atmospheric Research, Boulder, Colorado). In the international research programme TOGA COARE (TOGA: “Tropical Ocean and Global Atmosphere”; COARE: “Coupled Ocean-Atmosphere Response Experiment”), six of those ISSs were operated in the western Pacific, four of them on land and two of them on board of ships (PARSONS et al. 1994). In between, a modified version of such a 915-MHz profiler with spaced-antenna capability has been brought into operation at NCAR (COHN et al. 1994, COHN et al. 1997).

In Germany, UHF/VHF radars, in part with interferometry capability, are utilized at the Meteorologisches Institut and at the Max-Planck-Institut für Meteorologie in Hamburg, at the Meteorologisches Observatorium des Deutschen Wetterdienstes in Lindenberg, at the Institut für Meteorologie und Klimaforschung in Karlsruhe, at the Institut für Atmosphärenphysik in Kühlungsborn and at the Max-Planck-Institut für Aeronomie in Katlenburg-Lindau (e.g., HIRSCH 1994; PETERS et al. 1994; COHN & CHILSON 1995; HIRSCH 1996; CHILSON & SCHMIDT 1996; STEINHAGEN et al. 1997).

1.2.2 UHF/VHF scatter- and reflection mechanisms and problems of the interpretation of conventional Doppler spectra

A considerable portion of the pioneering work to understand the propagation of electromagnetic waves in turbulent media was done by TATARSKII (1961; 1971). See also GAGE & BALSLEY (1980), GAGE (1990) and DOVIAK & ZRNIĆ (1993, pp. 424ff.). DOVIAK & ZRNIĆ (1984) put forward a generalized theoretical formulation of the UHF/VHF backscattering from refractive-index irregularities in the optically clear atmosphere.
On the basis of measured UHF/VHF clear-air Doppler spectra it is possible to get insight into the spatial and temporal distribution of refractive-index irregularities in the air and, therefore, also into the spatial and temporal distribution of wind, waves, turbulence, humidity and temperature (siehe z.B. KLOSTERMEYER et al. 1988; GAGE 1990).

The classical and simultaneously simplest interpretation of the variance spectrum $S(\omega)$ of a clear-air Doppler signal is as follows (see, e.g., WOODMAN & GUILLÉN 1974, S. 496). The integral over $S(\omega)$ is a measure of the variance of the refractive index within the observation volume. The Doppler shift,

$$\omega_D = \frac{\int_{-\infty}^{\infty} S(\omega) \omega d\omega}{\int_{-\infty}^{\infty} S(\omega) d\omega}, \quad (1.1)$$

provides the “Doppler velocity”

$$v_D = -\frac{\omega_D}{2k} = -\frac{\omega_D \lambda}{4\pi}, \quad (1.2)$$

which is typically interpreted as the projection of the wind vector, spatially averaged over the observation volume, in the direction of the axis of the radar beam. (Here, $k$ and $\lambda$ mean radar wavenumber and radar wavelength, respectively.) Eventually, the second central moment of the Doppler spectrum is typically interpreted as a measure of the variance of the radial velocity within the observation volume as well as of the reciprocal of the life-time of the turbulent eddies at length scales close to the half radar wavelength.

For several reasons, however, the classical interpretation of the first three moments (i.e., the “zeroth”, the first, and the second moment) of the Doppler spectrum is valid only approximately.

Anisotropy of refractive-index irregularities because of the suppression of vertical motion in the case of stable stratification can lead to a pronounced aspect sensitivity of VHF echo-intensities in near-zenith beam-pointing directions (RÖTTGER & LIU 1978; GAGE & GREEN 1978; DOVIK & ZRNIĆ 1984) and to a significant difference between the Doppler velocity and the radial wind velocity averaged over the observation volume. Without taking into account aspect-sensitivity effects, there is a risk to misinterpret observed VHF echo-intensities and Doppler velocities. Without an aspect-sensitivity correction, one would, when applying the DBS technique (DBS: “Doppler beam swinging”) for small off-zenith beam-pointing angles, underestimate the true horizontal wind velocity systematically. There are some indications (MUSCHINSKI 1996b) that echo-intensity aspect sensitivity in combination with Kelvin-Helmholtz waves in the shear zones of jet streams can give rise to an apparent mean vertical wind velocity of several tens of cm/s. On the other hand, it has been demonstrated that under favourable conditions VHF radars can measure the mean vertical wind with an accuracy of about 1 cm/s (BALSLEY et al. 1988; MUSCHINSKI et al. 1996; MUSCHINSKI et al. 1997, 1998). Therefore, a quantitative
understanding of the aspect sensitivity is necessary, in particular for the accurate determination of the vertical wind.

There are also observations in the UHF regime which contain some indications for the relevance of aspect sensitivity (METCALF 1975); of course, UHF aspect sensitivity is not as obvious as VHF aspect sensitivity, but there might be differences between Doppler velocity and radial wind velocity also in the UHF regime (DOVIK, priv. comm., 1995; see also PALMER et al. 1991, p. 425).

In the free atmosphere, refractive-index irregularities are often confined within thin layers. That is, within the observation volume the irregularities can be inhomogeneously distributed in the form of "sheets" (DALAUDIER et al. 1994) or in the form of turbulent layers (e.g., METCALF & ATLAS 1973, VANZANDT et al. 1978, MUSCHINSKI & WODE 1997, 1998). Inhomogeneity of the refractive-index irregularities can also lead to misinterpretations of observed Doppler spectra ("finite range volume effects"; FUKAO et al. 1988a; FUKAO et al. 1988b), independently of the anisotropy.

An additional effect, which is relevant in particular for applications in climate research, is the so-called "downward bias" (NASTROM & VANZANDT 1994). The existence of a downward bias has been inferred from the fact that long-term averages (over up to several months) of the Doppler velocities observed with vertically pointing VHF radars amount often to about -5 cm/s. Subsiding motion of significantly more than 1 cm/s over that long time intervals, however, seem unrealistical, and NASTROM & VANZANDT (1994) explain those observations in terms of a non-zero correlation between vertical wind velocity and radar reflectivity due to gravity waves. NASTROM & VANZANDT's (1994) gravity-wave-induced downward bias has physically nothing to do with the downward bias due to Kelvin-Helmholtz waves presumed by MUSCHINSKI (1996b); but when analyzing observed Doppler measurements, it is probably hard to distinguish between those two effects without any a-priori assumptions.

1.2.3 Interferometry using UHF/VHF radars

In the conventional Doppler radar technique ("Doppler beam swinging", monostatic radar, single frequency), structure and dynamics of the refractive-index field within the observation volume are projected on a single (real) Doppler spectrum.

Increasingly, UHF/VHF are being operated in interferometric modes. An overview over history and recent developments of interferometric applications in probing the atmosphere using UHF/VHF radars are given, e.g., by KUDEKI (1994) and COHN & CHILSON (1995). Using frequency-domain interferometry (FDI) and interferometry with spaced antennas (spatial interferometry, SI), one gets, as compared to the standard Doppler technique, additional information concerning structure and dynamics within the observation volume, in particular in the customary case that the backscattering in the observation volume is dominated by one or a few turbulent or non-turbulent laminae, the inner structures of which may differ from each other with respect to the degree of anisotropy and turbulence intensity. SI provides the tilting-angles of those layers (or the orientations of the main axes of the three-dimensional wavenumber spectrum of the refractive-index irregularity field within the layers) with good accuracy. This is very useful for the correction of estimation errors in the determination of the vertical wind velocity (see, e.g.,
PALMER et al. 1991; PALMER et al. 1994). FDI, however, provides the thickness of layers if they are not too thick in comparison with the radial extent of the observation volume (KUDEKI & STITT 1987; FRANKE 1990; STITT & KUDEKI 1991).

It is to be expected that, in the long term, the combination of SI, FDI and DBS will become the standard technique in the UHF/VHF Doppler radar technique in atmospheric research and maybe even in routine monitoring of the atmosphere.

1.3 Problem and plan of this work

Based on a series of seminal studies (PEKERIS 1947, BOOKER & GORDON 1950, VILLARIS & WEISSKOPF 1954, TATARSKII 1961, GJESSING 1962, OTTERSTEN 1969, TATARSKII 1971, LIU & YEH 1980, DOVIAK & ZRNIĆ 1984), nowadays, the zeroth moment of the Doppler variance spectrum (i.e., the backscattering cross-sections of atmospheric refractive-index irregularities) in the case of volume scattering from refractive-index irregularities in the optically clear atmosphere are fairly well understood. With regard to the higher moments of the variance spectra, however, the theoretical developments are less thoroughly advanced. But there exist a large number of more or less semi-empirical studies that deal with the problem of how the spatial and temporal structure of the refractive-index irregularity field is projected on the temporal structure of the Doppler signals.

In the following sections, a theory of the moments of Doppler signals is presented. The concept relies on the assumption that the essential information of Doppler signals is contained in their first moments, such that in the forward calculation an explicit knowledge about the Doppler spectra is not necessary. It will be shown that the first moments of the Doppler spectra can be calculated from the spatial (not the temporal) structure of the irregularities of the refractive index and the irregularities of the three components of the wind vector as well as of the spatial structure of the covariances between wind- and refractive-index irregularities, and it will be explicitly shown how this can be done. Neither Taylor's hypothesis nor any assumption for the decorrelation of turbulent structures as a function of time will be used. The assumptions are:

- The refractive-index perturbations are very small as compared to unity
- The far-field condition is fulfilled
- The refractive index is a quasi-conservative quantity.

The concept will be worked out for the standard Doppler technique, for frequency-domain interferometry and for the spaced-antenna interferometry. Explicit equations are given for radars with Gaussian weighting functions.

In Sec. 2, the foundations of Fourier analysis are rederived for complex random functions in space and time. In Sec. 3, the so-called “sampling-functions” for the three above-mentioned sampling modes are defined and given in explicit form. In Sec. 4, the “atmospheric functions”, which describe the spatial structure of the refractive-index irregularities in the context with the spatial structure of the wind irregularities, are defined and given in explicit form. In Sec. 5, it is demonstrated how the moments of the Doppler spectra may be written as convolution products.
of the respective sampling functions and atmospheric functions. In Sec. 6, the zeroth and first moments of the Doppler variance spectra in the case of the standard Doppler technique are derived. In Sec. 7, the zeroth and first moments of the Doppler cross-spectra when applying frequency-domain interferometry are examined, and in Sec. 8 the zeroth and first moments of spaced-antenna Doppler cross-spectra are obtained.
2 Mathematical Foundations: Fourier Analysis

The theory of stochastic processes and the Fourier analysis form a conceptual and mathematical framework for the interpretation of meteorological and geophysical time series in general and of Doppler-radar signals in particular.

In this Section, the most important definitions and relationships of Fourier analysis are briefly recapitulated. A detailed presentation of the theory of stochastic processes is given, e.g., by Papoulis (1965); a well readable summary of Fourier analysis can be found, e.g., in Meyberg & Vachenauer (1991, p. 285–357). Here, we rederive all theorems and relationships, which will be made use of in later Sections, from the Fourier integral theorem as the starting point. Since we will deal in part with complex random functions, we have to make sure that all theorems hold not only for real functions but also for complex functions. Often, it is hard to find out which equations in a textbook on Fourier analysis hold for complex functions and which are restricted to real functions. In this Section, we make sure that all theorems hold for complex functions and that the terminology is consistent.

2.1 Fourier integral theorem and Fourier transformation

Under assumptions which are not stated here in detail (see, e.g., Papoulis 1965; Meyberg & Vachenauer 1991) but which are often fulfilled for meteorological and geophysical complex time series, the Fourier integral theorem holds (e.g., Meyberg & Vachenauer 1991, S. 337):

\[
I(t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( e^{j\omega t'} \int_{-\infty}^{\infty} e^{-j\omega t} I(t) \, dt \right) \, d\omega.
\]  

(2.1)

In compact notation, (2.1) reads

\[
I(t') = \mathcal{F}^{-1} \{ \mathcal{F}\{I(t)\}(\omega) \},
\]  

(2.2)

or even more compactly:

\[
I = \mathcal{F}^{-1} \mathcal{F} I.
\]  

(2.3)

The transformations \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) are called Fourier transformation,

\[
F(\omega) = \mathcal{F}\{I(t)\} = \int_{-\infty}^{\infty} e^{-j\omega t} I(t) \, dt,
\]  

(2.4)

and inverse Fourier transformation, respectively:
\[ I(t) = \mathcal{F}^{-1} \{ F(\omega) \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega) \, d\omega. \] (2.5)

The complex function \( F(\omega) \) is called the Fourier transform of \( I(t) \). The Fourier transformation is a projection from the time domain into the frequency domain; the inverse Fourier transformation, however, is a projection from the frequency domain into the time domain.

2.2 Parseval’s theorem and Doppler spectrum

Parseval’s theorem,

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} |I(t)|^2 \, dt, \] (2.6)

is also known as the “energy equation”, and it provides the correct normalization of the “energy density spectrum” \( |F(\omega)|^2 \) if the expectation value of the “energy” \( |I(t)|^2 \) is given in the time domain. In many practical cases, the quantity \( |I(t)|^2 \) is not an energy but the absolute value of the square of any quantity fluctuating around zero, for example of a pressure fluctuation, a velocity fluctuation or a temperature fluctuation. Nevertheless, the terms “energy spectrum” or “power spectrum” are also customary in this more general sense.

It is convention to introduce the quantity \( S(\omega) \), which is proportional to the function \( |F(\omega)|^2 \) and the integral of which is equal to the expectation value of \( \langle |I|^2 \rangle_t \), that is, to the variance of \( I \):

\[ \int_{-\infty}^{\infty} S(\omega) \, d\omega = \langle |I|^2 \rangle_t = \int_{-\infty}^{\infty} |I(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega, \] (2.7)

where

\[ S(\omega) = \frac{1}{2\pi} |F(\omega)|^2. \] (2.8)

\( S(\omega) \) is called the variance spectrum of the quantity \( I \). In the case that \( I \) is a Doppler signal, we call \( S(\omega) \) the “Doppler spectrum” for brevity.

In defining \( S(\omega) \), we do not follow the definitions used by DOVIK & ZRNIČ (1984), DOVIK & ZRNIČ (1993) and DOVIK et al. (1996). In those works, the integral over \( S(\omega) \) is set equal to \( R\langle |I|^2 \rangle \), where \( R \) is the system’s resistance. That is, both definitions differ from each other just by the factor \( R \). PAPOULIS (1965, p. 338), however, defines \( S(\omega) \) as identical to \( |F(\omega)|^2 \), without the factor \( 1/(2\pi) \).
The autocovariance function \( C(\tau) \) of a complex function \( I(t) \) is defined as follows:

\[
C(\tau) = \langle I^*(t)I(t + \tau) \rangle_t = \int_{-\infty}^{\infty} I^*(t)I(t + \tau) \, dt.
\] (2.9)

Now, we show that the square of the magnitude of the Fourier transform of \( I(t) \) is identical to the Fourier transform of \( C(\tau) \). First, consider

\[
|\mathcal{F}\{I(t)\}|^2 = \mathcal{F}\{I(t)\} \cdot (\mathcal{F}\{I(t)\})^* = \left( \int_{-\infty}^{\infty} e^{-j\omega t} I(t) \, dt \right) \left( \int_{-\infty}^{\infty} e^{-j\omega t'} I^*(t') \, dt' \right)^*.
\] (2.10)

Using

\[
\left( \int_{-\infty}^{\infty} e^{-j\omega t'} I^*(t') \, dt' \right)^* = \int_{-\infty}^{\infty} e^{+j\omega t'} I^*(t') \, dt',
\] (2.11)

we obtain after some rearranging

\[
|\mathcal{F}\{I(t)\}|^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j\omega(t-t')} I^*(t')I(t) \, dt \, dt'.
\] (2.12)

By introducing the time lag variable \( \tau = t - t' \) and after eliminating \( t \) by substitution, it follows

\[
|\mathcal{F}\{I(t)\}|^2 = \int_{-\infty}^{\infty} e^{-j\omega\tau} \left( \int_{-\infty}^{\infty} I^*(t')I(t' + \tau) \, dt' \right) \, d\tau = \int_{-\infty}^{\infty} e^{-j\omega\tau} C(\tau) \, d\tau,
\] (2.13)

hence

\[
|\mathcal{F}\{I(t)\}|^2 = \mathcal{F}\{C(\tau)\},
\] (2.14)

quod erat demonstrandum.

That is, \(|F(\omega)|^2\) is the Fourier transform of the autocovariance function \( C(\tau) \), and \( C(\tau) \) is the inverse Fourier transform of \(|F(\omega)|^2\). These two relations,

\[
|F(\omega)|^2 = \int_{-\infty}^{\infty} e^{-j\omega\tau} C(\tau) \, d\tau
\] (2.15)
and

\[ C(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{+j\omega\tau} |F(\omega)|^2 d\omega, \quad (2.16) \]

are known as Wiener-Khintchine-relations, after the seminal works by Wiener (1930) and Khintchine (1934); see also Parzen (1962, p. 110).

According to (2.8), one may write the Wiener-Khintchine-relations also in terms of the variance spectrum \( S(\omega) \):

\[ S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega\tau} C(\tau) d\tau \quad (2.17) \]

and

\[ C(\tau) = \int_{-\infty}^{\infty} e^{+j\omega\tau} S(\omega) d\omega. \quad (2.18) \]

2.4 The moments theorem

In the following Sections, it will become evident that the mathematical effort to calculate \( S(\omega) \) and \( C(\tau) \), respectively, can be very different. The difference can be especially significant when calculating the moments of \( S(\omega) \).

The \( n \)th moment of the variance spectrum \( S(\omega) \) is defined as

\[ M_n = \int_{-\infty}^{\infty} S(\omega)\omega^n d\omega. \quad (2.19) \]

As already mentioned in Sec. 1, the “zeroth” and the first moments of the Doppler spectrum \( S(\omega) \) (echo-intensity and Doppler shift) are classically interpreted as proportional to volume reflectivity and radial wind velocity, respectively, averaged over the observation volume. The second central moment (the spectral width) is classically interpreted as the variance of the radial wind velocity distribution within the observation volume. Therefore, the first moments of the Doppler spectrum are of great interest in radar meteorology.

The proof of the moments theorem is simple but original (compare, e.g., Papoulis 1965, p. 157; Parzen 1962, p. 17). \( C(\tau) \) may be written as a power series in \( \tau \) by expanding the exponential function in the Wiener-Khintchine-relation into a power series:
\[ C(\tau) = \int_{-\infty}^{\infty} \left( \sum_{n=0}^{\infty} \frac{(j\omega\tau)^n}{n!} \right) S(\omega) \, d\omega. \]  

Equation (2.20)

Rearranging the order of integration and summation leads to

\[ C(\tau) = \sum_{n=0}^{\infty} \left( \frac{j^n}{n!} \tau^n \int_{-\infty}^{\infty} S(\omega)\omega^n \, d\omega \right) = \sum_{n=0}^{\infty} \frac{j^n}{n!} M_n \tau^n. \]  

Equation (2.21)

Therefore, the representation of \( C(\tau) \) as a power series is

\[ C(\tau) = \sum_{n=0}^{\infty} a_n \tau^n, \]  

Equation (2.22)

where the coefficients \( a_n \) are related as follows to the moments \( M_n \).

\[ a_n = \frac{j^n}{n!} M_n. \]  

Equation (2.23)

Using complete induction, it is easy to show that the power series for the \( m \)th derivative of \( C(\tau) \) is given by

\[ \frac{\partial^m}{\partial \tau^m} C(\tau) = \sum_{n=m}^{\infty} \frac{j^n}{(n-m)!} M_n \tau^{n-m}, \]  

Equation (2.24)

and it is obvious that in the case \( \tau = 0 \) the sum contains just one non-zero summand:

\[ \frac{\partial^m}{\partial \tau^m} C(\tau = 0) = j^m M_m. \]  

Equation (2.25)

This is the moments theorem. Especially, for \( m = 0 \) one obtains

\[ C(0) = M_0. \]  

Equation (2.26)

This is the proof of Parseval's theorem, (2.6). That is, Parseval's theorem is a special case of the moments theorem. Furthermore, the moments theorem enables us to calculate the higher moments of the variance spectrum directly from the higher derivatives of the autocovariance function \( C(\tau) \) at zero lag (\( \tau = 0 \)). That is, it is possible to determine the Doppler shift and the spectral width of the Doppler spectrum from the derivatives of \( C(\tau) \) at zero lag, without calculating the Doppler spectrum explicitly by carrying out the Fourier transformation of the autocovariance function.


2.5 Central moments

Often not the (ordinary) moments but the central moments are of interest. For example, the spectral width is not proportional to the second moment but to the second central moment. The central moments are the (ordinary) moments of the centered variance spectrum

\[ S'(\Omega) = S(\omega) \]  

(2.27)

with the centered frequency coordinate

\[ \Omega = \omega - \omega_1, \]  

(2.28)

where \( \omega_1 \) is the center of gravity of the non-centered variance spectrum:

\[ \omega_1 = \frac{\int_{-\infty}^{\infty} S(\omega) \omega \, d\omega}{\int_{-\infty}^{\infty} S(\omega) \, d\omega} = \frac{M_1}{M_0}. \]  

(2.29)

Thus, the \( m \)th central moment is

\[ Z_m = \int_{-\infty}^{\infty} S'(\Omega) \Omega^m \, d\Omega = \int_{-\infty}^{\infty} S(\omega)(\omega - \omega_1)^m \, d\omega, \]  

(2.30)

and binomial theorem leads to

\[ Z_m = \int_{-\infty}^{\infty} S(\omega) \left\{ \sum_{n=0}^{m} (-1)^n \binom{m}{n} \omega^{m-n} \omega_1^n \right\} \, d\omega. \]  

(2.31)

The binomial coefficients are defined as

\[ \binom{m}{n} = \frac{m!}{n! (m-n)!}. \]  

(2.32)

It turns out that the \( m \)th central moment \( Z_m \) may be written in terms of the (ordinary) moments \( M_0, M_1, ..., M_m \):

\[ Z_m = \sum_{n=0}^{m} (-1)^n \binom{m}{n} \left( \frac{M_1}{M_0} \right)^n M_{m-n}. \]  

(2.33)

It is easy to verify
\[ Z_0 = M_0, \quad (2.34) \]
\[ Z_1 = 0 \quad (2.35) \]

and
\[ Z_2 = M_2 - \frac{M_1^2}{M_0}. \quad (2.36) \]

The moment theorem provides the helpful relationship
\[ Z_2 = \frac{C''(0)}{C'(0)} - C''(0), \quad (2.37) \]

where the number of the primes at \( C \) means the degree of the derivative.

### 2.6 Cross-covariance function and cross-spectrum

The cross-covariance function of two complex functions \( I_1(t) \) and \( I_2(t) \) is defined as
\[ C_{12}(\tau) = (I_1^* t \cdot I_2(t + \tau)) = \int_{-\infty}^{\infty} I_1^* t I_2(t + \tau) dt, \quad (2.38) \]

and the cross-spectrum of \( I_1 \) and \( I_2 \) is defined as
\[ S_{12}(\omega) = \frac{1}{2\pi} F_1^* (\omega) F_2(\omega), \quad (2.39) \]

where \( F_1(\omega) \) and \( F_2(\omega) \) are the Fourier transforms of \( I_1(t) \) and \( I_2(t) \), respectively. Note that the definitions of the autocovariance function and of the variance spectrum as given in (2.9) and (2.8) are special cases of the definitions of the cross-covariance function, (2.38), and of the cross-spectrum, (2.39), respectively.

Now, consider the Fourier transform of the cross-covariance function \( C_{12}(\tau) \):
\[
\mathcal{F}\{C_{12}(\tau)\} = \int_{-\infty}^{\infty} e^{-j\omega t} C_{12}(\tau) d\tau \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j\omega(t' - t)} I_1^* (t) I_2(t') dt dt' \\
= F_1^* (\omega) F_2(\omega). \quad (2.40)
\]
We find

\[ S_{12}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega \tau} C_{12}(\tau) d\tau, \]  

(2.41)

and the inverse relationship reads

\[ C_{12}(\tau) = \int_{-\infty}^{\infty} e^{j\omega \tau} S_{12}(\omega) d\omega. \]  

(2.42)

That is, there exist relationships between the cross-covariance function and the cross-spectrum, in analogy to the relationships between the autocovariance function and the variance spectrum known as the Wiener-Khintchine-relations. Generalized relations exist also for complex scalar fields in three-dimensional space. Let

\[ R_{12}(\vec{\delta}) = \langle n_1^*(\vec{\tau}) n_2(\vec{\tau} + \vec{\delta}) \rangle_{\vec{\tau}} \]

\[ = \iiint_{-\infty}^{\infty} n_1^*(\vec{\tau}) n_2(\vec{\tau} + \vec{\delta}) d^3\tau \]  

(2.43)

be the three-dimensional cross-covariance function of two complex fields \( n_1(\vec{\tau}) \) and \( n_2(\vec{\tau}) \) in three-dimensional space, and let

\[ \Phi_{12}(\vec{k}) = \frac{1}{(2\pi)^3} H^*_1(\vec{k}) H_2(\vec{k}) \]  

(2.44)

be the three-dimensional cross-spectrum of \( n_1(\vec{\tau}) \) and \( n_2(\vec{\tau}) \), where

\[ H_1(\vec{k}) = \iiint_{-\infty}^{\infty} e^{-j\vec{k} \cdot \vec{\tau}} n_1(\vec{\tau}) d^3\tau \]  

(2.45)

and

\[ H_2(\vec{k}) = \iiint_{-\infty}^{\infty} e^{-j\vec{k} \cdot \vec{\tau}} n_2(\vec{\tau}) d^3\tau \]  

(2.46)

are the three-dimensional Fourier transforms of \( n_1(\vec{\tau}) \) and \( n_2(\vec{\tau}) \), respectively. It is easy to show that, correspondingly to the one-dimensional case, the following relations between \( R_{12}(\vec{\delta}) \) and \( \Phi_{12}(\vec{k}) \) hold:
\[ \Phi_{12}(\tilde{k}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-j\tilde{k} \cdot \delta} R_{12}(\delta) d^3 \delta \] (2.47)

and

\[ R_{12}(\delta) = \iiint_{-\infty}^{\infty} e^{j\tilde{k} \cdot \delta} \Phi_{12}(\tilde{k}) d^3 k. \] (2.48)

As a special case, we obtain the following relations between the autocovariance function of a field \( n(\vec{r}) \),

\[ R_n(\delta) = \langle n^*(\vec{r}) n(\vec{r} + \delta) \rangle = \iiint_{-\infty}^{\infty} n^*(\vec{r}) n(\vec{r} + \delta) d^3 r, \] (2.49)

and the variance spectrum of \( n(\vec{r}) \),

\[ \Phi_n(\vec{k}) = \frac{1}{(2\pi)^3} |H_n(\vec{k})|^2 \] (2.50)

where

\[ H_n(\vec{k}) = \iiint_{-\infty}^{\infty} e^{-j\vec{k} \cdot \vec{r}} n(\vec{r}) d^3 r \] (2.51)

is the three-dimensional Fourier transform of \( n(\vec{r}) \):

\[ \Phi_n(\vec{k}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-j\vec{k} \cdot \delta} R_n(\delta) d^3 \delta \] (2.52)

(compare Tatarskii 1961, p. 17, eq. 1.24) and

\[ R_n(\delta) = \iiint_{-\infty}^{\infty} e^{j\vec{k} \cdot \delta} \Phi_n(\vec{k}) d^3 k \] (2.53)

(compare Tatarskii 1961, p. 16, eq. 1.22).
2.7 Cross-covariance function and cross-spectrum of a random function and its temporal or spatial derivative

Sometimes, one has to solve the problem of calculating the cross-covariance function or the cross-spectrum of a random function and its temporal or spatial derivative.

Let \( I(t) \) be a complex random function of time \( t \), and let \( F_I(\omega) \) be its Fourier transform. Then, \( I(t) \) may be written as the inverse Fourier transform of \( F_I(\omega) \),

\[
I(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} F_I(\omega) d\omega,
\]

and the derivative of \( I(t) \) may be written as follows:

\[
I'(t) = \frac{\partial I(t)}{\partial t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega e^{j\omega t} F_I(\omega) d\omega.
\]

Thus,

\[
F_{I'}(\omega) = j\omega F_I(\omega)
\]

is the Fourier transform of \( \partial I/\partial t \), and the cross-spectrum of \( I \) and \( \partial I/\partial t \) may be stated as

\[
S_{II'}(\omega) = \frac{1}{2\pi} F_I^*(\omega) F_{I'}(\omega) = \frac{j\omega}{2\pi} F_I^*(\omega) F_I(\omega) = j\omega S_I(\omega),
\]

where \( S_I(\omega) \) is the variance spectrum of \( I(t) \). Now, we can also give the cross-covariance function of \( I(t) \) and its derivative:

\[
C_{II'}(\tau) = \left< I^*(t) \frac{\partial I}{\partial t}(t + \tau) \right>_t = \int_{-\infty}^{\infty} e^{j\omega \tau} S_{II'}(\omega) d\omega = \int_{-\infty}^{\infty} j\omega e^{j\omega \tau} S_I(\omega) d\omega.
\]

In particular, we obtain the covariance of \( I(t) \) and \( \partial I/\partial t \):
\[ C_{IR}(0) = \left\langle I^*(t) \frac{\partial I}{\partial t}(t) \right\rangle_t \]
\[ = j \int_{-\infty}^{\infty} S_I(\omega) \omega d\omega \]
\[ = jM_1 = C'_I(0). \]  

(2.59)

Here, \( M_1 \) is the first moment of the variance spectrum of \( I(t) \), and \( C'_I(0) \) is the derivative of the autocovariance function of \( I(t) \) at zero lag (\( \tau = 0 \)).

Corresponding relations exist also for complex random functions in three-dimensional space, as is shown in the following. Let \( n(\vec{r}) \) be a complex random function in three-dimensional space and

\[ H_n(\vec{k}) = \int_{-\infty}^{\infty} e^{-j\vec{k} \cdot \vec{r}} n(\vec{r}) d^3r \]

(2.60)

the Fourier transform of \( n(\vec{r}) \). Then, \( n(\vec{r}) \) may be written as a three-dimensional inverse Fourier transform,

\[ n(\vec{r}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{j\vec{k} \cdot \vec{r}} H_n(\vec{k}) d^3k, \]

(2.61)

and we obtain the gradient of \( n(\vec{r}) \):

\[ \vec{\nabla} n(\vec{r}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} j\vec{k} e^{j\vec{k} \cdot \vec{r}} F_n(\vec{k}) d^3k \]

(2.62)

where

\[ \vec{\nabla} n(\vec{r}) = \left( \frac{\partial}{\partial r_1}, \frac{\partial}{\partial r_2}, \frac{\partial}{\partial r_3} \right) n(\vec{r}). \]

(2.63)

Thus, the Fourier transform of a component of \( \vec{\nabla} n \), for example of the component \( \nabla_1 n \), is given by

\[ H_{\nabla_1 n}(\vec{k}) = jk_1 H_n(\vec{k}), \]

(2.64)

and we obtain the cross-spectrum of \( n(\vec{r}) \) and \( \nabla_1 n(\vec{r}) \) as follows:
\[ \Phi_n \nabla \cdot \n(k) = \frac{1}{(2\pi)^3} H_n^*(k) H_v \cdot n(k) \]
\[ = \frac{j k_i}{(2\pi)^3} \left| H_n(k) \right|^2 = j k_i \Phi_n(k). \]  
(2.65)

Hence, the cross-covariance function of \( n(\vec{r}) \) and \( \nabla \cdot n(\vec{r}) \) is given by

\[ R_n \nabla \cdot n(\vec{\delta}) = \left\langle n^*(\vec{r}) \nabla \cdot n(\vec{r} + \vec{\delta}) \right\rangle_{\vec{r}} \]
\[ = \iiint_{-\infty}^{\infty} n^*(\vec{r}) \nabla \cdot n(\vec{r} + \vec{\delta}) d^3 r \]
\[ = \iiint_{-\infty}^{\infty} e^{j \vec{k} \cdot \vec{\delta}} \Phi_n \nabla \cdot n(k) d^3 k \]
\[ = \iiint_{-\infty}^{\infty} e^{j \vec{k} \cdot \vec{\delta}} j k_i \Phi_n(k) d^3 k. \]  
(2.66)

In particular, we have

\[ R_n \nabla \cdot n(0) = \iiint_{-\infty}^{\infty} j k_i \Phi_n(k) d^3 k. \]  
(2.67)

2.8 The correlation theorem for random functions of time and of three-dimensional space

First, we consider the cross-covariance function of two complex random functions of time, \( I_1(t) \) and \( I_2(t) \):

\[ C_{12}(\tau) = \int_{-\infty}^{\infty} I_1^*(t) I_2(t + \tau) dt. \]  
(2.68)

The Fourier transform of \( C_{12}(\tau) \) is

\[ \mathcal{F}\{C_{12}(\tau)\} \]
\[ = \int_{-\infty}^{\infty} e^{-j\omega \tau} \left( \int_{-\infty}^{\infty} I_1^*(t) I_2(t + \tau) dt \right) d\tau \]
\[ = \int_{-\infty}^{\infty} e^{-j\omega(t' - t)} \int_{-\infty}^{\infty} I_1^*(t) I_2(t') dt dt' \]
\[ \left( \int_{-\infty}^{\infty} e^{-j\omega t} I_1(t) dt \right)^* \left( \int_{-\infty}^{\infty} e^{-j\omega t'} I_2(t') dt' \right) = F_1^*(\omega) F_2(\omega), \] (2.69)

where

\[ F_1(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} I_1(t) dt \] (2.70)

and

\[ F_2(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} I_2(t) dt. \] (2.71)

Thus, the Fourier transform of the cross-covariance function \( C_{12}(\tau) \) of two complex random functions \( I_1(t) \) and \( I_2(t) \) is the (simple) product of the conjugate complex of the Fourier transform of \( I_1(t) \) and the Fourier transform of \( I_2(t) \), and we may write \( C_{12}(\tau) \) as inverse Fourier transform of \( F_1^*(\omega) \cdot F_2(\omega) \):

\[ C_{12}(\tau) = \mathcal{F}^{-1}\{F_1^*(\omega) F_2(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega \tau} F_1^*(\omega) F_2(\omega) d\omega = \int_{-\infty}^{\infty} e^{j\omega \tau} S_{12}(\omega) d\omega. \] (2.72)

Corresponding relations exist also for two complex random functions \( n_1 \) and \( n_2 \) in three-dimensional space. Let \( R_{12}(\vec{\delta}) \) be the cross-covariance function of the two complex functions \( n_1(\vec{r}) \) and \( n_2(\vec{r}) \) in three-dimensional space:

\[ R_{12}(\vec{\delta}) = \iiint_{-\infty}^{\infty} n_1^*(\vec{r}) n_2(\vec{r} + \vec{\delta}) d^3 r. \] (2.73)

Then, its Fourier transform is given by

\[ \mathcal{F}\{R_{12}(\vec{\delta})\} = H_1^*(\vec{k}) H_2(\vec{k}), \] (2.74)

where

\[ H_1(\vec{k}) = \iiint_{-\infty}^{\infty} e^{-j\vec{k} \cdot \vec{r}} n_1(\vec{r}) d^3 r \] (2.75)

and
\[ H_2(\vec{k}) = \iiint_{-\infty}^{\infty} e^{-j\vec{k} \cdot \vec{r}} n_2(\vec{r}) d^3r \] (2.76)

are the spatial Fourier transforms of \( n_1(\vec{r}) \) and \( n_2(\vec{r}) \), respectively. Again, as in the one-dimensional case, we may write the cross-covariance function as the inverse Fourier transform of the cross-spectrum \( \Phi_{12}(\vec{k}) \) (apart from a numerical factor \((2\pi)^3\):

\[ R_{12}(\vec{\delta}) = \mathcal{F}^{-1} \left\{ H_1^*(\vec{k}) H_2(\vec{k}) \right\} = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{j\vec{\delta} \cdot \vec{k}} H_1^*(\vec{k}) H_2(\vec{k}) d^3k = \iiint_{-\infty}^{\infty} e^{j\vec{\delta} \cdot \vec{k}} \Phi_{12}(\vec{k}) d^3k. \] (2.77)

This is the correlation theorem. Obviously, the correlation theorem is a generalization of the Wiener-Khintchine relation(s).

As a special case, we obtain for zero lag in three-dimensional space (\( \vec{\delta} = 0 \)):

\[ R_{12}(0) = \iiint_{-\infty}^{\infty} n_1(\vec{r}) n_2(\vec{r}) d^3r = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} H_1^*(\vec{k}) H_2(\vec{k}) d^3k = \iiint_{-\infty}^{\infty} \Phi_{12}(\vec{k}) d^3k. \] (2.78)

2.9 The Fourier transform of a real function

Often, integrations can be significantly simplified by making use of symmetry properties of the integrand. Let us consider the Fourier transform of a real function \( n(\vec{r}) \):

\[ H_n(\vec{k}) = \iiint_{-\infty}^{\infty} e^{-j\vec{k} \cdot \vec{r}} n(\vec{r}) d^3r. \] (2.79)

Hence, the complex conjugate of \( H_n(\vec{k}) \) is

\[ H_n^*(\vec{k}) = \iiint_{-\infty}^{\infty} e^{+j\vec{k} \cdot \vec{r}} n^*(\vec{r}) d^3r \]

\[ = \iiint_{-\infty}^{\infty} e^{-j(-\vec{k}) \cdot \vec{r}} n(\vec{r}) d^3r = \Phi_n(-\vec{k}), \] (2.80)

and we obtain the following symmetry relation for a real random function \( n(\vec{r}) \):

\[ H_n^*(\vec{k}) = H_n(-\vec{k}). \] (2.81)
3 The Sampling-Functions

3.1 Introductory remark

In the following, we will see that the moments of Doppler spectra may be written as six-dimensional convolution products of two functions: a “sampling-function”, \( G_{11}(\vec{\sigma}, \vec{\delta}) \) or \( G_{12}(\vec{\sigma}, \vec{\delta}) \), which defines the properties and the sampling-mode of the radar, and an “atmospheric function” that describes the spatial structure of the refractive-index irregularities \( R_n(\vec{\sigma}, \vec{\delta}, \tau=0) \) or its temporal change \( R'_n(\vec{\sigma}, \vec{\delta}, \tau=0) \) or \( R''_n(\vec{\sigma}, \vec{\delta}, \tau=0) \). In this Section, we will derive the sampling-functions, and in the next Section the atmospheric functions.

3.2 The monostatic radar in the standard operational mode

With regard to the terminology, we follow closely DOVIAK & ZRNIĆ (1984), DOVIAK & ZRNIĆ (1993) and DOVIAK et al. (1996). Let \( I(t) \) be the complex receiver current and \( R \) the receiver resistance. The received power, averaged over one period of the carrier frequency, is given by

\[
P_r = \frac{R}{2} I^* I
\]

(DOVIAK & ZRNIĆ 1993, p. 456). We call \( I(t) \) the “weather signal” or, more shortly, “the signal”.

DOVIAK & ZRNIĆ (1984; 1993, p. 456) have shown that, in the case of a monostatic Doppler radar, \( I(t) \) may be written as follows:

\[
I(t) = A \int |W(\vec{r})| f_2^2(\vec{r}) n'(\vec{r}, t) e^{-j2k_0 t} dV
\]

with the parameter

\[
A = \frac{g}{\lambda r_0^2} \left( \frac{P_t}{2R} \right)^{\frac{1}{2}}
\]

(3.3)

Here, \( \lambda \) is the radar wavelength,

\[
k_0 = \frac{2\pi}{\lambda}
\]

To be precise, the instantaneous receiver current is not \( I(t) \) but \( \exp(i\omega_0 t) \cdot I(t) \), where \( \omega_0 \) is the radar’s carrier frequency. It is, however, customary to split off and ignore the phase factor \( \exp(i\omega_0 t) \).

In a monostatic radar, the same antenna is used to transmit a radar pulse and, a short time later, to receive the echo of the transmitted pulse. When operating a monostatic radar, the radar is switched at the pulse repetition frequency between the transmitting and receiving mode by means of a quick switcher (e.g., CZECHOWSKY et al. 1984). A radar having, in addition to the transmitting antenna, one or more receiving antennas, is called a bistatic or multistatic radar, respectively.
is the corresponding wavenumber, \( g \) the antenna gain, \( r_0 \) the distance between the center of the antenna and the center of the observation volume, \( j \) is the imaginary unit, \( n'(r', t) \) is the field of the atmospheric refractive-index irregularities at time \( t \), and \( r_t \) is the distance between the center of the antenna and the integration point \( r' \), i.e.,

\[
r_t = | - r_0 + r' |,
\]

where \( r_0 \) is the vector from the center of the observation volume to the center of the antenna, see Fig. (3.1).

Fig. 3.1: Sketch of the coordinate system used to describe the spatial distribution of the refractive-index irregularities as observed by a monostatic clear-air Doppler radar (DOVIAK and ZRNIĆ 1984).
We call the function $|W(\vec{r})|f_0^2(\vec{r})$, which is normalized to unity, the “one-way weighting-function” or shortly the “weighting-function”. $|W(\vec{r})|$ is its radial part (the “receiver weighting-function”), and $f_0^2(\vec{r})$ is its transverse part (also known as “gain function” or “one-way antenna pattern”). The integration is carried out over all points for which the integrand contributes significantly to the integral. In practice, the weighting-function $|W(\vec{r})|f_0^2(\vec{r})$ drops rapidly to zero, so that it is customary to call the volume within the plane at that the weighting-function has values half of its maximum value (the “−6-dB plane”) the “observation volume”, the “control volume”, or the “resolution volume”. Doviak & Zrnić (1993) call it $V_6$. In the following, we prefer the term “observation volume”.

The center of the observation volume is by definition identical to the point for which the weighting-function has its maximum. It has turned out to be an advantage to choose the center of the observation volume as the origin of the $\vec{r}$-coordinate system (see Doviak & Zrnić 1984, Doviak & Zrnić 1993 and Doviak et al. 1996).

The measurement principle of coherently pulsed radars (“Doppler radars”) may be mathematically represented by a projection of a scalar field $n'(\vec{r}, t)$ (the refractive-index irregularity field at time $t$) in three-dimensional space on a single complex number $I(t)$ (real and imaginary part of the complex weather signal at time $t$):

$$I_1(t) = \iint_{-\infty}^{\infty} G_1(\vec{r})n'(\vec{r}, t)d^3r.$$  

(3.6)

(In order to make a formal distinction between the weighting-function and the weather signal for the standard mode and for the interferometric modes to be defined later, respectively, here we have indicated $G$ and $I$ with “1”.) From (3.2) we find

$$G_1(\vec{r}) = A \cdot |W(\vec{r})|f_0^2(\vec{r})e^{-j2kr_0t}.$$  

(3.7)

Doviak & Zrnić (1984) assumed that the weighting-function is Gaussian in all three dimensions and symmetrical about the beam axis:

$$|W(\vec{r})| = W(l) = \exp \left(\frac{-t^2}{4a^2}\right)$$  

(3.8)

and

$$f_0^2 = \exp \left(\frac{-\Theta^2}{4a^2}\right) = \exp \left(\frac{-t_1^2 + t_2^2}{4r_0^2a^2}\right) = \exp \left(\frac{-t_1^2 + t_2^2}{2\sigma_t^2}\right),$$  

(3.9)

where

$$a^2 = 2r_0^2c_0^2.$$  

(3.10)
According to Doviak and Zrnić (1984), we introduce the auxiliary variables $t_1, t_2, l$ and $t'_1, t'_2, l'$ as the components of the vectors $\vec{r}$ and $\vec{r'}$, respectively, i.e.,

$$\vec{r} = \begin{pmatrix} t_1 \\ t_2 \\ l \end{pmatrix}$$

(3.11)

and

$$\vec{r'} = \begin{pmatrix} t'_1 \\ t'_2 \\ l' \end{pmatrix},$$

(3.12)

respectively.

The angular width $\sigma_\theta$ may be stated in terms of the one-way half-power width, $\Theta_1$, which is a widely used measure of the beam width:

$$\sigma_\theta = \frac{\Theta_1}{4\sqrt{\ln 2}} = 0.30 \Theta_1$$

(3.13)

(Doviak & Zrnić 1984). The length $\sigma_r$ in (3.8), in the case of a receiver having a Gaussian shaped frequency response “matched” to a rectangular pulse of duration $T$ (see Doviak and Zrnić 1993, p. 116), is given by

$$\sigma_r = 0.35 \frac{cT}{2},$$

(3.14)

where $c = 3 \cdot 10^8 \text{ms}^{-1}$ is the velocity of light (Doviak & Zrnić 1984).

After expanding the phase angle $2k_0 r_1$ in (3.7) up to the second order with respect to the coordinates $t_1, t_2, l$, we obtain

$$\exp(-j2k_0 r_1) = \exp\left[-j2k_0 \left( r_0 + l + \frac{t_1^2 + t_2^2}{2r_0} \right) \right]$$

$$= \exp(-j2k_0 r_0) \exp\left[-j2k_0 \left( l + \frac{t_1^2 + t_2^2}{2r_0} \right) \right].$$

(3.15)

Here, $l$ is the radial coordinate as measured from the center of the observation volume, and $t_1$ and $t_2$ are the two coordinates perpendicular to the beam axis.

This provides

$$G_1(\vec{r}) = A \exp\left(-\frac{t_1^2 + t_2^2}{2\sigma_t^2}\right) \cdot \exp\left(-\frac{l^2}{4\sigma_l^2}\right) \cdot \exp\left[-j2k_0 \left( l + \frac{t_1^2 + t_2^2}{2r_0} \right) \right] \cdot \exp(-j2k_0 r_0)$$

(3.16)
as the one-way weighting function of a monostatic Doppler radar.

The autocovariance function of the weather signal turns out to be

\[ C_{11}(\tau) = \int_{-\infty}^{\infty} I_1^*(t) I_1(t + \tau) dt \]

\[ = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} G_1^*(r) n'(r, t) d^3r \right) \left( \int_{-\infty}^{\infty} G_1(r') n'(r', t) d^3r' \right) dt \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_1^*(r) G_1(r') \left( \int_{-\infty}^{\infty} n'(r, t) n'(r', t + \tau) dt \right) d^3r d^3r' \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{11}(\sigma, \bar{\sigma}) R_n(\sigma, \bar{\sigma}, \tau) d^3\sigma d^3\bar{\sigma} \quad (3.17) \]

with the so-called “two-way sampling-function”

\[ G_{11}(\sigma, \bar{\sigma}) = G_1^*(r) G_1(r') \]

and the seven-dimensional space-time autocovariance function of the atmospheric refractive-index field,

\[ R_n(\sigma, \bar{\sigma}, \tau) = \langle n'(r, t) n'(\bar{r}, t + \tau) \rangle_t \]

\[ = \int_{-\infty}^{\infty} n'(r, t) n'(\bar{r}, t + \tau) dt. \quad (3.19) \]

Here, we have introduced the sum- and difference-coordinates

\[ \bar{\sigma} = \left( \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{array} \right) = \frac{r + \bar{r}}{2} = \frac{1}{2} \left( \begin{array}{c} t_1 + t'_1 \\ t_2 + t'_2 \\ l + l' \end{array} \right) \]

\[ = \left( \begin{array}{c} \delta_1 \\ \delta_2 \\ \delta_3 \end{array} \right) = r - \bar{r} = \left( \begin{array}{c} t_1 - t'_1 \\ t_2 - t'_2 \\ l - l' \end{array} \right) \]

(\textit{compare Tatarskii 1961, pp. 52ff.}, and we have assumed that the refractive index is a real quantity, i.e., \((n')^* = n'\).

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After inserting (3.7) into (3.18) and after some elementary manipulations, we find

\[ G_{11}(\sigma, \delta) = A^2 \exp \left( -\frac{\sigma_1^2 + \sigma_2^2 + \delta_1^2}{\sigma_t^2} \right) \exp \left( -\frac{\sigma_3^2 + \delta_1^2}{2\sigma_r^2} \right) \exp \left[ +j2k_0 \left( \frac{\delta_3 + \sigma_1\delta_1 + \sigma_2\delta_2}{r_0} \right) \right]. \]

(3.22)

In the following, we will see that the determination of the integral in (3.17) may be considerably simplified if one, by making use of the convolution theorem, performs the integration in wavenumber space:

\[ H_{11}(\sigma, \delta) = \mathcal{F}_\delta \{ G_{11}(\sigma, \sigma) \} \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j\vec{k} \cdot \sigma} G_{11}(\sigma, \sigma) d^2 \sigma. \]

(3.23)

Thus,

\[ H_{11}(\sigma, \delta) = A^2 \exp \left( -\frac{\sigma_1^2 + \sigma_2^2}{\sigma_t^2} \right) \exp \left( -\frac{\sigma_3^2 + \delta_1^2}{2\sigma_r^2} \right) \cdot \exp \left( -\frac{\delta_3^2}{8\sigma_r^2} \right) \]
\[ \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( -j\vec{k} \cdot \sigma \right) \cdot \exp \left( -\frac{\delta_2^2 + \delta_3^2}{4\sigma_t^2} \right) \cdot \exp \left( -\frac{\delta_3^2}{8\sigma_r^2} \right) \cdot \exp \left[ +j2k_0 \left( \frac{\delta_3 + \sigma_1\delta_1 + \sigma_2\delta_2}{r_0} \right) \right] d^2 \sigma. \]

(3.24)

We recognize the factor in front of the integral as being identical to \( A^2 f_0^4(\sigma) |W(\sigma)|^2 \), and we obtain

\[ H_{11}(\sigma, \delta) = A^2 f_0^4(\sigma) |W(\sigma)|^2 \]
\[ \times \int_{-\infty}^{\infty} \exp \left[ -\frac{\delta_1^2}{4\sigma_t^2} - \left( jk_1 - j \frac{2k_0\sigma_1}{r_0} \right) \delta_1 \right] d\delta_1 \]
\[ \times \int_{-\infty}^{\infty} \exp \left[ -\frac{\delta_2^2}{4\sigma_t^2} - \left( jk_2 - j \frac{2k_0\sigma_2}{r_0} \right) \delta_2 \right] d\delta_2 \]
\[ \times \int_{-\infty}^{\infty} \exp \left[ -\frac{\delta_3^2}{8\sigma_r^2} - \left( jk_3 - j2k_0 \right) \delta_3 \right] d\delta_3. \]

(3.25)

The three integrals are solved using the identity
\[
\int_{-\infty}^{\infty} \exp \left( -a^2 x^2 - bx \right) \, dx = \sqrt{\frac{\pi}{a^2}} \exp \left( \frac{b^2}{4a^2} \right),
\]

which gives

\[
H_{11}(\vec{\sigma}, \vec{k}) = 4(2\pi)^{\frac{3}{2}} \sigma^2 \sigma_r A^2 f_\phi^4(\vec{\sigma}) |W(\vec{\sigma})|^2 \times \exp \left[ -\sigma_I^2 \left( k_1 - \frac{2k_0 \sigma_1}{r_0} \right)^2 - \sigma_t^2 \left( k_2 - \frac{2k_0 \sigma_2}{r_0} \right)^2 - 2\sigma_t^2 (k_3 - 2k_0)^2 \right].
\]

We call \(H_{11}(\vec{\sigma}, \vec{k})\) the “spectral two-way sampling-function” for the monostatic Doppler radar operated in the standard mode.\(^4\) It is easy to see that \(H_{11}(\vec{\sigma}, \vec{k})\) is a real function, i.e.,

\[
H_{11}(\vec{\sigma}, \vec{k}) = H_{11}(\vec{\sigma}, \vec{k})
\]

is fulfilled. Furthermore, we notice that \(H_{11}(\vec{\sigma}, \vec{k})\) has a sharp maximum at \(k_3 = 2k_0\), implying that the radar sees essentially just refractive-index irregularities at wavelengths close to the so-called Bragg-wavenumber

\[
k_B = 2k_0 = \frac{4\pi}{\lambda}.
\]

### 3.2.1 Simplifying the spectral sampling-function using Dirac’s delta-function

As mentioned above, the spectral sampling-function contains factors of the functional form

\[
f(k) = \exp \left[ -\sigma^2 (k - k_{\text{max}})^2 \right].
\]

It is easy to see that \(f(k)\) has a maximum at \(k = k_{\text{max}}\) and has the width

\[
\Delta k = \frac{2}{\sigma},
\]

where we define the width as the difference between the two \(k\)-values at that \(f(k)\) has the value \(e^{-1}\). In the sampling-functions, \(\sigma\) is a measure of the transverse \((\sigma_I)\) or longitudinal \((\sigma_t)\) size of the observation volume; hence, the normalized width of the spectral sampling-function, i.e., the ratio between the width \(\Delta k\) and the radar wavenumber \(k_0\) is given by

\(^4\)We emphasize that \(H_{11}(\vec{\sigma}, \vec{k})\) is the spectral two-way sampling-function for the monostatic Doppler radar operated in the standard mode in order to distinguish \(H_{11}(\vec{\sigma}, \vec{k})\) from the “spectral cross-sampling-function” \(H_{11}^{\text{FDI}}(\vec{\sigma}, \vec{k})\) for the monostatic Doppler radar operated in the frequency-domain interferometry (FDI) mode, which we will introduce in the next subsection.
If the respective $a$ is sufficiently large as compared to $\lambda$ and if the spatial refractive-index spectrum $\Phi_\alpha(k_1,k_2,k_3)$, which is convolved with the sampling-function for the respective dimension, is sufficiently smooth at wavenumbers around the sampling-function’s maximum, then the factors $f(k)$ may be approximated by Dirac-delta-functions $\delta_D(k - k_{\text{max}})$. When applying this approximation, it has to be made sure that the integral is conserved:

$$\int_{-\infty}^{\infty} \exp \left[ -\sigma^2 (k - k_{\text{max}})^2 \right] dk = \int_{-\infty}^{\infty} c_\delta \cdot \delta_D(k - k_{\text{max}}) dk.$$  \hfill (3.33)

The coefficient $c_\delta$ comes out as

$$c_\delta = \sqrt{\frac{\pi}{\sigma^2}}. \hfill (3.34)$$

That is, the weight of the delta-function is proportional to the width of the Gaussian function to be approximated.

Now, we make use of this procedure and simplify the spectral sampling-function $H_{11}(\vec{\sigma}, \vec{k})$, which at the position $\vec{\sigma}$ has a sharp $\vec{k}$-space maximum at the wave-vector

$$\vec{k}_B(\vec{\sigma}) = 2k_0 \begin{pmatrix} \frac{\sigma_1}{\sigma_0} \\ \frac{\sigma_2}{\sigma_0} \\ 1 \end{pmatrix}. \hfill (3.35)$$

We call $\vec{k}_B(\vec{\sigma})$ the “local Bragg wave-vector”. After elementary manipulations, we obtain

$$H_{11}(\vec{\sigma}, \vec{k}) = (2\pi)^3 A^2 f_\alpha^4(\vec{\sigma})|W(\vec{\sigma})|^2 \delta_D(\vec{k} - \vec{k}_B(\vec{\sigma})). \hfill (3.36)$$

### 3.3 Frequency-domain interferometry (FDI)

In contrast to the standard mode, one measures quasi-simultaneously two or more weather signals $I_1(t), I_2(t), ...$ when operating a monostatic Doppler radar in the frequency-domain-interferometry (FDI) mode.

Now, let $I_1(t)$ be the weather signal measured at the radar wavenumber $k_0$, and let $I_2(t)$ be the signal measured at another wavenumber $k_0 + \Delta k$:

$$I_1(t) = \iint_{-\infty}^{\infty} G_1(\vec{r}) n'(\vec{r}, t) d^3r \hfill (3.37)$$
and

\[ I_2(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_2(\vec{r}) n'(\vec{r}, t) d^3r, \quad (3.38) \]

where

\[ G_1(\vec{r}) = A \cdot \exp \left( -\frac{t_1^2 + t_2^2}{2\sigma_1^2} \right) \cdot \exp \left( -\frac{l_1^2}{4\sigma_1^2} \right) \cdot \exp \left[ -j2k_0 \left( l + \frac{t_1^2 + t_2^2}{2r_0} \right) \right] \cdot \exp (-j2k_0 r_0) \quad (3.39) \]

and

\[ G_2(\vec{r}) = A \cdot \exp \left( -\frac{t_1^2 + t_2^2}{2\sigma_1^2} \right) \cdot \exp \left( -\frac{l_2^2}{4\sigma_2^2} \right) \cdot \exp \left[ -j2(k_0 + \Delta k) \left( l + \frac{t_1^2 + t_2^2}{2r_0} \right) \right] \cdot \exp (-j2(k_0 + \Delta k) r_0), \quad (3.40) \]

are the two spectral one-way sampling-functions. The cross-covariance function of the two weather signals is (compare Sec. 3.2) given by

\[ C_{12}(\tau) = \int_{-\infty}^{\infty} I_1^*(t) I_2(t + \tau) dt \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{12}(\vec{\sigma}, \vec{\delta}) R_{11}(\vec{\sigma}, \vec{\delta}, \tau) d^3\delta d^3\sigma, \quad (3.41) \]

where we identify the function

\[ G_{12}(\vec{\sigma}, \vec{\delta}) = G_1^*(\vec{r}) G_2(\vec{r}'), \quad (3.42) \]

which we call the “FDI cross-sampling-function”.

If one first writes \( G_1^* \) in terms of \( t_1, t_2, l \) and \( G_2^* \) in terms of \( t_1', t_2', l' \) and then returns to sum- and difference coordinates, one gets

\[ G_{12}(\vec{\sigma}, \vec{\delta}) = A^2 \cdot \exp \left[ -\frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2\sigma_1^2} - \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2\sigma_1^2} \right] \cdot \exp \left[ -j2\Delta k \left( r_0 + \sigma_3 - \frac{\sigma_3}{2} + \frac{\sigma_3^2}{4} + \frac{\sigma_3^2}{4} - \sigma_1 \delta_1 - \sigma_2 \delta_2 \right) \right] \]

\[ \times \exp \left[ -j2\Delta k \left( r_0 + \sigma_3 - \frac{\sigma_3}{2} + \frac{\sigma_3^2}{4} + \frac{\sigma_3^2}{4} - \sigma_1 \delta_1 - \sigma_2 \delta_2 \right) \right] \]

\[ = G_{11}(\vec{\sigma}, \vec{\delta}) \cdot \exp \left[ -j2\Delta k \left( r_0 + \sigma_3 - \frac{\sigma_3}{2} + \frac{\sigma_3^2}{4} + \frac{\sigma_3^2}{4} - \sigma_1 \delta_1 - \sigma_2 \delta_2 \right) \right]. \quad (3.43) \]
Thus, the cross-sampling-function for the monostatic FDI may be written as the product of the monostatic standard two-way sampling-function $G_{11}(\bar{\sigma}, \bar{\delta})$ and a phase factor $e^{i\varphi}$, where $\varphi$ is a function in the six-dimensional $\sigma$-$\delta$-space.

Now, we derive the spectral cross-sampling-function

$$H_{12}(\bar{\sigma}, \bar{k}) = \iiint e^{-j\bar{\delta} \cdot \bar{k}} G_{12}(\bar{\sigma}, \bar{\delta}) d^3 \delta,$$

i.e. the Fourier transform of $G_{12}(\bar{\sigma}, \bar{\delta})$ with respect to the lag-space ($\delta$) coordinates.

Again, we make use of

$$\int_{-\infty}^{\infty} \exp \left( -a^2 x^2 - bx \right) dx = \sqrt{\frac{\pi}{a^2}} \exp \left( \frac{b^2}{4a^2} \right),$$

where $a^2$ and $b$ may be complex, and we find after some rearranging:

$$H_{12}(\bar{\sigma}, \bar{k}) = 4(2\pi)^{\frac{3}{2}} \sigma_t^2 \sigma_r \cdot \frac{1}{1 + j \frac{\Delta k \sigma_t^2}{r_0}} A^2 f_0^4(\sigma)|W(\sigma)|^2$$

$$\times \exp \left[ -j2k \left( r_0 + \sigma_3 + \frac{\sigma_t^2 + \sigma_r^2}{2r_0} \right) \right]$$

$$\times \exp \left[ -\frac{\sigma_t^2}{1 + j \frac{\Delta k \sigma_t^2}{r_0}} \frac{(k_1 - (2k_0 + \Delta k) \frac{\sigma_t}{r_0})^2}{1 + j \frac{\Delta k \sigma_t^2}{r_0}} \right]$$

$$-2\sigma_r^2 (\sigma_t - (2k_0 + \Delta k))^2 \right].$$

(3.46)

It is easy to see that, in the case $\Delta k = 0$, the spectral FDI cross-sampling-function $H_{12}(\bar{\sigma}, \bar{k})$ is identical to the standard spectral two-way sampling-function $H_{11}(\bar{\sigma}, \bar{k})$.

3.3.1 Simplifying the spectral FDI cross-sampling-function

As we have shown in Sec. 3.2.1 when simplifying the spectral standard two-way sampling-function $H_{11}(\bar{\sigma}, \bar{k})$, factors of the functional form $\exp(-\sigma^2 k^2)$ can be approximated by Dirac delta-functions. Now, we apply this procedure in order to simplify the spectral FDI cross-sampling-function $H_{12}(\bar{\sigma}, \bar{k})$.

First, we assume $\Delta k \ll 2k_0$, such that

$$2k_0 + \Delta k \approx 2k_0$$

(3.47)
is fulfilled, and the three Gaussian factors in (3.46) may be simplified as follows:

\[
\exp \left[ -\sigma_t^2 (k_1 - 2k_0 \sigma_1 r_0 )^2 - \sigma_t^2 (k_2 - 2k_0 \sigma_2 r_0 )^2 - \sigma_t^2 (k_3 - 2k_0 )^2 \right] \\
\approx \sqrt{\frac{\pi}{\sigma_t^2}} \cdot \sqrt{\frac{\pi}{\sigma_t^2}} \cdot \sqrt{\frac{\pi}{2\sigma_t^2}} \cdot \delta_D (k_1 - 2k_0 \sigma_1 r_0 ) \cdot \delta_D (k_2 - 2k_0 \sigma_2 r_0 ) \cdot \delta_D (k_3 - 2k_0 )
\]

(3.48)

where

\[
\sigma_t^2 = \frac{\sigma_t^2}{1 + j\frac{\Delta k \sigma_t}{r_0}}.
\]

After inserting into (3.46) we obtain

\[
H_{12}(\bar{\sigma}, \bar{k}) = (2\pi)^2 A^2 f_\phi(\bar{\sigma}) |W(\bar{\sigma})|^2 \delta_D (\bar{k} - \bar{k}_B(\bar{\sigma})) \\
\times \exp \left[ -j2\Delta k \left( r_0 + \sigma_3 + \frac{\sigma_1^2 + \sigma_2^2}{2r_0} \right) \right] \\
= H_{11}(\bar{\sigma}, \bar{k}) \cdot \exp \left[ -j2\Delta k \left( r_0 + \sigma_3 + \frac{\sigma_1^2 + \sigma_2^2}{2r_0} \right) \right],
\]

(3.50)

where \( H_{11}(\bar{\sigma}, \bar{k}) \) is the simplified spectral standard two-way sampling-function (Sec. 3.2.1). Obviously, \( H_{12}(\bar{\sigma}, \bar{k}) \) and \( H_{11}(\bar{\sigma}, \bar{k}) \) differ just by the phase factor \( \exp(-j2\Delta k r_t(\bar{\sigma})) \), where

\[
r_t(\bar{\sigma}) = r_0 + \sigma_3 + \frac{\sigma_1^2 + \sigma_2^2}{2r_0}
\]

(3.51)

is the distance between the center of the antenna and the point \( \bar{\sigma} \) within the observation volume.

In a monostatic frequency-domain interferometry (FDI) one measures with the same antenna quasi-simultaneously two different weather signals \( I_1(t) \) and \( I_2(t) \) at two slightly different wavenumbers \( k_0 \) and \( k_0 + \Delta k \), respectively. The phase differences between \( I_1(t) \) and \( I_2(t) \) contain information about the radial distribution of the refractive-index variance within the observation volume. If this radial distribution has a pronounced maximum at a certain range, it is possible to determine unambiguously the radial coordinate of the position of the scattering center within the observation volume from the phase of the zeroth moment of the FDI Doppler cross-spectrum.

3.4 Spatial interferometry (SI)

The spaced-antenna interferometry ("spatial interferometry", SI, or "spaced-antenna technique") is, in a certain sense, the transverse counterpart of the frequency-domain interferometry (FDI). While FDI provides information about the radial (but not the transverse) component
of the position of the scattering center within the observation volume, SI provides information about the two transverse (but not the radial) components of the position of the scattering center. That is, FDI and SI complement ideally. The first FDI/SI experiment was carried out in February 1989 at the Jicamarca Radio Observatory, Peru, by STITT & KUDEKI (1991).

In this Subsection, we derive the physical-space and spectral SI cross-sampling-functions \(G_{12}(\vec{\sigma}, \bar{\delta})\) and \(H_{12}(\vec{\sigma}, \bar{k})\), respectively. In doing so, our starting-point is the work by DOVIAK et al. (1996), but later, we will simplify the formalism in a way that enables us to derive explicitly the general form of the SI sampling-functions, without restricting ourselves to the special cases of homogeneity and/or isotropy as in DOVIAK et al. (1996). In a recent work, Holloway et al. (1997) extended DOVIAK et al.’s (1996) theory to the somewhat more general case of vertical anisotropy.

We consider the weather signal \(I_1(t)\) which is measured by a receiving antenna \(R1\) at the position \((\vec{r}_0 + \vec{a}_1)\). Again, \(\vec{r}_0\) is the position of the (single) transmitting antenna, where, as before, the origin of the coordinate system is at the center of the observation volume. The vector

\[
\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}
\]

(3.52)
is the position of the receiving antenna \(R1\) with respect to the transmitting antenna, see Fig. (3.2).

Here, we allow, in contrast to DOVIAK et al. (1996), the vector \(\vec{a}_1\) to be three-dimensional in order to not unnecessarily restrict the analysis to vertically pointing spaced-antenna configurations. As in the monostatic case, the instantaneous weather signal received by a slightly displaced receiving antenna may be written as a volume integral over the instantaneous refractive-index irregularity field \(n'(\vec{r}, t)\) (see DOVIAK et al. 1996):

\[
I_1(t) = \frac{2\pi \sqrt{g_T g_1}}{k_0 r_0^2} \left( \frac{P_T}{2R} \right)^{\frac{1}{2}} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int |W(\vec{r})| f_{\Theta T}(\vec{r}) f_{\Theta R1}(\vec{r}) n'(\vec{r}, t) \exp \left[-jk_0(r_T + r_1)\right] d^3r. \quad (3.53)
\]

Here, \(g_T\) and \(g_1\) are the gains of the transmitting antenna \(T\) and of the receiving antenna \(R1\), respectively, and the antenna patterns of \(T\) and \(R1\) are given by

\[
f_{\Theta T} = \exp \left( -\Theta_0^2 \frac{\Theta_T^2}{8\sigma_{\Theta T}^2} \right) = \exp \left( -\frac{t_0^2 + t_T^2}{4\sigma_{\Theta T}^2} \right) \quad (3.54)
\]

and
Fig. 3.2: Sketch of the coordinate system used to describe the spatial distribution of the refractive-index irregularities as observed by a spaced-antenna configuration, compare DOVIAK et al. (1996).

\[ f_{\theta R1} = \exp \left( -\frac{(t_1 - x_1)^2 + (t_2 - y_1)^2}{4\sigma_{tR}^2} \right), \]  

respectively, where

\[ \sigma_{tT} = \sqrt{2} r_0 \sigma_{\theta T} \]  

and

\[ \sigma_{tR} = \sqrt{2} r_0 \sigma_{\theta R} \]
define the transverse diameter of the effective observation volume.

As DOVIAK et al. (1996) did, we assume that the gain (and, therefore, also $\sigma_\Theta R$) is the same for all receiving antennas.

For many SI configurations

$$|\vec{a}| \ll \sigma_T$$

(3.58)

is fulfilled, so that we can considerably simplify $f_{\Theta R1}(\vec{r})$:

$$f_{\Theta R1}(\vec{r}) = \exp \left( -\frac{t_1^2 + t_2^2}{4\sigma_{T2}^2} \right),$$

(3.59)

and we obtain the composite one-way antenna pattern

$$f_\Theta^2(\vec{r}) = f_{\Theta T}(\vec{r}) f_{\Theta R1}(\vec{r}) = \exp \left( -\frac{t_1^2 + t_2^2}{2\sigma_t^2} \right),$$

(3.60)

where

$$\frac{1}{2\sigma_t^2} = \frac{1}{4\sigma_{TT}^2} + \frac{1}{4\sigma_{TR}^2}.$$  (3.61)

In the case $\sigma_{TT} = \sigma_{TR}$, we have $\sigma_t = \sigma_{TT} = \sigma_{TR}$. Thus, in the case of a small spacing, $|\vec{a}_1| \ll \sigma_T$, we may write $I_1(t)$ as follows:

$$I_1(t) = A \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |W(\vec{r})| f_\Theta^2(\vec{r}) n'(\vec{r}, t) \exp [-jk_0(r_T + r_1)] d^3r,$$

(3.62)

where

$$|W(\vec{r})| = \exp \left( -\frac{t^2}{4\sigma_t^2} \right)$$

(3.63)

is the radial one-way weighting-function,

$$f_\Theta^2(\vec{r}) = \exp \left( -\frac{t_1^2 + t_2^2}{2\sigma_t^2} \right)$$

(3.64)

is the composite one-way antenna pattern with

$$\frac{1}{\sigma_t^2} = \frac{1}{2\sigma_{TT}^2} + \frac{1}{2\sigma_{TR}^2},$$

(3.65)
and where

\[ A = \frac{g}{\lambda r_0} \left( \frac{P_t}{2R} \right)^{\frac{1}{2}} \]  

(3.66)

is a constant into which several radar parameters have been absorbed. The composite gain \( g \) is the geometric mean of the gains of the transmitting antenna and the receiving antenna:

\[ g = \sqrt{g_T g_R}. \]  

(3.67)

Apart from the phase factor \( \exp \left[ -jk_0(r_T + r_1) \right] \), the expression for \( I_1(t) \) in (3.62) is identical to the corresponding expressions for the monostatic radar in the standard and FDI modes, respectively (see Secs. 3.2 and 3.3).

Now, we expand the optical path \( r_T + r_1 \) up to the second order with respect to the coordinates \( t_1, t_2, l \):

\[ r_T + r_1 = \left( r_0 + l + \frac{t_1^2 + t_2^2}{2r_0} \right) + \left( r_0 + (l - z_1) + \frac{(t_1 - x_1)^2 + (t_2 - y_1)^2}{2r_0} \right). \]  

(3.68)

We neglect the terms \( x_1^2/(2r_0) \) and \( y_1^2/(2r_0) \) to obtain

\[ r_T + r_1 = 2 \left( r_0 + l + \frac{t_1^2 + t_2^2}{2r_0} \right) - \left( z_1 + \frac{x_1 t_1 + y_1 t_2}{r_0} \right), \]  

(3.69)

and we find

\[ I_1(t) = \int_\infty^{-\infty} \int_\infty^{-\infty} G_1(\vec{r}) n'(\vec{r}, t) d^3r, \]  

(3.70)

where

\[
G_1(\vec{r}) = \exp \left[ -\frac{t_1^2 + t_2^2}{2\sigma_i^2} - \frac{l^2}{4\sigma_r^2} \right] \times \exp \left[ -j2k_0 \left( r_0 + l + \frac{t_1^2 + t_2^2}{2r_0} - \frac{z_1}{2} - \frac{x_1 t_1 + y_1 t_2}{2r_0} \right) \right] \]  

(3.71)

is the composite one-way SI sampling-function.

Hence, the SI cross-sampling-function is given by
\[ G_{g_1}^{g_2}(r, r') = G_1^1(r)G_2^1(r') = A^2 \exp \left[ \frac{-t_1^2 + t_1'^2 + t_2^2 + t_2'^2}{2 \sigma_1^2} - \frac{t^2 + t'^2}{4 \sigma_r^2} \right] \times \exp \left[ jk_0 \left( r_T - r_T' + r_1 - r_2' \right) \right], \quad (3.72) \]

where

\[
G(r, r') = \frac{A}{\sqrt{2 \pi} \sigma_r} \exp \left[ -\frac{t^2 + t'^2}{2 \sigma_r^2} \right] \times \exp \left[ jk_0 \left( r_T - r_T' + r_1 - r_2' \right) \right],
\]

\[
r_T + r_1 - r_T' - r_2' = \left( l - l' + \frac{t_2 - t_1'}{2 \tau_0} \right) - \left( z_1 - z_2 + \frac{x_1 t_1 - x_2 t_1' + y_1 t_2 - y_2 t_2'}{\tau_0} \right). \quad (3.73)
\]

Here,

\[
\delta_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}
\]

is the position of the second receiving antenna \( R_2 \). Now, we return to sum- and difference-coordinates and obtain

\[
G_{12}^{\delta_1}(\sigma, \delta) = A^2 \exp \left[ \frac{-\sigma_1^2 + \delta_1^2 + \sigma_2^2 + \delta_2^2}{2 \sigma_r^2} - \frac{\sigma_3^2 + \delta_3^2}{2 \sigma_r^2} \right] \times \exp \left[ j2k_0 \left( z_{12} + \frac{\Delta x_{12}}{2} + \frac{\Delta y_{12}}{2} \right) \right], \quad (3.75)
\]

that is

\[
G_{12}^{\delta_1}(\sigma, \delta) = G_{11}(\sigma, \delta) \exp \left[ j2k_0 \left( \frac{\Delta x_{12}}{2} + \frac{\Delta y_{12}}{2} \right) \right], \quad (3.76)
\]

where we have introduced the following vectors which define the relative and absolute position of the two receiving antennas:

\[
\begin{pmatrix} \Delta x_{12} \\ \Delta y_{12} \\ \Delta z_{12} \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}
\]

and

\[
G_{12}(\sigma, \delta) = G_{11}(\sigma, \delta) \exp \left[ j2k_0 \left( \frac{\Delta x_{12}}{2} + \frac{\Delta y_{12}}{2} \right) \right], \quad (3.77)
\]

are valid.
respectively. We perform the Fourier transformation of \( G_{12}^{SL}(\vec{\sigma}, \vec{\delta}) \) with respect to the lag-space coordinates and obtain the spectral SI cross-sampling-function:

\[
H_{12}^{SL}(\vec{\sigma}, \vec{k}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j\vec{k} \cdot \vec{\delta}} G_{12}^{SL}(\vec{\sigma}, \vec{\delta}) d^3 \delta
\]

where

\[
\begin{align*}
B(\vec{\sigma}) &= |W(\vec{\sigma})|^2 f_{\delta}^{\vec{\delta}}(\vec{\sigma}) = \exp \left[ -\frac{\sigma_1^2 + \sigma_2^2}{\sigma_t^2} - \frac{\sigma_3^2}{2\sigma_t^2} \right] \\
&= 4(2\pi)^{\frac{3}{2}} \sigma_t^2 \sigma_r A^2 B(\vec{\sigma}) \exp \left[ jk_0 \left( \Delta z_{12} + \frac{\Delta x_{12} \sigma_1 + \Delta y_{12} \sigma_2}{r_0} \right) \right] \\
&\times \exp \left[ -\sigma_t^2 \left( k_1 - 2k_0 \frac{\sigma_1 - \bar{z}_{12}}{r_0} \right)^2 - \sigma_t^2 \left( k_2 - 2k_0 \frac{\sigma_2 - \bar{y}_{12}}{r_0} \right)^2 \right. \\
&\left. - 2\sigma_t^2 (k_3 - 2k_0)^2 \right],
\end{align*}
\]

(3.79)

is the composite two-way weighting-function.

### 3.4.1 Simplifying the spectral SI cross-sampling-function

If the assumptions \( \sigma_t \gg \lambda \) and \( \sigma_r \gg \lambda \) are fulfilled, we again may replace the Gaussian functions in (3.79) with Dirac delta-functions. Assuming small antenna spacings,

\[
|a_1| \ll \sigma_t
\]

(3.81)

and

\[
|a_2| \ll \sigma_t,
\]

(3.82)

we may neglect \( \bar{x}_{12} \) and \( \bar{y}_{12} \) in (3.79), thus obtaining

\[
\begin{align*}
H_{12}^{SL}(\vec{\sigma}, \vec{k}) &= (2\pi)^3 A^2 B(\vec{\sigma}) \delta_D(\vec{k} - \vec{k}_B(\vec{\sigma})) \\
&\times \exp \left[ jk_0 \left( \Delta z_{12} + \frac{\Delta x_{12} \sigma_1 + \Delta y_{12} \sigma_2}{r_0} \right) \right]
\end{align*}
\]

(3.80)
\[ H_{11}(\sigma, \bar{k}) \cdot \exp \left[ j k_0 \left( \Delta z_{12} + \frac{\Delta x_{12} \sigma_1 + \Delta y_{12} \sigma_2}{r_0} \right) \right]. \] (3.83)

Similar to the FDI cross-sampling-functions in the case of zero frequency spacing ($\Delta k = 0$), in the case of zero antenna spacing ($\Delta x_{12} = \Delta y_{12} = \Delta z_{12} = 0$) also the SI cross-sampling-functions are identical to the standard two-way sampling-functions.
4 The Time-Lag Derivatives of the Local Space-Time Autocovariance Function of the Refractive Index

As shown in the previous Section, the temporal auto- and cross-covariance functions of the weather signal(s) may be written as follows:

\[ C_{12}(\tau) = \iiint_{-\infty}^{\infty} G_{12}(\sigma, \delta) R_n(\sigma, \delta, \tau) \, d^3 \delta \, d^3 \sigma, \]  

(4.1)

where

\[ R_n(\sigma, \delta, \tau) = \langle n'(\sigma, t) n'(\sigma + \delta, t + \tau) \rangle_t \]  

(4.2)

is the seven-dimensional local space-time autocovariance function of the refractive index. Now, we consider the \( m \)th \( \tau \)-derivative of \( C_{12}(\tau) \) at zero time-lag \( (\tau=0) \):

\[ \left. \frac{\partial^m}{\partial \tau^m} C_{12}(\tau) \right|_{\tau=0} = \iiint_{-\infty}^{\infty} G_{12}(\sigma, \delta) \left. \frac{\partial^m}{\partial \tau^m} R_n(\sigma, \delta, \tau) \right|_{\tau=0} \, d^3 \delta \, d^3 \sigma, \]  

(4.3)

or in a more compact notation:

\[ C_{12}^{(m)}(0) = \iiint_{-\infty}^{\infty} G_{12}(\sigma, \delta) R_n^{(m)}(\sigma, \delta, 0) \, d^3 \delta \, d^3 \sigma. \]  

(4.4)

Once we know the \( m \)th \( \tau \)-derivative of \( R_n(\sigma, \delta, \tau) \), we know also \( C_{12}^{(m)}(0) \) and, therefore, also the \( m \)th moment of the Doppler spectrum since the function \( G_{12}(\sigma, \delta) \) is known (see Sec. 3) and does not depend on \( \tau \). We obtain

\[ \left. \frac{\partial^m}{\partial \tau^m} R_n(\sigma, \delta, \tau) \right|_{\tau=0} = \left. \frac{\partial^m}{\partial \tau^m} \left( n'(\sigma, t) n'(\sigma + \delta, t + \tau) \right) \right|_{\tau=0} = \left. \left( n'(\sigma, t) \frac{\partial^m}{\partial \tau^m} n'(\sigma + \delta, t + \tau) \right) \right|_{\tau=0} = \left. \left( n'(\sigma, t) \frac{\partial^m}{\partial \tau^m} n'(\sigma + \delta, t) \right) \right|_{\tau=0}, \]  

(4.5)

i.e., we need to know the covariance of the refractive-index irregularities at \( \sigma \) and the \( m \)th local temporal derivative of the refractive-index irregularities at \( \sigma + \delta \).

\(^5\) Here we give explicitly only the cross-covariance function \( C_{12}(\tau) \) since the autocovariance function \( C_{11}(\tau) \) is a special case of \( C_{12}(\tau) \).
Before we examine this problem in detail, in Sec. 4.1 we will give a short review of the mechanisms which generate atmospheric refractive-index irregularities. In Sec. 4.2, we follow TATARSKII (1961) and write the refractive-index irregularities $n'$ in terms of conserved quantities and their fluctuations, and we will show that $n'$ may be well approximated as the fluctuations of the generalized potential refractive index. At the end of this Section, we will show that the lag-space Fourier transform of the refractive-index autocovariance function $R_n(\vec{\sigma}, \vec{\delta}, \tau)$ at zero time-lag ($\tau=0$) and the lag-space Fourier transform of the $\tau$-derivative of $R_n(\vec{\sigma}, \vec{\delta}, \tau)$ at zero time-lag may be written in terms of the spatial variance- and cross-spectra of the wind-vector- and refractive-index irregularities.

4.1 Generation and modification of refractive-index irregularities in the optically clear atmosphere

4.1.1 Turbulence, waves and differential advection

Small-scale refractive-index irregularities in the optically clear atmosphere can be generated and modified by a variety of different mechanisms. They can be roughly divided into three categories: turbulence, waves, and differential advection.

Turbulent fluctuations of the refractive index are practically ubiquitous in the atmosphere (see, e.g., GOSSARD 1960). The statistical properties of atmospheric turbulence, in particular in the planetary boundary layer, have been investigated in a vast amount of experimental studies using in-situ sondes (e.g., ROTH 1964; KAIMAL et al. 1972; WYNGAARD et al. 1978; LENSCHOW et al. 1980; YAGLOM 1981; HAUF 1984; ENGELBART 1989; KOTTMEIER & ENGELBART 1992; QUANTE et al. 1996; GRUNWALD et al. 1996; MAI et al. 1996; KRAMM et al. 1996; WILLIAMS et al. 1996; BANGE et al. 1997).

Nowadays, it is possible to study atmospheric turbulence also on the basis of large-eddy simulations (LES) (see, e.g., MOENG & WYNGAARD 1988, SCHMIDT & SCHUMANN 1989, MAISON 1994 and MUSCHINSKI 1996a). YAGLOM (priv. comm., May 1997) expects more accurate results for the values of the quasi-universal turbulence coefficients (Kolmogorov-coefficient, von-Kármán-coefficient etc.; see, e.g., MUSCHINSKI & ROTH 1993) in the future from numerical simulations rather than from experiments in the laboratory or in the atmosphere. See also the recent LES of the logarithmic wind-profile and of the von-Kármán-coefficient by CAI (1996).

Waves are omnipresent phenomena in the stably stratified atmosphere (GOSSARD & HOOKE 1975), and there are many indications that gravity-wave breaking is one of the most important mechanisms of turbulence generation in the atmosphere (see, e.g., FRITTS & RASTOGI 1985, WEINSTOCK 1987, NASTROM & EATON 1993, MUSCHINSKI 1997).

A very interesting mechanism for the generation of horizontally elongated refractive-index discontinuities, which has nearly been ignored by the clear-air-radar community, is differential advection. If initially more or less randomly oriented inhomogeneities (generated by convection or by convectively or dynamically unstable waves) are horizontally advected by a large-scale, vertically inhomogeneous horizontal wind, those inhomogeneities are stretched out to thin laminae (e.g., SCHMIDT 1916, PALUCH et al. 1992). In the case of stable stratification and weak "background turbulence", such laminae may have life-times on the order of several days. Whether the
layers observed with VHF-FDI (e.g., MUSCHINSKI et al. 1996 and MUSCHINSKI et al. 1998) and the meter-scale (DALAUDIER et al. 1994) and sub-meter-scale laminae (MUSCHINSKI & WODE 1997, 1998) observed with in-situ sondes are to be attributed to differential advection, is still an open question.

All those processes may contribute to the generation of atmospheric small-scale refractive-index irregularities, the space-time autocovariance function of which is known only for some ideal cases (see GURVICH 1997).

4.1.2 Fossil and active turbulence in the stably stratified atmosphere

Turbulence in stably stratified shear flows is sporadically generated by dynamic or convective instability, and it is organized in thin layers (see, e.g., the review article by FRITTS & RASTOGI 1985). Since in stable stratification (large gradient-Richardson-number) the turbulent Prandtl-number is larger than 1 (WITTICH & ROTH 1984; SCHUMANN & GERZ 1995) the velocity irregularities decay more rapidly than the irregularities of passive scalars (e.g., the potential temperature, the specific humidity, and other constituents of the air), see also DOVIAK & ZRNIĆ (1993, p. 476). After some time, the irregularities of passive scalars are advected as “fossil turbulence” by the local velocity field, which is no longer turbulent. Later, the refractive-index perturbations $n'$ decay due to molecular diffusion, and the fine-scale structure of the $n'$-field vanishes at a time-scale $\tau$ which depends on the length scale (METCALF & ATLAS 1973, p. 28; BATECHELOR 1953, p. 93):

$$\tau = \frac{1}{2DK^2}.$$ (4.6)

Here, $K = 2\pi/\Lambda$ is the wavenumber, and $D$ is the molecular diffusivity of the respective passive scalar. That is, the small-scale irregularities disappear more rapidly than the large-scale irregularities. Clear-air radars “see” just the irregularities at wavenumbers close to the radar’s Bragg-wavenumber $k_B = 4\pi/\lambda$. Hence, the life-time of the fossil irregularities seen by the radar is a function of the radar’s wavelength $\lambda$:

$$\tau_r = \frac{1}{2Dk_B^2} = \frac{\lambda^2}{32\pi^2D}.$$ (4.7)

From the assumption that a clear-air radar in the lower troposphere sees essentially water-vapor irregularities ($D = 2.4 \cdot 10^{-5} \text{m}^2 \text{s}^{-1}$, see METCALF & ATLAS 1973), we obtain estimates for the life-times as a function of the radar wavelength: $\tau_r = 1.3 \text{ s}$ for $\lambda = 10 \text{ cm}$, $\tau_r = 130 \text{ s}$ for $\lambda = 1 \text{ m}$ and $\tau_r = 79 \text{ min}$ for $\lambda = 6 \text{ m}$, respectively.

This has an important implication for the interpretation of UHF/VHF Doppler-radar observations: it is tempting to presume that short-wavelength radars (wavelength around 10 cm) are sensitive mostly to active turbulence, whereas for VHF radars fossil turbulent irregularities
might be more relevant. Possibly, the long life-time of “fossil turbulence” at Bragg wavelengths \((\lambda/2)\) around 3 m (life-time \(\tau\), around 1 h) is the explanation for the observational fact that there are practically always VHF-radar echoes from the stably stratified atmosphere (DOVIAK & ZRNIĆ 1993, p. 476). Direct observational evidence for the relevance of “fossil turbulence” for clear-air-radar echoes, however, is still lacking (DOVIAK, priv. comm., March 1996).

4.2 The refractive index as a function of pressure, generalized potential temperature, and specific humidity

In the troposphere and stratosphere, where the effect of electrically charged particles on clear-air-radar echoes is negligible, the refractive index \(n\) is a function of the pressure \(p\), the (absolute) temperature \(T\) and the water-vapor partial pressure \(e\). A good approximation is

\[
    n = 1 + a \cdot \frac{p}{T} + b \cdot \frac{e}{T^2},
\]

where the constants \(a\) and \(b\) are given by

\[
    a = 7.76 \cdot 10^{-7} \frac{K}{Pa},
\]

and

\[
    b = 3.73 \cdot 10^{-3} \frac{K^2}{Pa},
\]

see DOVIAK & ZRNIĆ (1993, p. 16, eq. 2.19).

Here we give the instructive example by DOVIAK & ZRNIĆ (1993, p. 17): for \(e = 10\, hPa\), \(p = 1000\, hPa\) and \(T = 300\, K\), the refractive index amounts to

\[
    n = 1 + 2.59 \cdot 10^{-4} + 4.14 \cdot 10^{-5} = 1 + 3.00 \cdot 10^{-4}. \quad (4.11)
\]

In this case, which is typical for the lower troposphere during a summer day in the temperate latitudes, the contribution from the air density \((2.59 \cdot 10^{-4})\) is about 6 times as much as the humidity contribution \((4.15 \cdot 10^{-5})\). Both contributions together, however, amount to a relative refractive-index change of 0.03% as compared to the vacuum refractive index. Hence, typical variations of temperature and humidity lead only to \(n\)-variations in the part-per-million range.

\[
\text{METCALF & ATLAS (1973, p. 28) exclude explicitly the importance of fossil turbulence for clear-air-radar echoes at radar wavelengths of 10 cm: "... and the time scale } T = 1.31 \text{ s. Thus the observation of a continuous echo layer which is not due to specular reflection implies continuing production of refractivity fluctuations by continuing turbulence. 'Fossil turbulence' must therefore be excluded as a source of clear-air echoes." On the other hand, one has to bear in mind that atmospheric VHF radars, which are sensitive to irregularities with molecular-diffusion life-times on the order of 1 h, were not known in the early seventies.}
\]
Now, we introduce the generalized potential temperature $\Theta$ (Ottersten 1969, see also Muschinski 1997):

$$\Theta = T \cdot \left( \frac{p_i}{p} \right)^\kappa.$$  \hspace{1cm} (4.12)

Here, $T$ is the actual temperature, $p_i$ the temporal average of the air pressure in the center of the $n'$ layer under observation, $p$ the actual pressure, and $\kappa$ is given by

$$\kappa = \frac{c_p - c_v}{c_p} = \frac{R_d}{c_p} = 0.286.$$ \hspace{1cm} (4.13)

Here, $c_p = 1005 \text{ J/(kg} \cdot \text{K)}$ and $c_v = 718 \text{ J/(kg} \cdot \text{K)}$ are the specific heats of dry air at constant pressure and at constant volume, respectively, and $R_d = 287 \text{ J/(kg} \cdot \text{K)}$ is the specific gas constant for dry air. The numerical values have been taken from Houghton (1977, p. 164).

Using the hydrostatic equation,

$$\frac{\partial p}{\partial z} = -\rho g$$ \hspace{1cm} (4.14)

($g = 9.807 \text{ m} \cdot \text{s}^{-2}$ is the acceleration due to gravity), and the gas equation for dry air,

$$p = \rho R_d T$$ \hspace{1cm} (4.15)

($\rho$ is the air density), and after differentiating with respect to the altitude $z$ we obtain:

$$\frac{\partial \Theta}{\partial z} = \frac{\partial T}{\partial z} + \gamma_a,$$ \hspace{1cm} (4.16)

where

$$\gamma_a = \frac{g}{c_p} = 9.8 \cdot 10^{-3} \frac{\text{K}}{\text{m}}$$ \hspace{1cm} (4.17)

is the adiabatic lapse rate.

In meteorology, it is customary to decompose a fluctuating quantity into its (temporal) mean and its fluctuation:

$$\Theta = \overline{\Theta} + \Theta' = (\overline{T} + T') \cdot \left( \frac{p_i}{\overline{p} + p'} \right)^\kappa$$ \hspace{1cm} (4.18)

where

$$\overline{\Theta} = \overline{T} \cdot \left( \frac{p_i}{\overline{p}} \right)^\kappa.$$ \hspace{1cm} (4.19)
In the case of a thin layer, \( (p_l/p)^k = 1 \) is a good approximation, hence

\[
\Theta = \bar{T}. \tag{4.20}
\]

Calculating the difference between (4.18) and (4.19) and assuming that the fluctuations are small as compared to the mean values provides

\[
\Theta' = T' - \frac{1}{c_p \rho} p'. \tag{4.21}
\]

Neglecting the pressure-fluctuation contribution, we have approximately:

\[
\Theta' = T'. \tag{4.22}
\]

The advantage of using the generalized potential temperature \( \Theta \) is that \( \Theta \) is a conservative quantity (as is the ordinary potential temperature) and that nevertheless, at any altitude, the mean values \( \Theta \) and \( \bar{T} \) are equal to each other. (The latter condition is not fulfilled for the ordinary potential temperature, where \( p_l \) is specified as the pressure at a reference level near the ground.)

Now, we write the water-vapor pressure \( e \) in (4.8) in terms of the specific humidity \( q \). For the water vapor and for dry air (the air without the water vapor), the following gas equations hold:

\[
e = \rho_w R_w T \tag{4.23}
\]

and

\[
p_d = \rho_d R_d T. \tag{4.24}
\]

The specific humidity is

\[
q = \frac{\rho_w}{\rho_w + \rho_d} = \frac{1}{1 + \frac{\rho_w}{\rho_d} \frac{p_d}{e}}. \tag{4.25}
\]

Since

\[
p = p_d + e, \tag{4.26}
\]

we have approximately

\[
e = \frac{R_w}{R_d} qp = 1.608 qp \tag{4.27}
\]
if $e \ll p_d$ is fulfilled. From (4.8), (4.9) and (4.10), it follows

$$n = 1 + c_d \frac{p}{\Theta^2} + c_w \frac{pq}{\Theta^2},$$

(4.28)

where

$$c_d = a = 7.76 \cdot 10^{-7} \text{KPa}^{-1}$$

(4.29)

and

$$c_w = \frac{R_W}{R_d} = 6.00 \cdot 10^{-3} \text{KPa}^{-1}.$$

(4.30)

are universal parameters.

4.3 Local temporal changes of the refractive index

Ignoring the contribution of pressure fluctuations, (4.28) enables us to write the local temporal refractive-index derivatives in terms of the two conserved quantities $\Theta$ and $q$ as follows:

$$\frac{\partial n}{\partial t} = -\frac{c_d p}{\Theta^2} \frac{\partial \Theta}{\partial t} - \frac{2 c_w q p}{\Theta^2} \frac{\partial \Theta}{\partial t} + \frac{c_w q p}{\Theta^2} \frac{\partial q}{\partial t},$$

(4.31)

It is easy to verify that the ratio $T_1/T_2$ of the first and second terms on the right-hand side amounts to about $0.02/q$ if we assume $\Theta \approx 300 \text{K}$. For example, we have $q \approx 0.02$ in the case of a temperature of $30^\circ \text{C}$ and a relative humidity of 70%. That is, the second term dominates over the first term only under very humid and hot conditions. The ratio of the absolute values of the third term, $T_3$, and the first term, $T_1$, is given by

$$\frac{T_3}{T_1} = \frac{c_w}{c_d} \frac{\partial q}{\partial t}.$$

(4.32)

The condition that $T_1$ and $T_3$ are equal to each other is fulfilled if the changes $\Delta q$ and $\Delta \Theta$ are related like

$$\frac{\Delta q}{\Delta \Theta} = \frac{c_d}{c_w} = \frac{1}{7.73 \cdot 10^3 \text{K}}.$$

(4.33)

That is, a temperature fluctuation $\Delta \Theta = 1 \text{K}$ would lead to the same refractive-index fluctuation as a humidity fluctuation $\Delta q = 1.3 \cdot 10^{-4}$, which at $p = 1000 \text{hPa}$ corresponds to a water-vapor
fluctuation of $\Delta e = 0.2 \text{hPa}$. In the case of an air temperature of $20^\circ \text{C}$, this corresponds to a relative-humidity fluctuation of just about 1%. Therefore, it is to be expected that, in a not too dry planetary boundary layer at mid-latitude summer-time temperatures, the humidity term $T_3$ dominates over the term $T_1$.

Finally, we consider the ratio of $T_3$ and $T_2$:

$$\frac{T_3}{T_2} = \frac{1}{2} \frac{\partial^2 q}{\partial \theta^2}.$$

(4.34)

The absolute values of $T_3$ and $T_2$ are equal to each other if the relative change of $q$ is twice as much as the relative change of $\theta$. That is, in the case $\Delta \theta = 3\text{K}$ (ca. 1% relative change), $T_3$ dominates over $T_2$ if the relative humidity change $\Delta q/q$ exceeds 2%.

4.4 The first time-lag derivative of the local space-time autocovariance function of the refractive index

At the beginning of this Section, we have shown that $C_{12}(0), C'_{12}(0), C''_{12}(0)$ etc. may be written as weighted integrals over $R_n(\bar{\sigma}, \bar{\delta}, 0), R'_n(\bar{\sigma}, \bar{\delta}, 0), R''_n(\bar{\sigma}, \bar{\delta}, 0)$ etc. Therefore, in order to specify models for $C_{12}(\tau)$ and its derivatives at zero time-lag ($\tau=0$), we have to specify models for $R_n(\bar{\sigma}, \bar{\delta}, \tau)$ and its $\tau$-derivatives at zero time-lag.

Assuming that $n$ is conserved within the observation volume (compare Secs. 4.2 and 4.3), we may write the local temporal derivative of the $n$-fluctuation as follows (e.g., ETLING 1996, p. 83):

$$\frac{\partial n'}{\partial t} = \frac{\partial n}{\partial t} = -\vec{v} \cdot \nabla n.$$

(4.35)

Here, $\vec{v} = (v_1, v_2, v_3)$ is the field of the instantaneous wind vector within the observation volume. Moreover, (4.3) leads to

$$\frac{\partial}{\partial \tau} R_n(\bar{\sigma}, \bar{\delta}, \tau) \bigg|_{\tau=0} = \left< n'(\bar{\sigma}, t) \frac{\partial}{\partial t} n'(\bar{\sigma} + \bar{\delta}, t) \right>_t$$

$$= \left< n'(\bar{\sigma}, t) \left( -\vec{v}(\bar{\sigma} + \bar{\delta}, t) \cdot \nabla n(\bar{\sigma} + \bar{\delta}, t) \right) \right>_t$$

$$= -\left< n'(\bar{\sigma}, t) \left( [\nabla n(\bar{\sigma} + \bar{\delta}) + \vec{v}'(\bar{\sigma} + \bar{\delta}, t)] \cdot [\nabla n(\bar{\sigma} + \bar{\delta}) + \nabla n'(\bar{\sigma} + \bar{\delta}, t)] \right) \right>_t$$

$$= -\left< n'(\bar{\sigma}, t) \left[ \vec{v}(\bar{\sigma} + \bar{\delta}) \cdot \nabla n(\bar{\sigma} + \bar{\delta}) - \vec{v}'(\bar{\sigma} + \bar{\delta}, t) \cdot \nabla n(\bar{\sigma} + \bar{\delta}) \right] \right>_t$$

$$= -\vec{v}(\bar{\sigma} + \bar{\delta}) \cdot \left< n'(\bar{\sigma}, t) \nabla n'(\bar{\sigma} + \bar{\delta}, t) \right>_t$$

$$-\nabla n(\bar{\sigma} + \bar{\delta}) \cdot \left< n'(\bar{\sigma}, t) \nabla' (\bar{\sigma} + \bar{\delta}, t) \right>_t.$$
The first of the four scalar products vanishes since

\[ \langle n'(\bar{\sigma}, t) \rangle_t = 0. \]  

That is, there remain three scalar products in the expression for \( R'_n(\bar{\sigma}, \bar{\delta}, 0) \). Eventually, we obtain nine terms:

\[
\frac{\partial}{\partial \tau} R_n(\bar{\sigma}, \bar{\delta}, \tau) \bigg|_{\tau=0} = -\sum_{i=1}^{3} \bar{v}_i(\bar{\sigma} + \bar{\delta}) \left\langle n'(\bar{\sigma}, t) \frac{\partial n'_i}{\partial \sigma_i} (\bar{\sigma} + \bar{\delta}, t) \right\rangle_t
- \sum_{i=1}^{3} \frac{\partial n_i}{\partial \sigma_i} (\bar{\sigma} + \bar{\delta}) \left\langle n'(\bar{\sigma}, t) v'_i(\bar{\sigma} + \bar{\delta}, t) \right\rangle_t
- \sum_{i=1}^{3} \left\langle n'(\bar{\sigma}, t) v'_i \frac{\partial n'_i}{\partial \sigma_i} (\bar{\sigma} + \bar{\delta}, t) \right\rangle_t.
\]  

That is, we have to deal with three different types of cross-covariance functions, which now we write as inverse Fourier transforms of spatial cross-spectra. From (2.66) we get

\[
\left\langle n'(\bar{\sigma}, t) \frac{\partial n'_i}{\partial \sigma_i} (\bar{\sigma} + \bar{\delta}, t) \right\rangle_t = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i k \cdot \delta} j k_i \Phi_n(\bar{\sigma}, \bar{k}) d^3 k,
\]  

and (2.48) provides

\[
\left\langle n'(\bar{\sigma}, t) v'_i(\bar{\sigma} + \bar{\delta}, t) \right\rangle_t = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i k \cdot \delta} \Phi_{nv_i}(\bar{\sigma}, \bar{k}) d^3 k.
\]  

Also using (2.48), we obtain for the third type:

\[
\left\langle n'(\bar{\sigma}, t) v'_i \frac{\partial n'_i}{\partial \sigma_i} (\bar{\sigma} + \bar{\delta}, t) \right\rangle_t = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i k \cdot \delta} \Phi_{n(v_i \nabla_i n')} (\bar{\sigma}, \bar{k}) d^3 k.
\]  

In order to further simplify \( R'_n(\bar{\sigma}, \bar{\delta}, 0) \), we make use of the following approximations for the mean fields of wind and refractive index:

\[
\bar{v}_i(\bar{\sigma} + \bar{\delta}) \approx \bar{v}_i(\bar{\sigma})
\]  

and

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\[ \frac{\partial \Omega_i}{\partial \sigma_i} (\bar{\sigma} + \tilde{\delta}) \approx \frac{\partial \Omega_i}{\partial \sigma_i} (\bar{\sigma}), \]  
(4.43)
i.e., we assume that \( \bar{\nu}_i \) and \( \frac{\partial \Omega_i}{\partial \sigma_i} \) are sufficiently smooth functions of \( \bar{\sigma} \). Then, from the foregoing, it follows:

\[ R'_n(\bar{\sigma}, \tilde{\delta}, 0) = -\sum_{i=1}^{3} \left\{ \bar{v}_i(\bar{\sigma}) \int \int \int e^{iK \cdot \tilde{\delta}} jk_i \Phi_n(\bar{\sigma}, \tilde{k}) d^3 k + \frac{\partial \Omega_i}{\partial \sigma_i} (\bar{\sigma}) \int \int \int e^{iK \cdot \tilde{\delta}} \Phi_{nu_i}(\bar{\sigma}, \tilde{k}) d^3 k + \int \int \int e^{iK \cdot \tilde{\delta}} \Phi_{n(v_i \nabla n)}(\bar{\sigma}, \tilde{k}) d^3 k \right\}. \]  
(4.44)

Finally, we obtain

\[ R'_n(\bar{\sigma}, \tilde{\delta}, 0) = \frac{1}{(2\pi)^3} \int \int \int e^{iK \cdot \tilde{\delta}} \times \left\{ - (2\pi)^3 \sum_{i=1}^{3} \left( \bar{v}_i(\bar{\sigma}) jk_i \Phi_n(\bar{\sigma}, \tilde{k}) + \frac{\partial \Omega_i}{\partial \sigma_i} (\bar{\sigma}) \Phi_{nu_i}(\bar{\sigma}, \tilde{k}) + \Phi_{n(v_i \nabla n)}(\bar{\sigma}, \tilde{k}) \right) \right\} d^3 k, \]  
(4.45)

that is, the Fourier transform of \( R'_n(\bar{\sigma}, \tilde{\delta}, 0) \) with respect to the lag-space coordinates \( \tilde{\delta} \) may be written as

\[ \mathcal{F}_{\bar{\delta}} \left\{ R'_n(\bar{\sigma}, \tilde{\delta}, 0) \right\} = - (2\pi)^3 \sum_{i=1}^{3} \left( \bar{v}_i(\bar{\sigma}) jk_i \Phi_n(\bar{\sigma}, \tilde{k}) + \frac{\partial \Omega_i}{\partial \sigma_i} (\bar{\sigma}) \Phi_{nu_i}(\bar{\sigma}, \tilde{k}) + \Phi_{n(v_i \nabla n)}(\bar{\sigma}, \tilde{k}) \right). \]  
(4.46)
5 The Moments of the Variance- and Cross-Spectra of the Doppler Signals

The starting-point of our analysis was the finding that the values of the autocovariance function of the Doppler signal at zero time-lag, i.e., \(C_{11}(0)\), and the values of its \(\tau\)-derivatives at zero time-lag, i.e., \(C'_{11}(0)\), \(C''_{11}(0)\), may be written as 6D (six-dimensional) integrals over 6D products of a 6D "two-way sampling-function" \(G_{11}(\bar{\sigma}, \bar{\delta})\) and the 6D space-time refractive-index autocovariance function at zero time-lag, \(R_n(\bar{\sigma}, \bar{\delta}, 0)\), or its \(\tau\)-derivatives at zero time-lag, respectively, i.e., \(R'_n(\bar{\sigma}, \bar{\delta}, 0)\), \(R''_n(\bar{\sigma}, \bar{\delta}, 0)\) ("atmospheric functions"):

\[
C_{11}(0) = \iiint_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{11}(\bar{\sigma}, \bar{\delta}) R_n(\bar{\sigma}, \bar{\delta}, 0) d\bar{\sigma} d\bar{\delta} d^3 \sigma,
\]

\[
C'_{11}(0) = \iiint_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{11}(\bar{\sigma}, \bar{\delta}) R'_n(\bar{\sigma}, \bar{\delta}, 0) d\bar{\sigma} d\bar{\delta} d^3 \sigma,
\]

\[
C''_{11}(0) = \iiint_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{11}(\bar{\sigma}, \bar{\delta}) R''_n(\bar{\sigma}, \bar{\delta}, 0) d\bar{\sigma} d\bar{\delta} d^3 \sigma. \tag{5.1}
\]

Corresponding equations hold for interferometric modes, i.e., for the cross-covariance functions and its derivatives at zero time-lag:

\[
C_{12}(0) = \iiint_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{12}(\bar{\sigma}, \bar{\delta}) R_n(\bar{\sigma}, \bar{\delta}, 0) d\bar{\sigma} d\bar{\delta} d^3 \sigma,
\]

\[
C'_{12}(0) = \iiint_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{12}(\bar{\sigma}, \bar{\delta}) R'_n(\bar{\sigma}, \bar{\delta}, 0) d\bar{\sigma} d\bar{\delta} d^3 \sigma,
\]

\[
C''_{12}(0) = \iiint_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{12}(\bar{\sigma}, \bar{\delta}) R''_n(\bar{\sigma}, \bar{\delta}, 0) d\bar{\sigma} d\bar{\delta} d^3 \sigma. \tag{5.2}
\]

The moments theorem, (2.25), allows the moments of the variance- and cross-spectra to be determined from measured values of \(C_{11}(0)\), \(C'_{11}(0)\), \(C''_{11}(0)\) and \(C_{12}(0)\), \(C'_{12}(0)\), \(C''_{12}(0)\), respectively. The sampling-functions \(G_{11}(\bar{\sigma}, \bar{\delta})\) and \(G_{12}(\bar{\sigma}, \bar{\delta})\) as well as their Fourier transforms are known. In Sec. 3, we have explicitly derived the most important of them. Furthermore, we have shown in Sec. 4 how the atmospheric functions may be written as inverse Fourier transforms of functions that may be formulated in terms of the spatial variance- and cross-spectra of the wind- and refractive-index irregularities. We have carried out the analysis explicitly and in detail for \(R'_n(\bar{\sigma}, \bar{\delta}, 0)\). It would be straightforward to do so also for \(R''_n(\bar{\sigma}, \bar{\delta}, 0)\) but because of the lengthy analysis, we have not done it here. Since spatial wind- and refractive-index irregularities are customarily described in wavenumber space (and not in lag space) it is helpful to rewrite the lag-space convolution products as wavenumber-space convolution products by using the convolution theorem.
For the “zeroth” moment of the variance spectrum, $S_{11}(\omega)$, we obtain:

$$C_{11}(0) = M_{11}^{(0)} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{11}(\sigma, \tilde{\sigma}) \mathcal{F}_{\xi} \{ R_n(\sigma, \tilde{\sigma}, 0) \} (\sigma, \tilde{\sigma}) d^3k d^3\sigma$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{11}(\sigma, \tilde{\sigma}) \Phi_n(\sigma, \tilde{\sigma}) d^3k d^3\sigma.$$  \hspace{1cm} (5.3)

$C_{11}'(0)$ provides the first moment of $S_{11}(\omega)$:

$$C_{11}'(0) = jM_{11}^{(1)} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{11}(\sigma, \tilde{\sigma}) \mathcal{F}_{\xi} \{ R_n'(\sigma, \tilde{\sigma}, 0) \} (\sigma, \tilde{\sigma}) d^3k d^3\sigma$$

$$= -\sum_{i=1}^{3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{11}(\sigma, \tilde{\sigma}) \left( \overline{\nu_i}(\sigma) j k_i \Phi_n(\sigma, \tilde{\sigma}) \right) d^3k d^3\sigma + \frac{\partial \overline{\nu_i}}{\partial \sigma_i}(\sigma) \Phi_n(\sigma, \tilde{\sigma}) + \Phi_n(\sigma, \tilde{\sigma}) \right) d^3k d^3\sigma., \hspace{1cm} (5.4)$$

that is, we may write the first moment of the Doppler spectrum $S_{11}(\omega)$ as follows:

$$M_{11}^{(1)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{11}(\sigma, \tilde{\sigma}) \left\{ -\overline{\nu_1}(\sigma) k_1 + \overline{\nu_2}(\sigma) k_2 + \overline{\nu_3}(\sigma) k_3 \right\} \Phi_n(\sigma, \tilde{\sigma})$$

$$+ j \frac{\partial \overline{\nu_1}}{\partial \sigma_1}(\sigma) \Phi_n(\sigma, \tilde{\sigma}) + j \frac{\partial \overline{\nu_2}}{\partial \sigma_2}(\sigma) \Phi_n(\sigma, \tilde{\sigma}) + j \frac{\partial \overline{\nu_3}}{\partial \sigma_3}(\sigma) \Phi_n(\sigma, \tilde{\sigma})$$

$$+ j \Phi_n(\sigma, \tilde{\sigma}) \left\{ \Phi_n(\sigma, \tilde{\sigma}) + \Phi_n(\sigma, \tilde{\sigma}) \right\} d^3k d^3\sigma. \hspace{1cm} (5.5)$$

Eventually, we obtain for the second moment of $S_{11}(\omega)$:

$$C_{11}''(0) = -M_{11}^{(2)} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{11}(\sigma, \tilde{\sigma}) \mathcal{F}_{\xi} \{ R_n''(\sigma, \tilde{\sigma}, 0) \} (\sigma, \tilde{\sigma}) d^3k d^3\sigma.$$  \hspace{1cm} (5.6)

As mentioned above, it is possible to determine the Fourier transform of $R_n''(\sigma, \tilde{\sigma}, 0)$ in a similar manner as the Fourier transform of $R_n'(\sigma, \tilde{\sigma}, 0)$. This, however, will not be done in this work.

The moments of the cross-spectra for interferometric applications are obtained by replacing the spectral two-way sampling-function $H_{11}(\sigma, \tilde{\sigma})$ with the spectral cross-sampling-function $H_{12}(\sigma, \tilde{\sigma})$ specified by the respective interferometric sampling-mode. For the sake of completeness, in the following we give the equations for the three first moments of the Doppler cross-spectrum $S_{12}(\omega)$:

$$\textit{Equation for } S_{12}(\omega) \textit{ moments.}$$

$$\textit{Equation for } S_{12}(\omega) \textit{ moments.}$$

$$\textit{Equation for } S_{12}(\omega) \textit{ moments.}$$

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\[ M_{12}^{(0)} = \int_{-\infty}^{\infty} S_{12}(\omega) d\omega = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{12}(\vec{\sigma}, \vec{k}) \Phi_n(\vec{\sigma}, \vec{k}) d^3 k d^3 \sigma \]  

(5.7)

is the zeroth moment of \( S_{12}(\omega) \),

\[ M_{12}^{(1)} = \int_{-\infty}^{\infty} S_{12}(\omega) \omega d\omega = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{12}(\vec{\sigma}, \vec{k}) \left\{ -(\nu_1(\vec{\sigma})k_1 + \nu_2(\vec{\sigma})k_2 + \nu_3(\vec{\sigma})k_3) \Phi_n(\vec{\sigma}, \vec{k}) \right. \\
+ j \frac{\partial \overline{\nu_1}(\vec{\sigma})}{\partial \sigma_1} \Phi_{n v_1}(\vec{\sigma}, \vec{k}) + j \frac{\partial \overline{\nu_2}(\vec{\sigma})}{\partial \sigma_2} \Phi_{n v_2}(\vec{\sigma}, \vec{k}) + j \frac{\partial \overline{\nu_3}(\vec{\sigma})}{\partial \sigma_3} \Phi_{n v_3}(\vec{\sigma}, \vec{k}) \\
+ \left. j \Phi_{n \nu v_1}(\vec{\sigma}, \vec{k}) + j \Phi_{n \nu v_2}(\vec{\sigma}, \vec{k}) + j \Phi_{n \nu v_3}(\vec{\sigma}, \vec{k}) \right\} d^3 k d^3 \sigma. \]  

(5.8)

is the first moment of \( S_{12}(\omega) \), and

\[ M_{12}^{(2)} = \int_{-\infty}^{\infty} S_{12}(\omega) \omega^2 d\omega = -\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{12}(\vec{\sigma}, \vec{k}) \mathcal{F}_g \left\{ R_{n}(\vec{\sigma}, \vec{\delta}, 0) \right\} d^3 k d^3 \sigma \]  

(5.9)

is the second moment of \( S_{12}(\omega) \).

Now, it is formally clear how the spatial variance- and cross-spectra of wind- and refractive-index fluctuations as well as the mean wind- and refractive-index fields project into the first moments of the Doppler variance- and cross-spectra.
6 The Standard Doppler Radar

6.1 The zeroth moment

In this Section, we develop and discuss models for the zeroth and the first moment of the Doppler spectrum \( S_{11}(\omega) \), i.e., of the variance spectrum of the weather signal measured with a standard Doppler radar (monostatic, one-frequency mode).

Earlier, we have shown that the zeroth moment of the Doppler spectrum may be written as follows:

\[
M_{11}^{(0)} = \frac{1}{(2\pi)^3} \iiint H_{11}(\bar{\sigma}, \bar{k}) \Phi_n(\bar{\sigma}, \bar{k}) d^3 k d^3 \sigma, \tag{6.10}
\]

where

\[
H_{11}(\bar{\sigma}, \bar{k}) = (2\pi)^3 A^2 f^B_{\bar{\sigma}}(\bar{\sigma}) |W(\bar{\sigma})|^2 \delta_D(\bar{k} - \bar{k}_B(\bar{\sigma})) \tag{6.11}
\]

is the standard two-way spectral sampling-function, and

\[
\bar{k}_B(\bar{\sigma}) = 2k_0 \begin{pmatrix} \frac{\sigma_1}{r_0} \\ \frac{\sigma_2}{r_0} \\ 1 \end{pmatrix} \tag{6.12}
\]

is the local Bragg wave-vector.

In the case of isotropic refractive-index fluctuations, we have

\[
\Phi_n(\bar{k}) = \Phi_n(k), \tag{6.13}
\]

where \( k \) is the magnitude of the wave-vector \( \bar{k} \):

\[
k = |\bar{k}| = \sqrt{k_1^2 + k_2^2 + k_3^2}. \tag{6.14}
\]

In this case, we obtain

\[
M_{11}^{(0)} = \frac{1}{(2\pi)^3} \iiint (2\pi)^3 A^2 f^B_{\bar{\sigma}}(\bar{\sigma}) |W(\bar{\sigma})|^2 \delta_D(k - \bar{k}_B(\bar{\sigma})) \Phi_n(\bar{\sigma}, k) d^3 k d^3 \sigma \\
= A^2 \iiint f^B_{\bar{\sigma}}(\bar{\sigma}) |W(\bar{\sigma})|^2 \Phi_n(\bar{\sigma}, 2k_0) d^3 \sigma, \tag{6.15}
\]

where \( \Phi_n(2k_0) \) may still be a function of the location \( \bar{\sigma} \) within the observation volume.
6.1.1 Volume reflectivity and the refractive-index turbulence structure parameter

In order to derive the constant of proportionality between $\Phi_n(\sigma, 2k_0)$ and the local volume reflectivity $\eta(\sigma)$, we consider the contribution $dP_r$ that is made by the scattering volume $dV = d\sigma_1 d\sigma_2 d\sigma_3$ at the center of the observation volume, i.e., $\sigma = 0$, to the total received power $P_r$. We obtain

$$dP_r = \frac{R}{2} dM^{(0)}_{11} = \frac{R}{2} A^2 (2\pi)^3 \Phi_n(0, 2k_0) dV,$$

(6.16)

because the weighting-function $f_{\sigma}(\sigma)|W(\sigma)|^2$ has the value 1 at $\sigma = 0$.

Now,

$$S_i = \frac{P_I g}{4\pi r_0^2}$$

(6.17)

is the incoming radiation intensity at the the center of the observation volume (DOVIAK & ZRNIĆ 1993, p. 34, eq. 3.4). The part of the intensity scattered back to the antenna by the volume element $dV$ is given by

$$dS_r = S_i d\sigma_b$$

(6.18)

where $d\sigma_b$ is the back-scattering cross-section of the volume element $dV$. The corresponding contribution to the received power is

$$dP_r = dS_r A_e,$$

(6.19)

where $A_e$ is the effective antenna area:

$$A_e = \frac{g\lambda^2}{4\pi}.$$

(6.20)

This results in

$$dP_r = \frac{P_I g^2 \lambda^2}{(4\pi)^3 r_0^2} d\sigma_b.$$

(6.21)

Since the volume reflectivity $\eta$ is defined as the backscattering cross-section per scattering volume,

$$\eta = \frac{\sigma_b}{dV},$$

(6.22)

we obtain the important relation

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\[ \eta(\sigma) = 8\pi^2 k_0^4 \Phi_n(\sigma, 2k_0). \] (6.23)

In the case of locally isotropic and fully developed turbulence, the structure function of \( n' \) follows a Kolmogorov-type \( \delta^{\frac{4}{3}} \)-law,

\[ \left< \left( n'(\sigma, \delta) - n'(\sigma) \right)^2 \right> = C_n^2 \delta^{\frac{4}{3}}, \] (6.24)

where \( C_n^2 \) is known as refractive-index turbulence structure parameter. Tatarskii (1961, p. 48, eq. 3.24) showed that (6.24) determines unambiguously the three-dimensional refractive-index variance spectrum:

\[ \Phi_n(k) = \Phi_n(k) = \frac{\Gamma \left( \frac{8}{3} \right) \sin \left( \frac{\pi}{3} \right)}{4\pi^2} C_n^2 k^{-\frac{11}{3}} = 0.0330 C_n^2 k^{-\frac{11}{3}}. \] (6.25)

Inserting this result into (6.23) provides

\[ \eta(\sigma) = \frac{\Gamma \left( \frac{8}{3} \right) \sqrt{3} (4\pi)^{\frac{1}{3}}}{16} C_n^2 (\sigma) \lambda^{-\frac{1}{3}} = 0.3787 C_n^2 (\sigma) \lambda^{-\frac{1}{3}}. \] (6.26)

This is one of the most important theoretical results in radar meteorology (compare Tatarskii 1961, p. 76, eq. 4.45; Atlas et al. 1966; Ottersten 1969; Muschinski 1997; Gossard et al. 1998). In a classical experiment using a calibrated S-band radar and a helicopter-borne refractometer (Kropflı et al. 1968), Eq. (6.26) was empirically verified with an accuracy of about ±3 dB.

6.1.2 Aspect sensitivity and anisotropy of refractive-index irregularities

It has been known for about two decades that VHF-radar echo-intensities are aspect sensitive at beam-pointing directions close to the zenith (Rottger & Liu 1978, Gage & Green 1978), indicating that the spatial variance spectrum of the refractive-index irregularities, \( \Phi_n(k) \), is strongly anisotropic. Doviak & Zrnić (1984) investigated theoretically the VHF aspect sensitivity, assuming that the radar weighting-function is Gaussian in three dimensions and that \( R_n(\sigma, \delta) \) is homogeneous, i.e., independent of \( \sigma \), axially symmetric, and Gaussian in the components of the lag-space vector \( \delta \). They assumed “vertical anisotropy” (i.e., axial symmetry about a vertical axis), and they introduced the angle \( \Psi \) as the angle between the axis of isotropy (the axis normal to the “plane of isotropy”) beam axis define an angle \( \Psi \). Often, the axis of isotropy is vertical and thus, in this case, \( \Psi \) is the radar beam's zenith angle. A lengthy analysis (Doviak & Zrnić 1984, more detailed in Steffens 1995) provides the result that under those assumptions the backscattered power observed at near-zenith angles is a Gaussian function of \( \Psi \). This behavior has been found in measurements of the aspect sensitivity (Doviak & Zrnić

\[ Here, we make use of the identity \sin \left( \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2}. \]
1984, HOCKING et al. 1986), such that the Gaussian model for \( \Phi_n(\vec{k}) \) is often relied on when interpreting VHF Doppler-radar data.

The model by DOVIAK & ZRNIC (1984), however, has several limitations: (i) it is assumed that the refractive-index irregularities' coherence length in the direction of the axis of anisotropy is small as compared to the radar wavelength (this assumption is relaxed in section 6 of DOVIAK & ZRNIC ET AL. 1996); (ii) it is assumed that \( R_n(\vec{\sigma}, \vec{\delta}) \) is homogeneous. Using the theoretical framework developed in the present study, however, one could also examine the interesting case that \( \Psi \) and the coherence lengths vary within the observation volume. It is to be expected that more realistical models for the distribution of \( \Psi \) and the coherence lengths lead to models for the echo-intensity aspect-angle profiles in which the transition from the aspect sensitive range to the non-aspect-sensitive ("isotropic") aspect-angle range is not as sudden as in DOVIAK & ZRNIC (1984) model. The model used in section 6 in DOVIAK ET AL. (1996), however, should soften this "sudden" transition.

6.1.3 Backscattering from Gaussian and from Kolmogorov-type refractive-index irregularities — a comparison

While the \( k^{-\frac{5}{3}} \)-law for \( \Phi_n(k) \) follows from the classical theory of isotropic turbulence (see, e.g., BATCHELOR 1953, TATARSKII 1961), a rigorous theoretical justification for the Gaussian model is lacking. In this sense, the Gaussian model is \textit{ad hoc}; see also DOVIAK ET AL. (1996, sections 4.3.3 and 6) and HOLLOWAY ET AL. (1997, p. 1913, col. 1).

In this Subsection, we calculate the volume reflectivity \( \eta \) in the case of Gaussian \( n' \)-irregularities, and we compare the result with the \( C_n^2 \)-model for \( \eta \). Now, we consider the isotropic Gaussian refractive-index autocovariance function

\[
R_n(\vec{\sigma}, \vec{\delta}) = R_n(|\vec{\delta}|) = \langle n'^2 \rangle \cdot \exp \left[ -\frac{\delta_1^2 + \delta_2^2 + \delta_3^2}{2\rho^2} \right],
\]

(6.27)

where \( \langle n'^2 \rangle \) is the refractive-index variance, and \( \rho \) is the coherence length. Then the spectrum is given by

\[
\Phi_n(\vec{k}) = \langle n'^2 \rangle \frac{\rho^3}{(2\pi)^\frac{3}{2}} \exp \left[ -\frac{\rho^2}{2} (k_1^2 + k_2^2 + k_3^2) \right] = \langle n'^2 \rangle \frac{\rho^3}{(2\pi)^\frac{3}{2}} \exp \left( -\frac{\rho^2}{2} k_0^2 \right),
\]

(6.28)

and we obtain the following equation for the volume reflectivity:

\[
\eta = 8\pi^2 k_0^4 \Phi_n(2k_0) = \sqrt{8\pi} \langle n'^2 \rangle \rho^3 k_0^4 \exp \left( -2k_0^2 \rho^2 \right).
\]

(6.29)

We notice that \( \eta \) as a function of the radar wavenumber \( k_0 \) behaves differently, depending on whether \( 2k_0^2 \rho^2 \) is small or large as compared to 1. For a given \( k_0 \), there exists a critical coherence length \( \rho_c \) which marks the boundary between the range where \( \eta \) increases proportionally to \( \rho^3 \) and the range where \( \eta \) decreases exponentially with \( \rho^2 \):
that is, \[ \rho_c = \frac{\lambda}{\sqrt{8\pi}} = 0.199\lambda. \] (6.31)

In the case of small coherence lengths, \( \rho < \rho_c \), we have \( \exp(-2k_0\rho^2) \approx 1 \), and it follows that \( \eta \) is proportional to the “coherence volume” \( \rho^3 \) and proportional to \( k_0^3 \), that is, proportional to \( \lambda^{-4} \). Hence, in the case of small coherence lengths, refractive-index irregularities with a Gaussian spectrum behave like Rayleigh-scatterers (see also DOVIK ET AL. 1996, section 5.3, p. 174, col. 1). In the opposite case of large coherence lengths, \( \rho \gg \rho_c \), the Gaussian term \( \exp(-2k_0^2\rho^2) \) dominates, and \( \eta \) depends extremely sensitively on the product of \( k_0 \) and \( \rho \). For a given radar wavelength, a doubling of \( \rho \) implies a decrease of \( \eta \) by the factor \( \exp(6k_0^2\rho^2) \), i.e., in the case \( k_0\rho = 2 \) by the factor \( \exp(24) \) or by 208 dB; compare BRIGGS & VINCENT (1973). For a larger value of \( k_0\rho \) the sensitivity of \( \eta \) to a moderate change of \( \rho \) would be even more dramatic. Observed fluctuations of \( \eta \), however, are typically on the order of 10 or 20 dB (see, e.g., MUSCHINSKI 1997). That is, the Gaussian model implies for the case \( \rho > \rho_c \) a very large dependence of \( \eta \) on \( \rho \), which is not in line with UHF/VHF-radar observations. (There is no reason to expect that \( \rho \) is sufficiently constant.) According to the \( k^{-\frac{11}{3}} \)-model, however, see Eq. (6.26), \( \eta \) is directly proportional to \( C_n^2 \) and is independent of the outer scale of turbulence, \( \rho \). Therefore, for the interpretation of absolute volume reflectivities in the case of fully developed turbulence, the \( k^{-\frac{11}{3}} \)-model seems to provide more realistical results than the Gaussian model; compare DOVIK ET AL. (1996, section 6). See also DOVIK ET AL.’s (1996, section 3.3.) discussion on the error that can be made when applying the Gaussian model.

6.2 The first moment

6.2.1 The Doppler shift in the case of backscattering from a point scatterer

Certainly, a very important application of Doppler radars is to measure the radial velocity of the target. During the early years of the Doppler-radar technique, the most important targets were point targets, i.e., objects that covered just a small fraction of the observation volume (airplanes, raindrops).

Now, we derive the relationship which allows the target’s radial velocity \( v_r \) to be retrieved from the observed Doppler shift \( \omega_D \). We consider a point target which at time \( t = 0 \) is in the center of the observation volume. At time \( t \), let its radial distance to the center of the antenna be given by

\[ r_t(t) = r_0 + v_r t, \] (6.32)

implying a positive radial velocity \( v_r \) in the case that the target moves away from the antenna. The signal may be written as
\[ I_1(t) = |I_1| \cdot e^{-j[2k_0(r_0 + v_r t) + \varphi_0]}, \]  

(6.33)

where \( \varphi_0 \) is the initial phase angle. (We assume that \( |I_1| \) changes very much less rapidly with time than the phase factor does, and thus we neglect the time dependence of \( |I_1| \).) Then, the autocovariance function is

\[ C_{11}(\tau) = \langle I_1^*(t)I_1(t + \tau) \rangle_t = |I_1|^2 e^{-j2k_0v_r \tau}. \]  

(6.34)

Its derivative is

\[ C'_{11}(\tau) = -j2k_0v_r C_{11}(\tau), \]  

(6.35)

and we obtain for \( \tau = 0 \):

\[ v_r = -\frac{1}{j2k_0} \frac{C_{11}'(0)}{C_{11}(0)} = -\frac{1}{j2k_0} \frac{jM^{(1)}_{11}}{M^{(0)}_{11}}. \]  

(6.36)

Now, we have the relation between \( v_r \) and \( \omega_D \):

\[ v_r = -\frac{1}{2k_0} \frac{M^{(1)}_{11}}{M^{(0)}_{11}} = -\frac{\omega_D}{2k_0}. \]  

(6.37)

A positive Doppler shift implies a target motion towards the antenna, and a negative Doppler shift implies a target motion away from the antenna.

### 6.2.2 The Doppler shift in the case of backscattering from a continuous refractive-index field

In the previous Subsection, we have shown that the Doppler shift provides directly the target’s radial velocity:

\[ \omega_D = \frac{M^{(1)}_{11}}{M^{(0)}_{11}} = -2k_0v_r \]  

(6.38)

if the target is a point scatterer.

In this Subsection, we will examine the question whether this simple equation holds also in the case of backscattering from a continuous refractive-index field, and if it does, under what assumptions.

After inserting the simplified standard two-way sampling-function, eq. (3.36),
\[
H_{11}(\tilde{\sigma}, \tilde{k}) = (2\pi)^3 A^2 f_\delta^4(\tilde{\sigma})|W(\tilde{\sigma})|^2 \delta_D(\tilde{k} - \tilde{k}_B(\tilde{\sigma})),
\]

where

\[
\tilde{k}_B(\tilde{\sigma}) = 2k_0 \begin{pmatrix}
\frac{a_1}{r_0} \\
\frac{a_2}{r_0} \\
1
\end{pmatrix}
\]

is the Bragg wave-vector, into the equations for \(M^{(0)}_{11}\), Eq. (5.3), and for \(M^{(1)}_{11}\), Eq. (5.5), respectively, we obtain the following equations for \(M^{(0)}_{11}\) and for \(M^{(1)}_{11}\), respectively:

\[
M^{(0)}_{11} = (2\pi)^3 A^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_\delta^4(\tilde{\sigma})|W(\tilde{\sigma})|^2 \Phi_n(\tilde{\sigma}, \tilde{k}_B(\tilde{\sigma})) d^3 \sigma
\]

and

\[
M^{(1)}_{11} = (2\pi)^3 A^2 \sum_{i=1}^{3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_\delta^4(\tilde{\sigma})|W(\tilde{\sigma})|^2 \\
\times \left(-\nabla_i(\tilde{\sigma}) k_{Bi}(\tilde{\sigma}) \Phi_n(\tilde{\sigma}, \tilde{k}_B(\tilde{\sigma}))
\right. \\
\left. + j \frac{\partial \Phi_n}{\partial \sigma_i}(\tilde{\sigma}) \Phi_n_{vi}(\tilde{\sigma}, \tilde{k}_B(\tilde{\sigma})) + j \Phi_n(\nabla_i(\tilde{\sigma}), \tilde{k}_B(\tilde{\sigma})) \right) d^3 \sigma,
\]

where \(k_{Bi}(\tilde{\sigma})\) is the \(i\)th component of the local Bragg wave-vector \(\tilde{k}_B(\tilde{\sigma})\). \(M^{(1)}_{11}\) is the first moment of the signal's variance spectrum, i.e., it is the first moment of a real function, and thus, \(M^{(1)}_{11}\) is real. Hence, the real parts of the generally complex cross-spectra \(\Phi_{nvi}(\tilde{\sigma}, \tilde{k})\) and \(\Phi_n(\nabla_i, \nabla_i)(\tilde{\sigma}, \tilde{k})\) cannot contribute to \(M^{(1)}_{11}\). By definition, a cross-spectrum consists of a co-spectrum and a quadrature spectrum, i.e., we have

\[
\Phi_{nvi}(\tilde{\sigma}, \tilde{k}) = C_{nvi}(\tilde{\sigma}, \tilde{k}) + j Q_{nvi}(\tilde{\sigma}, \tilde{k})
\]

and

\[
\Phi_n(\nabla_i, \nabla_i)(\tilde{\sigma}, \tilde{k}) = C_n(\nabla_i, \nabla_i)(\tilde{\sigma}, \tilde{k}) + j Q_n(\nabla_i, \nabla_i)(\tilde{\sigma}, \tilde{k}),
\]

respectively. Hence, only the quadrature spectra can contribute to \(M^{(1)}_{11}\), and we obtain for the first moment:

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\[ M_{11}^{(1)} = - (2\pi)^3 A^2 \sum_{i=1}^{3} \iint_{-\infty}^{\infty} \hat{f}_B(\hat{\sigma}) |W(\hat{\sigma})|^2 \times \left( \bar{v}(\hat{\sigma}) k_B i(\hat{\sigma}) \Phi_n(\hat{\sigma}, \tilde{k}_B(\hat{\sigma})) + \frac{\partial n}{\partial \sigma_i}(\hat{\sigma}) Q_{nv_i}(\hat{\sigma}, \tilde{k}_B(\hat{\sigma})) + Q_{\bar{n}(v_i \nu_i)}(\hat{\sigma}, \tilde{k}_B(\hat{\sigma})) \right) d^3 \sigma. \] (6.45)

Three terms, which are all weighted by the radar's two-way weighting function, contribute to the first moment of the Doppler spectrum: the first term is the radial velocity, weighted by the value of the local spatial refractive-index variance spectrum at the local Bragg wave-vector; the second term shows that the quadrature spectrum of the refractive-index flux may contribute to \( M_{11}^{(1)} \) even if there is no mean wind, i.e., if \( \bar{v}(\hat{\sigma}) = 0 \); the third term implies that also a triple correlation \( \langle n'(\hat{\sigma})(\vec{v} \cdot \tilde{n}'(\hat{\sigma} + \vec{d})) \rangle \) may have an effect of the first moment of the Doppler spectrum.

If we neglect the last two terms, we obtain the following equation for the Doppler shift:

\[ \omega_D = -2k_0 \int_{-\infty}^{\infty} \hat{f}_B(\hat{\sigma}) |W(\hat{\sigma})|^2 \left( \bar{v}(\hat{\sigma}) \cdot \tilde{k}_B(\hat{\sigma}) \right) \Phi_n(\hat{\sigma}, \tilde{k}_B(\hat{\sigma})) d^3 \sigma. \] (6.46)

Moreover, we have

\[ \bar{v}(\hat{\sigma}) \cdot \tilde{k}_B(\hat{\sigma}) = 2k_0 \bar{v}(\sigma) \cdot \left( \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_0 \\ 1 \end{array} \right) = 2k_0 v_r(\hat{\sigma}), \] (6.47)

where \( v_r(\hat{\sigma}) \) is the local radial velocity. We obtain

\[ \omega_D = -2k_0 \int_{-\infty}^{\infty} \hat{f}_B(\hat{\sigma}) |W(\hat{\sigma})|^2 \Phi_n(\hat{\sigma}, \tilde{k}_B(\hat{\sigma})) d^3 \sigma. \] (6.48)

That is, in the case \( v_r(\hat{\sigma}) = v_r = const. \), the Doppler shift is related to the radial velocity like

\[ \omega_D = -2k_0 v_r. \] (6.49)

This is the same relation as we derived for a point scatterer.

In the following, we analyse and discuss two important special cases: a) backscattering from locally isotropic refractive-index irregularities; b) backscattering from thin, highly anisotropic refractive-index laminae.
6.2.3 Special case: the Doppler shift in the case of backscattering from locally isotropic turbulent refractive-index irregularities

We consider the case that the observation volume is filled by refractive-index irregularities which have no special direction at length scales comparable to the Bragg-wavelength $\lambda/2$, i.e., the spectrum of which is locally isotropic with respect to the Bragg-wavenumber:

$$\Phi_n(\vec{\sigma}, \vec{k}_B(\vec{\sigma})) = \Phi_n(\vec{\sigma}, |\vec{k}_B(\vec{\sigma})|) = \Phi_n(\vec{\sigma}, 2k_0) = \frac{\Gamma\left(\frac{8}{3}\right)\sin\left(\frac{\pi}{3}\right)}{4\pi^2}(2k_0)^{-\frac{5}{2}} C_n^2(\vec{\sigma}), \quad (6.50)$$

and we obtain the Doppler shift

$$\omega_D = -2k_0 \frac{\int \int \int B(\vec{\sigma})C_n^2(\vec{\sigma})v_r(\vec{\sigma})d^3\sigma}{\int \int \int B(\vec{\sigma})C_n^2(\vec{\sigma})d^3\sigma}, \quad (6.51)$$

where

$$B(\vec{\sigma}) = f_0^2(\vec{\sigma})|W(\vec{\sigma})|^2 \quad (6.52)$$

is the radar weighting-function. Hence, the observed Doppler velocity turns out as the spatially averaged radial wind velocity within the observation volume, weighted with the product of radar weighting-function and refractive-index turbulence structure parameter, $B(\vec{\sigma})C_n^2(\vec{\sigma})$.

It is known that in the stably stratified atmosphere, $C_n^2$ is usually strongly inhomogeneous with respect to the vertical direction: the small-scale turbulence is organized in thin layers or "sheets" (VANZANDT et al. 1978; MUSCHINSKI 1997; MUSCHINSKI & WODE 1997, 1998; GURVICH 1997). This may result in a strongly asymmetric effective weighting-function $B(\vec{\sigma})C_n^2(\vec{\sigma})$, in particular for non-vertical beam directions, and, therefore, in a systematic difference between the radial velocity retrieved from the Doppler shift (the Doppler velocity) and the spatially mean radial wind velocity (e.g., FUKAO et al. 1988a, FUKAO et al. 1988b). That is, there is a need for additional information about the distribution of $C_n^2$ within the observation volume or, more generally, about the effective weighting-function defined by (6.46). Such information can be gained from interferometric (FDI and/or SI) measurements, as will be shown in the next two Sections.

Another difficulty arises if $C_n^2(\vec{\sigma})$ and $v_r(\vec{\sigma})$ are correlated. In the case of a vertically pointing high-gain radar (small beam-width), $v_r(\vec{\sigma})$ is approximately equal to the vertical wind velocity $w(\vec{\sigma})$. When averaging over many realizations of $C_n^2(\vec{\sigma})$ und $w(\vec{\sigma})$, one obtains the following mean Doppler velocity (the "measured" mean vertical wind velocity):

$$\bar{w}_m = \frac{C_n^2w}{C_n^2} = \frac{C_n^2w + C_n^2w'}{C_n^2} = \bar{w} + \frac{C_n^2w'}{C_n^2}. \quad (6.53)$$
That is, there may be a finite systematic difference between measured and true mean vertical velocity:

$$\Delta \bar{w} = \frac{C_n^2 u''}{C_n^2}. \quad (6.54)$$

NASTROM & VANZANDT (1994) investigated $\Delta \bar{w}$ quantitatively for the case of a monochromatic gravity wave. For realistic parameters of the gravity wave, $\Delta \bar{w}$ is on the order of several cm s$^{-1}$. According to that result, one should expect that by using UHF/VHF radars it is possible to measure directly synoptic-scale vertical velocities, which are typically on the order of 1 cm s$^{-1}$, only if the gravity-wave amplitude is below a certain limit. MUSCHINSKI et al. (1998) compared $\bar{w}$-measurements using the standard Doppler method, with the temporal rates of change of long-living (up to more than 1 day) quasi-horizontal layers of refractive-index irregularities over the Harz Mountains, Germany. Both time series were retrieved from a 70-hours-long experiment using the SOUSY VHF Radar. The radar was operated in an FDI mode, such that layer altitudes could be retrieved from the data, independently of the Doppler shifts retrieved from the same data set. Both height-time cross-sections (layer altitudes and Doppler velocities) were compared with a height-time cross-section of the vertical velocity generated by means of the weather-forecasting model “BLM” (by the Geophysikalischer Beratungsdienst der Bundeswehr, Traben-Trarbach, Germany), which had been initialized with the routine radiosonde data of the period of the experiment.

In the lower free troposphere, which during the first half of the observation period was characterized by a shallow high-pressure area with small horizontal wind velocities, all three data sets showed negative vertical velocities of a few centimeters per second. In the middle and upper troposphere, however, the horizontal wind velocities were significantly higher, in particular during the second half of the observation period. In that portions of the height-time section, there was no significant agreement between the different temporally averaged vertical velocity profiles, and the BLM vertical velocity profiles showed a strong sensitivity to the geographical position, i.e., the correlation between the BLM-generated vertical velocity profiles at adjacent horizontal grid points, which were about 60 km apart from each other, was very low.

To what extent MUSCHINSKI et al.’s (1998) results are sensitive to the weather conditions is difficult to say. A future, similar experiment, which should be carried out over a longer time period (on the order of one month or even longer) that includes different synoptic situations, could provide more insight.

Less encouraging than tropospheric mean vertical wind measurements using VHF radars are still the attempts to directly observe mean vertical velocities in the planetary boundary layer using UHF radars operating at frequencies of about 1 GHz. ANGEVINE (1997) observed that the difference between the Doppler velocity $-\bar{w}/(2k_0)$ and the synoptic-scale vertical velocity $\bar{w}$ exhibits a pronounced diurnal cycle: at night-time, the differences are not significant; during day-time, however, he observed systematically mean downward velocities of up to 30 cm s$^{-1}$, in contrast to the expectation that mean vertical velocities should be on the order of only a few cm s$^{-1}$. The majority of the experts in the field appears to believe that this discrepancy is to be attributed mainly to the presence of biological targets (birds, insects), which at UHF frequencies might contribute significantly to the radar echoes via Rayleigh scattering (see WILCZAK et
al. 1995). Possibly, also a finite covariance $C_{21} w'$ in the convective boundary layer (CBL) contributes to the observed Doppler velocities. The space-time distribution of $\Delta w = \frac{C_{21} w'}{C_n}$ within the CBL could be investigated by using high-resolution in-situ turbulence measurements or by means of high-resolution large-eddy simulations (LES). It is to be expected that universal profiles of $\Delta w$ exist.

6.2.4 Special case: The Doppler shift in the case of backscattering from thin refractive-index laminae

In the stably stratified atmosphere, thin refractive-index laminae are quasi-ubiquitous. In those layers or sheets, the refractive-index variance spectrum $\Phi_n(\vec{\sigma}, \vec{k})$ can be highly anisotropic even at rather large wavenumbers. At a location $\vec{\sigma}$, the value of $\Phi_n(\vec{\sigma}, \vec{k})$ on the surface of the wavenumber-space sphere with radius $|\vec{k}| = 2k_0$ has a maximum in the direction in that the refractive-index coherence length has a minimum. This is the direction of the normal vector $\vec{n}$ of the laminae at the location $\vec{\sigma}$. At the location $\vec{\sigma}$, the iso-surfaces of $\Phi_n(\vec{\sigma}, \vec{k})$ in $\vec{k}$-space are cigar- or needle-shaped (prolate spheroids), the major axis of which is parallel to $\vec{n}$; see DOVIAK & ZRNIĆ (1984, Fig. 2).

Now, we consider the case that the laminae have the same normal vector $\vec{n}$ at all locations within the observation volume, and we assume that the direction of $\vec{n}$ is not too far from the zenith, i.e., the laminae are nearly horizontally oriented. Let $\vec{v}(\vec{\sigma}) = \text{const.}$ be the homogeneous wind field. From (6.46), we recognize that only that locations $\vec{\sigma}$ contribute significantly to the Doppler shift at which

$$\vec{k}_B(\vec{\sigma}) \parallel \vec{n}$$

is fulfilled. (Here, $\parallel$ stands for “is parallel to”.) At these locations, the vector towards the antenna center is perpendicular to the lamina plane (“mirror points”). In the case of very strong anisotropy at the Bragg wavenumber, we obtain from (6.48) the Doppler shift

$$\omega_D = -2k_0 \vec{v} \cdot \vec{n} = -2k_0 v_{\perp},$$

where $v_{\perp}$ is the velocity component parallel to $\vec{n}$. Hence, in the case of “specular reflection”, the radar does not see the projection of the wind velocity vector on the beam axis but the projection of the wind velocity vector on the vector $\vec{n}$ normal to the laminae. This effect has been investigated soon after the discover of the aspect sensitivity (see, e.g., GAGE et al. 1981), and later, algorithms have been put forward that allow the true radial velocity to be estimated from the Doppler shift, taking account for the aspect sensitivity (e.g., HOCKING et al. 1986). MUSCHINSKI (1996b) has emphasized that a skewed distribution of laminae tilting-angles, as it is to be expected in regions with Kelvin-Helmholtz instability, may lead to a systematic estimation error of mean vertical velocities observed using VHF radars, and that this error may be on the order of 10 cm s$^{-1}$. 

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7 Monostatic Frequency-Domain Interferometry

7.1 Introductory remarks

Frequency-domain interferometry using clear-air radars was first realized by Kudeki & Stitt (1987). Meanwhile, FDI has been implemented at quite a number of VHF radars (see, e.g., Palmer et al. 1990, Liu & Pan 1993, Chilson & Schmidt 1996).

Using the SOUSY VHF Radar, Chilson et al. (1997) were the first to observe upper-tropospheric Kelvin-Helmholtz billows in an FDI mode. Based on the same data set, Muschinski et al. (1998) compared observations of synoptic-scale vertical velocities (retrieved from standard Doppler and FDI observations) with vertical-velocity estimates generated by a weather-forecasting model initialized with routine radiosonde data.

When applying FDI, the radar is operated quasi-simultaneously and phase-coherently at two or more carrier frequencies. In the monostatic case, the simplified spectral FDI cross-sampling-function (see Sec. 3.3) is given by

\[
H_{12}(\vec{\sigma}, \vec{k}) = (2\pi)^3 A^2 B(\vec{\sigma}) \delta_D(\vec{k} - \vec{k}_B(\vec{\sigma})) \exp \left[ -j2\Delta k \left( r_0 + \sigma_3 + \frac{\sigma_1^2 + \sigma_2^2}{2r_0} \right) \right]
\]

\[
= H_{11}(\vec{\sigma}, \vec{k}) \cdot \exp(-j2\Delta k r_1(\vec{\sigma})),
\]

(7.1)

where

\[
B(\vec{\sigma}) = \int_{W(\vec{\sigma})}^{|W(\vec{\sigma})|^2}
\]

(7.2)

is the two-way radar weighting-function,

\[
\vec{k}_B(\vec{\sigma}) = 2k_0 \begin{pmatrix} \frac{\sigma_1}{r_0} \\ \frac{\sigma_2}{r_0} \\ 1 \end{pmatrix}
\]

(7.3)

is the local Bragg wave-vector,

\[
H_{11}(\vec{\sigma}, \vec{k}) = (2\pi)^3 A^2 B(\vec{\sigma}) \delta_D(\vec{k} - \vec{k}_B(\vec{\sigma}))
\]

(7.4)

is the simplified spectral standard two-way sampling-function, and

\[
r_1(t) = r_0 + \sigma_3 + \frac{\sigma_1^2 + \sigma_2^2}{2r_0}.
\]

(7.5)

is the distance between the location \( \vec{\sigma} \) and the center of the antenna.

The magnitudes of \( H_{11}(\vec{\sigma}, \vec{k}) \) and \( H_{12}(\vec{\sigma}, \vec{k}) \) are equal to each other:
\[ |H_{12}(\bar{\sigma}, \bar{k})| = |H_{11}(\bar{\sigma}, \bar{k})| = H_{11}(\bar{\sigma}, \bar{k}). \quad (7.6) \]

i.e., \( H_{12}(\bar{\sigma}, \bar{k}) \) differs from \( H_{11}(\bar{\sigma}, \bar{k}) \) just by a phase factor \( \exp(-j2\Delta kr_t(\bar{\sigma})) \).

### 7.2 The zeroth moment

In this Subsection, we investigate the zeroth moment of the FDI cross-spectrum \( S_{12}(\omega) \) as given by (5.7):

\[
M_{12}^{(0)} = \frac{1}{(2\pi)^{3}} \iiint_{-\infty}^{\infty} H_{12}(\bar{\sigma}, \bar{k}) \Phi_{n}(\bar{\sigma}, \bar{k}) d^{3}k d^{3}\sigma
\]

\[
= A^{2} \int_{-\infty}^{\infty} B(\bar{\sigma}) \exp(-j2\Delta kr_{t}(\bar{\sigma})) \Phi_{n}(\bar{\sigma}, \bar{k}_{B}(\bar{\sigma})) d^{3}\sigma. \quad (7.7)\]

#### 7.2.1 Locally isotropic refractive-index irregularities

First, we examine the case of a locally isotropic \( n' \)-field. Let \( \Phi_{n}(\bar{\sigma}, \bar{k}) \) be homogeneous with respect to the transversal (horizontal in the case of a vertically pointing beam) coordinates \( \sigma_{1} \) and \( \sigma_{2} \) but inhomogeneous with respect to the radial coordinate \( \sigma_{3} \). Let \( \Phi_{n}(\bar{\sigma}, 2k_{0}) \) be a Gaussian function of \( \sigma_{3} \), having a maximum at \( \sigma_{3} = z_{l} \): \[
\Phi_{n}(\bar{\sigma}, 2k_{0}) = \Phi_{n}^{max}(2k_{0}) \cdot \exp \left( -\frac{(\sigma_{3} - z_{l})^{2}}{2\sigma_{l}^{2}} \right). \quad (7.8)\]

Here, \( \sigma_{l} \) is a measure of the thickness of the layer in which the refractive-index irregularities are concentrated. We obtain the zeroth moment of the FDI Doppler cross-spectrum as follows:

\[
M_{12}^{(0)} = A^{2} \iiint_{-\infty}^{\infty} B(\bar{\sigma}) \exp \left[ -j2\Delta k \left( r_{0} + \sigma_{3} + \frac{\sigma_{l}^{2} + \sigma_{2}^{2}}{2r_{0}} \right) \right] \]

\[
\exp \left( -\frac{(\sigma_{3} - z_{l})^{2}}{2\sigma_{l}^{2}} \right) \Phi_{n}^{max}(2k_{0}) d^{3}\sigma. \quad (7.9)\]

By applying again

\[
\int_{-\infty}^{\infty} \exp \left( -a^{2}x^{2} - bx \right) dx = \sqrt{\frac{\pi}{a^{2}}} \exp \left( \frac{b^{2}}{4a^{2}} \right) \quad (7.10)\]
we find

\[ M_{12}^{(0)} = A^2 \exp \left( -\frac{z_f^2}{2\sigma_t^2} \right) \Phi_n^{\text{max}}(2k_0) \]
\[ \times \exp(-j2\Delta k r_0) \pi \sigma_t^2 \sqrt{\pi \sigma_r^2} \exp \left[ \frac{\left( -\frac{z_f}{\sigma_t} + j2\Delta k \right)^2}{4\frac{1}{2\sigma_t^2}} \right]. \]  

(7.11)

Here, we have introduced the auxiliary variables

\[ \frac{1}{\sigma_t^2} = \frac{1}{\sigma_i^2} + \frac{j\Delta k}{r_0} \]  

(7.12)

and

\[ \frac{1}{2\sigma_r^2} = \frac{1}{2\sigma_i^2} + \frac{1}{2\sigma_t^2}, \]  

(7.13)

where the indices “t”, “r”, and “l” stand for “transverse”, “radial” and “layer”, respectively. We obtain the zeroth moment of the variance spectrum \( S_{11}(\omega) \) by replacing the FDI cross-sampling-function \( H_{12}(\tilde{\sigma}, \tilde{k}) \) with the standard two-way sampling-function \( H_{11}(\tilde{\sigma}, \tilde{k}) \):

\[ M_{11}^{(0)} = A^2 \exp \left( -\frac{z_f^2}{2\sigma_t^2} \right) \Phi_n^{\text{max}}(2k_0) \pi \sigma_t^2 \sqrt{\pi \sigma_r^2} \exp \left[ \frac{\left( -\frac{z_f}{\sigma_t} \right)^2}{4\frac{1}{2\sigma_t^2}} \right]. \]  

(7.14)

We define the complex correlation coefficient of the two Doppler signals as follows:

\[ c_{12}(0) = \frac{C_{12}(0)}{\sqrt{C_{11}(0)C_{22}(0)}} = \frac{M_{12}^{(0)}}{\sqrt{M_{11}^{(0)}M_{22}^{(0)}}}, \]  

(7.15)

and because of

\[ \Delta k \ll 2k_0, \]  

(7.16)

that is, because of

\[ M_{11}^{(0)} \approx M_{22}^{(0)}, \]  

(7.17)

we obtain
Eventually, we find
\[ c_{12}(0) = \exp\left[-j2\Delta k\left(r_0 + \frac{\sigma_r^2}{\sigma_l^2} + z_l\right)\right] \exp\left(-2\sigma_r^2 \Delta k^2\right) \cdot \frac{1}{1 + \frac{j\Delta k \sigma_l^2}{r_0}}. \] (7.19)

Since \( \Delta k, \sigma_l \) and \( r_0 \) are known, radial location \( z_l \) as well as thickness \( \sigma_l \) of the layer may be retrieved from measured data of \( c_{12}(0) \). The phase of \( c_{12}(0) \) provides \( z_l \), whereas the magnitude of \( c_{12}(0) \) yields \( \sigma_l \).

Now, we consider two special cases: a) a very thin layer, \( \sigma_l \ll \sigma_r \); b) a quasi-homogeneous distribution of refractive-index irregularities, i.e., \( \sigma_l \gg \sigma_r \). We begin with the thin layer,

\[ \sigma_l \ll \sigma_r, \] (7.20)

and we obtain

\[ c_{12}(0) = \exp\left[-j2\Delta k(r_0 + z_l)\right] \exp(-2\sigma_r^2 \Delta k^2) \cdot \frac{1}{1 + \frac{j\Delta k \sigma_l^2}{r_0}}. \] (7.21)

In the opposite, quasi-homogeneous case,

\[ \sigma_l \gg \sigma_r, \] (7.22)

we have

\[ c_{12}(0) = \exp\left(-j2\Delta kr_0\right) \exp(-2\sigma_r^2 \Delta k^2) \cdot \frac{1}{1 + \frac{j\Delta k \sigma_l^2}{r_0}}. \] (7.23)

Two-frequency FDI allows one layer per observation volume to be fairly precisely localized, as long as there is only one layer in the observation volume. The determination of \( z_l \), however, is unambiguous only in the \( z_l \)-range in that \( \exp[-j2\Delta k(r_0 + z_l)] \) is unambiguous. This leads to the unambiguity condition

\[ -\pi \leq 2\Delta kz_l < \pi, \] (7.24)

which is equivalent to

\[ |z_l| \leq \frac{\pi}{2\Delta k}. \] (7.25)
In order to ensure an unambiguous determination of \( z_l \) within the interval \(-\Delta h/2, +\Delta h/2\), we obtain a rule for the (maximum) wavenumber-spacing \( \Delta k \):

\[
\Delta h = \frac{\pi}{\Delta k}.
\]  

(7.26)

For a given \( \Delta h \), this rule provides a (maximum) relative wavenumber-spacing \( \Delta k/k_0 \):

\[
\frac{\Delta k}{k_0} = \frac{1}{k_0} \frac{\pi}{\Delta h} = \frac{\lambda}{2\Delta h}.
\]  

(7.27)

The rule for \( \Delta k/k_0 \) is equivalent to the condition that, along the optical path \( 2\Delta h \), the optical-path difference in units of the two respective wavelengths must not be larger than 1. The factor 2 comes from the fact that during the measurement the path \( \Delta h \) is traversed twice by the electromagnetic wave. A typical case for MST radars is \( \lambda = 6 \text{ m} \) and \( \Delta h = 300 \text{ m} \) (\( \tau = 2 \mu\text{s} \)), that is \( \Delta k/k_0 = 0.01 \). Therefore, the relative frequency-spacing at \( \nu = 50 \text{ MHz} \) amounts to \( \Delta \nu = 0.5 \text{ MHz} \) (e.g., CHILSON et al. 1997, MUSCHINSKI et al. 1998). It is customary to choose \( \Delta h \) identical to the conventional range resolution \( \Delta r/2 \), where \( c \) is the speed of light, and \( \tau \) is the pulse duration.

Now, we discuss Eq. (7.21), i.e., FRANKE’s (1990) equation for the complex correlation coefficient \( c_{12}(0) \) in the case of a thin layer, having a Gaussian profile and fulfilling \( \sigma_l \ll \sigma_r \).

We consider the phase factor \( \exp[-j2\Delta k(r_0 + z_l)] \). If the phase of \( c_{12}(0) \) can be measured with an accuracy of \( \Delta \varphi \), the accuracy of the retrieved \( z_l \)-value is given by

\[
\Delta z_l = \frac{\Delta \varphi}{2\pi} \Delta h.
\]  

(7.28)

In order to obtain an accuracy of \( \Delta z_l = \Delta h/10 \), a phase accuracy of \( \Delta \varphi \approx 30^\circ \) is required. For \( \Delta h = 300 \text{ m} \), a helpful rule of thumb is \( \Delta z_l/\Delta h \approx 1 \text{ m deg}^{-1} \).

Since any uncertainty of \( r_0 \) provides a phase uncertainty, the estimates for the radial location of a scattering layer should be interpreted as an estimate relative to the center of the respective observation volume. However, if it is guaranteed that intrinsic phase uncertainties do not drift, time series of \( z_l \) represent fairly reliably temporal changes of the altitude of a layer. In this case, for VHF-FDI measurements in the free troposphere a \( z_l \)-resolution of about 20 m is achievable (see, e.g., CHILSON et al. 1997, MUSCHINSKI et al. 1998). The last term in (7.21) describes an additional constant phase angle, which, however, is sufficiently small such that it may be neglected in many applications.

The magnitude of \( c_{12}(0) \) (the “coherence”) is given by

...
\[ |c_{12}(0)| = \frac{\exp(-2\sigma_l^2 \Delta k^2)}{1 + \left(\frac{\Delta k \sigma_t^2}{r_0}\right)^2}. \tag{7.29} \]

There are two effects which lead to a decrease of the coherence \(|c_{12}(0)|\): (i) a finite layer thickness \(\sigma_l\); (ii) a finite curvature of the wave-fronts within the observation volume.\(^{11}\) First, ignore the curvature effect, i.e., the deviation of the denominator from 1, and consider the term \(\exp(-2\sigma_l^2 \Delta k^2)\). The sensitivity of \(|c_{12}(0)|\) to \(\sigma_l\) is drastically reduced if \(\sigma_l\) is considerably smaller than the critical value

\[ \sigma_c = \frac{1}{\sqrt{2} \Delta k} = \frac{\Delta h}{\sqrt{2} \pi} = \frac{\Delta h}{4.4}. \tag{7.30} \]

For small \(\sigma_l\) and small \(\sigma_t\), the following approximation holds:

\[ |c_{12}(0)| \approx \left(1 - 2\sigma_l^2 \Delta k^2\right) \cdot \left(1 - \frac{\Delta k^2 \sigma_t^4}{r_0^2}\right) \approx 1 - \left(2\sigma_l^2 + \frac{\sigma_t^2}{r_0^2}\right) \Delta k^2. \tag{7.31} \]

The curvature effect and the effect of a finite \(\sigma_l\) on the coherence are equal to each other if \(\sigma_l\) is given by

\[ \sigma_l = \frac{1}{\sqrt{2} \tau_0} = \frac{\sqrt{2}}{16 \cdot \ln 2} r_0 \Theta_1^2. \tag{7.32} \]

For the SOUSY VHF Radar (\(\Theta_1 = 5^\circ\)), we obtain \(\sigma_l = 9.7\) m at \(r_0 = 10\) km. That is, for not too large \(\Theta_1\) and not too large \(r_0\), the curvature effect is important only for very thin layers. We conclude that the achievable resolution for the determination of \(\sigma_l\) depends essentially on the accuracy \(\Delta|c_{12}(0)|\) with which the coherence can measured. If \(\Delta|c_{12}(0)|\) is simultaneously the smallest resolvable deviation of the coherence from 1 (as is fulfilled in the case of a very thin layer), we obtain

\[ 1 - \Delta|c_{12}(0)| = 1 - 2\sigma_l^2 \Delta k^2, \tag{7.33} \]

i.e., the finite resolution of the coherence leads to the the following value of the smallest resolvable layer thickness:

\[ \sigma_l = \frac{\sqrt{\Delta|c_{12}(0)|}}{\sqrt{2} \Delta k} = \frac{\sqrt{\Delta|c_{12}(0)|}}{\sqrt{2} \pi} \Delta h. \tag{7.34} \]

\(^{11}\)Of course, also a finite tilting-angle between the layer plane the plane normal to the beam, or any undulation of the layer within the observation volume would give rise to a decrease of the coherence. Those effects, however, are a priori excluded by the model that we are discussing here.
For $\Delta |c_{12}(0)| = 0.1$, this leads to $\sigma_l \approx \Delta h/14$, implying $\sigma_l \approx 20$ m for $\Delta h = 300$ m.

The discussion would be incomplete if we would not discuss the physical assumptions that our model for $c_{12}(0)$ relies on.

### 7.2.2 Comparison with Franke's (1990) model for the FDI coherence

The model for the FDI coherence $c_{12}(0)$ that was presented and discussed in the previous Sub-section is similar to Franke's (1990) model. The following model assumptions are common to both models: (i) the radar weighting-function $B(\vec{\sigma})$ is Gaussian; (ii) the refractive-index irregularities are locally isotropic; (iii) the refractive-index irregularities are concentrated in a Gaussian-weighted layer, which is transverse to the beam axis. There is, however, an important discrepancy with respect to model assumption (iii): Franke (1990) assumed that $R_n(\sigma_3, \vec{\delta})$ is Gaussian with respect to $\sigma_3$:

\[
R_n(\sigma_3, \vec{\delta}) = \langle n'^2 \rangle_{\text{max}} \cdot R_{n\delta}(\vec{\delta}) \exp \left[ -\frac{(\sigma_3 - z_l)^2}{2\sigma_l^2} \right], \tag{7.35}
\]

whereas we have assumed that $\Phi_n(\sigma_3, 2k_0)$ is a Gaussian function with respect to $\sigma_3$:

\[
\Phi_n(\sigma_3, 2k_0) = \Phi_n^{\text{max}}(2k_0) \exp \left[ -\frac{(\sigma_3 - z_l)^2}{2\sigma_l^2} \right]. \tag{7.36}
\]

How are both assumptions related to each other? Franke (1990) obtained his model for $c_{12}(0)$ by carrying out the integration in $\sigma$-$\delta$-space, whereas the model developed in Sec. 7.2.1 was derived by an integration in $\sigma$-$k$-space. Franke (1990) assumed that $\langle n'^2 \rangle$, i.e., the refractive-index variance, may be a function of $\sigma_3$, without the “inner structure” $R_{n\delta}(\vec{\sigma}, \vec{\delta})$ being a function of $\sigma_3$. Radial homogeneity (i.e., independence of $\sigma_3$) of $R_{n\delta}(\vec{\delta})$, however, implies radial homogeneity of the local outer scales of the refractive-index irregularities. This is not in line with phenomenological laws for the behaviour of turbulent fluctuations in the vicinity of geometrical boundaries. It is known that in the atmospheric surface layer, fluxes and variances may be considered constant as a function of the distance to the geometrical boundary (here the ground), whereas the outer scale of turbulence, $\rho$ (the mixing-length), is proportional to that distance. Thus, in the vicinity of a wall or of the ground, we expect $\langle n'^2 \rangle = \text{const.}$ and $\rho \propto z$, where $z$ is the distance to the wall or to the ground. For a turbulent layer within a stably stratified environment, we expect a similar behaviour. The upper and lower boundaries of the turbulent layer play the role of two geometrical boundaries.\(^{12}\) It is to be expected that $\langle n'^2 \rangle$ is approximately constant within the layer, even close to the boundaries, but that, as is known from the plane Couette flow (see, e.g., Muschinski 1992), close to the boundaries the local outer scale $\rho$ is roughly proportional to the distance to the nearest of the two boundaries. High-resolution in-situ observations of a 7-m-thick turbulent layer in the free troposphere above the Arctic sea ice (Muschinski & Wode 1997, Muschinski & Wode 1998) provide support to the notion that $\langle n'^2 \rangle$ is nearly constant within the layer and that $\rho$ has a maximum at the center of the

\(^{12}\)I am grateful to Professor Roth, Hanover, Germany, who brought this point to my attention many years ago.
layer and decreases towards the boundaries, in contrast to the respective model assumptions by Franke (1990).

What kind of profile for \( \Phi_n(\sigma_3, 2k_0) \), however, is to be expected then? From Sec. 6.1.1, we know

\[
\Phi_n(\sigma_3, 2k_0) \propto C_n^2(\sigma_3). \tag{7.37}
\]

From the definition of the refractive-index turbulence structure parameter \( C_n^2 \),

\[
C_n^2 = \frac{\left\langle (n'(\bar{\sigma} + \delta) - n'(\bar{\sigma}))^2 \right\rangle}{\delta^4}, \tag{7.38}
\]

and from the assumption that \( n'(\bar{\sigma} + \delta) \) and \( n'(\bar{\sigma}) \) are not correlated in the case \( |\delta| \geq \rho \), we obtain an elementary relationship between \( C_n^2 \), the outer scale \( \rho \), and the refractive-index variance \( \langle n'^2 \rangle \):

\[
C_n^2 \approx \frac{2\langle n'^2 \rangle}{\rho^3}. \tag{7.39}
\]

This equation may be interpreted locally, implying that close to the boundaries, where \( \rho \) is small, \( C_n^2 \) has fairly sharp maxima. Obviously, the notion of a single Gaussian function describing the \( C_n^2 \)-profile across a turbulent layer does not fit into this picture. Rather, it is to be expected that a turbulent layer is characterized by a pair of \( C_n^2 \)-maxima. Using two-frequency FDI, it is not possible to empirically decide whether an observed layer is in fact a double layer with respect to \( C_n^2 \) or not. This is, however, in principle possible by using three-frequency FDI. In principle, it is also possible to observe more than two layers by using FDI with more than three frequencies, but for practical reasons (frequency allocation, robustness of the algorithm in the presence of noise) the three-frequency FDI appears to be the most promising approach at the moment.

Muschinski & Wode (1998) have pointed out that FMCW-radar observations (Richter 1969) often show double layers (see, e.g., Fig. 3 in Atlass et al. 1970; the upper part of Fig. 2.18 in Gossard 1990, p. 493; Fig. 7 in Metcalf 1975; Gossard et al. 1970; Plate 3 in Eaton et al. 1995). It seems promising to seek for such pairs of reflective layers in the atmosphere on the basis of a correspondingly modified model for the three coherences \( c_{12}(0) \), \( c_{13}(0) \) und \( c_{23}(0) \) (three-frequency FDI).

### 7.2.3 Anisotropic refractive-index irregularities

In the foregoing, we have assumed that \( \Phi_n(\vec{\sigma}, \vec{k}) \) is locally isotropic at the Bragg wavenumber \( |\vec{k}| = 2k_0 \). Because of the well-known aspect sensitivity at VHF frequencies, this is an unrealistic assumption, especially in the case of a vertically pointing beam. As discussed already in Sec. 6, the effect of vertical anisotropy may be represented by replacing the (nominal) radar weighting-function \( B(\sigma) = f_0^2(\sigma)|W(\sigma)|^2 \) by an effective weighting-function \( B_{\text{eff}}(\sigma) \) in a way that the effective beam width is reduced as compared to the nominal beam width (e.g., Hocking et al. 1986), implying also a decrease of the transversal size of the effective observation volume.
This leads to a reduction of the annoying curvature effect for the determination of \( \sigma_l \) and \( z_t \) according to (7.21), implying that the curvature factor \( (1 + j \Delta k \sigma_l^2 / r_0)^{-1} \) in (7.21) may often be neglected, in particular for high-gain radars. When using lower-gain radars, i.e., if the curvature effect has a non-negligible effect on phase and magnitude of the coherence, however, the change of the curvature due to aspect sensitivity has to be taken into account. In several theoretical studies, the effects of aspect sensitivity, of the functional form of \( \Phi_n(\tilde{\sigma}, \tilde{k}) \) as function of \( |\tilde{k}| \), as well as the effects of finite layer tilting-angles have been investigated in more detail (LIU & PAN 1993, CHU & CHEN 1995, CHEN et al. 1997). These studies, however, do not discuss the above-mentioned problem of the non-Gaussian \( C_n^2 \)-profile across a turbulent layer.

### 7.3 The first moment

In Sec. 6, we have examined the Doppler shift \( \omega_D \) for the standard sampling-mode. Neglecting the effects of the cross-spectra \( \Phi_{nn}(\tilde{\sigma}, \tilde{k}) \) and \( \Phi_{n(\nu_1,\nu_2)}(\tilde{\sigma}, \tilde{k}) \), we have obtained the Doppler shift for the standard operational mode (monostatic, single frequency) as follows:

\[
\omega_D = \frac{M_{11}^{(1)}}{M_{11}^{(0)}} = -2k_0 \frac{\iiint \bar{\nu}_r(\tilde{\sigma}) B(\tilde{\sigma}) \Phi_n(\tilde{\sigma}, \tilde{k}_B(\tilde{\sigma})) d^3 \sigma}{\iiint B(\tilde{\sigma}) \Phi_n(\tilde{\sigma}, \tilde{k}_B(\tilde{\sigma})) d^3 \sigma}.
\]  
(7.40)

As the FDI counterpart of \( \omega_D \), we define

\[
\omega_{FDI}^D = - \frac{M_{12}^{(1)}}{\sqrt{M_{11}^{(0)} M_{22}^{(0)}}}.
\]  
(7.41)

In the case \( \Delta k \ll 2k_0 \), again we have

\[
M_{11}^{(0)} \approx M_{22}^{(0)},
\]  
(7.42)

and we obtain

\[
\omega_{FDI} = -2k_0 \frac{\iiint \bar{\nu}_r(\tilde{\sigma}) \exp[-j2\Delta kr_t(\tilde{\sigma})] B(\tilde{\sigma}) \Phi_n(\tilde{\sigma}, \tilde{k}_B(\tilde{\sigma})) d^3 \sigma}{\iiint B(\tilde{\sigma}) \Phi_n(\tilde{\sigma}, \tilde{k}_B(\tilde{\sigma})) d^3 \sigma}
\]  
(7.43)

where

\[
r_t(\tilde{\sigma}) = r_0 + \sigma_3 + \frac{\sigma_1^2 + \sigma_2^2}{2r_0}.
\]  
(7.44)

Obviously, \( \omega_{FDI}^D \) provides a weighted spatial average of the complex radial velocity \( \bar{\nu}_r(\sigma) \exp[-j2\Delta kr_t(\sigma)] \), where \( B(\tilde{\sigma}) \Phi_n(\tilde{\sigma}, \tilde{k}_B(\tilde{\sigma})) \) is the weighting-function. Now, we consider
the case that $\Phi_n(\tilde{\sigma}, \vec{k}_B(\tilde{\sigma}))$ has a pronounced maximum at a single location $\tilde{\sigma}_0$ and negligible at other locations $\tilde{\sigma}$ within the observation volume:

$$\Phi_n(\tilde{\sigma}, \vec{k}_B(\tilde{\sigma})) = \Phi_0 \delta_D(\tilde{\sigma} - \tilde{\sigma}_0). \quad (7.45)$$

In this case, we obtain

$$\omega_{FDI}^D = -2k_0 v_r(\tilde{\sigma}_0) \exp[-j2\Delta kr_t(\tilde{\sigma}_0)]. \quad (7.46)$$

That is, in this case the magnitude of the generalized complex Doppler shift measured in the FDI mode is equal to the Doppler shift in the standard case, i.e., $-2k_0 v_r$, but its phase angle $\exp[-j2\Delta kr_t(\tilde{\sigma}_0)]$ contains additional information about the radial location of the scattering center.
8 Spatial Interferometry

8.1 Introductory remarks

In Sec. 7, we have seen that one may gain information about the position of a scattering-center within the observation volume by evaluating the coherence between two Doppler signals $I_1(t)$ and $I_2(t)$, which are quasi-simultaneously and phase-coherently measured with the same antenna at two slightly different carrier frequencies. This is the basic idea of frequency-domain interferometry (FDI).

When applying spatial interferometry (SI; also known as the “spaced-antenna technique” or SA technique; in this work, however, we avoid the somewhat vague term “radar interferometry” which has also been used by some authors), one measures simultaneously and phase-coherently $N$ different Doppler signals $I_1(t), I_2(t), ..., I_N(t)$ at the same carrier frequency but at $N$ (at least two) different receiving antennas $R_1, R_2, ..., R_N$, which are displaced with respect to each other and with respect to the transmitting antenna.

Radar measurements using spaced antennas have been known for a long time (BRIGGS et al. 1950). Soon after WOODMAN & GUILLEN (1974) with his classical work had initiated the worldwide development and deployment of MST radars, spaced-antenna interferometry has also been applied at MST radars (e.g., RÖTTGER & VINCENT 1978, VINCENT & RÖTTGER 1980, LARSEN & RÖTTGER 1989, VAN BAELEN 1991). Meanwhile, also spaced-antenna UHF wind-profilers have been brought into operation (DOVIAK et al. 1994, COHN & CHILSON 1995, LATAITIS et al. 1995, COHN et al. 1997), even as FMCW (frequency-modulated continuous-wave) radar and in combination with RASS (radio-acoustic sounding system), see HIRSCH (1994) and HIRSCH (1996).

8.2 The zeroth moment

In Sec. 3.4.1, we have derived the simplified spectral SI cross-sampling-function:

$$H_{12}^{SI}(\sigma, k) = H_{11}(\sigma, k) \cdot \exp\left[jk_0 \left(\Delta z_{12} + \frac{\Delta x_{12} \sigma_1 + \Delta y_{12} \sigma_2}{\tau_0}\right)\right], \quad (8.1)$$

where

$$H_{11}(\sigma) = (2\pi)^3 A^2 B(\sigma) \delta_D(k - k_B(\sigma)) \quad (8.2)$$

is the simplified spectral standard two-way sampling-function,

$$B(\sigma) = |W(\sigma)|^2 f_\phi^2(\sigma) \quad (8.3)$$

is the radar weighting-function, and
\[ \vec{k}_B(\vec{\sigma}) = 2k_0 \begin{pmatrix} \frac{\sigma_1}{r_0} \\ \frac{\sigma_2}{r_0} \\ 1 \end{pmatrix} \]  

(8.4)

is the local Bragg wave-vector. Now, we consider the simple situation that the positions of the receiving antennas \( R_1 \) and \( R_2 \) have the same \( y \)- and \( z \)-coordinates. That is, the antenna-spacing is

\[ \Delta x_{12} = x_2 - x_1. \]  

(8.5)

Hence, from (5.7) the zeroth moment of the cross-spectrum \( S_{12}(\omega) \) is given by

\[ M^{(0)}_{12} = \iiint_{-\infty}^{\infty} H_{12}^{\text{SI}}(\vec{\sigma}, \vec{k}) \Phi_n(\vec{\sigma}, k_0) d^3k d^3\sigma \]

\[ = (2\pi)^3 A^2 \iiint_{-\infty}^{\infty} B(\vec{\sigma}) \Phi_n(\vec{\sigma}, \vec{k}_B(\vec{\sigma})) \exp \left( jk_0 \frac{\Delta x_{12} \sigma_1}{r_0} \right) d^3\sigma. \]  

(8.6)

As an instructive example we first consider a well-localized scattering-center at the position

\[ \vec{\sigma}_s = \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix}. \]  

(8.7)

We assume that the refractive-index irregularities are locally isotropic at \( \vec{\sigma}_s \), such that we may write \( \Phi_n(\vec{\sigma}, \vec{k}_B(\vec{\sigma})) \) as follows:

\[ \Phi_n(\vec{\sigma}, \vec{k}_B(\vec{\sigma})) = \Phi_n(2k_0) \cdot \delta_D(\vec{\sigma} - \vec{\sigma}_s), \]  

(8.8)

and we obtain

\[ M^{(0)}_{12} = (2\pi)^3 A^2 B(\vec{\sigma}_s) \Phi_n(2k_0) \exp \left( jk_0 \frac{\Delta x_{12} x_s}{r_0} \right). \]  

(8.9)

Thus, the value of \( x_s \), i.e., the \( \sigma_1 \)-coordinate of the position of the scattering-center, may be retrieved from the phase of \( M^{(0)}_{12} \) (compare Liu et al. 1990, p. 555, eq. 28). Of course, a similar relation exists for the retrieval of the \( \sigma_2 \)-coordinate \( y_s \) by evaluating the cross-spectrum \( S_{13}(\omega) \) measured with a receiver pair \( R_1 \) and \( R_3 \) that has the same \( x \)- and \( z \)-coordinates but is spaced with respect to the \( y \)-direction.
8.2.1 Unambiguity of the position retrieval

The interval of unambiguity for the $x_s$-retrieval is determined by the antenna-spacing $\Delta x_{12}$, similarly as the interval of unambiguity for the $x_t$-retrieval using FDI is given by the wavenumber-spacing $\Delta k$. In order to guarantee unambiguity for $x_s$ in an interval

$$\frac{\Delta t}{2} \leq x_s < \frac{\Delta t}{2} \quad (8.10)$$

(“t” means “transverse”), the antenna-spacing $\Delta x_{12}$ has to fulfil the condition

$$-\pi \leq k_0 \frac{\Delta x_{12}}{r_0} \frac{\Delta t}{2} < \pi, \quad (8.11)$$

that is

$$|\Delta x_{12}| \leq \frac{2\pi r_0}{k_0} \frac{\Delta t}{\lambda} = \frac{\lambda}{\Delta\Theta}, \quad (8.12)$$

where the angular interval of unambiguity is defined by the angle

$$\Delta\Theta = \frac{\Delta t}{r_0}. \quad (8.13)$$

It is useful to express $\Delta\Theta$ in units of the beamwidth $\Theta_1$:

$$b = \frac{\Delta\Theta}{\Theta_1}, \quad (8.14)$$

such that we eventually obtain

$$|\Delta x_{12}| \leq \frac{\lambda}{b\Theta_1}. \quad (8.15)$$

For example, in the case of the parameters $b = 3$ and $\Theta_1 = 10^\circ$ we obtain the condition $|\Delta x_{12}| \leq 7.6 \lambda$. For a 915-MHz SA wind-profiler ($\lambda = 32.8 \text{cm}$), the antenna-spacing $|\Delta x_{12}|$ must not be larger than 2.5 m. Since $|\Delta x_{12}|$ determines simultaneously the sensitivity of the phase of $M_{12}^{(0)}$ as a function of $x_s$, in order to maximize the sensitivity it is necessary to choose $\Delta x_{12}$ as large as possible, that is

$$|\Delta x_{12}| \approx \frac{\lambda}{b\Theta_1}. \quad (8.16)$$
8.2.2 Localization of a three-dimensional Gaussian distribution of refractive-index irregularities

In general, it is not reasonable to a priori assume that the backscattering-center is concentrated within a very small, point-like portion of the observation volume. As a somewhat more realistical model, we now consider a three-dimensional distribution that is centered at \( s \):  
\[
\Phi_n(\vec{s}, 2k_0) = \Phi_n^{\max}(2k_0) \cdot \exp \left( -\frac{(\sigma_1 - x_s)^2}{2\rho_x^2} - \frac{(\sigma_2 - y_s)^2}{2\rho_y^2} - \frac{(\sigma_3 - z_s)^2}{2\rho_z^2} \right). \tag{8.17}
\]

Here, \( \rho_x, \rho_y \) and \( \rho_z \) are length scales which characterize the extension of the "turbulent spot" with respect to the \( \sigma_1 \)-, the \( \sigma_2 \)-, and the \( \sigma_3 \)-direction, respectively. As we did already in the FDI analysis, again we consider the complex correlation coefficient between the two signals \( I_1(t) \) and \( I_2(t) \):  
\[
c_{12}^{\text{SI}}(0) = \frac{M_{12}^{(0)}}{\sqrt{M_{11}^{(0)} M_{22}^{(0)}}} = \frac{\iint B(\vec{s}) \Phi_n(\vec{s}, 2k_0) \exp \left( jk_0 \frac{\Delta x_1 \sigma_1}{\sqrt{r_0}} \right) d\vec{s}}{\iint B(\vec{s}) \Phi_n(\vec{s}, 2k_0) d\vec{s}}. \tag{8.18}
\]

We recognize that for the functional form of \( \Phi_n(\vec{s}, 2k_0) \) as specified above, the integrations over \( \sigma_2 \) and \( \sigma_3 \) in the nominator and the denominator, respectively, cancel out, and we obtain  
\[
c_{12}^{\text{SI}}(0) = \frac{\int_{-\infty}^{\infty} \exp \left( -\frac{\sigma_1^2}{\sigma_1'^2} \right) \exp \left( -\frac{(\sigma_1 - x_s)^2}{2\rho_x^2} \right) d\sigma_1}{\int_{-\infty}^{\infty} \exp \left( -\frac{\sigma_1^2}{\sigma_1'^2} \right) \exp \left( -\frac{(\sigma_1 - x_s)^2}{2\rho_x^2} \right) d\sigma_1} = \exp \left( -\frac{k_0^2 \Delta x_1^2 \sigma_1'^2}{4r_0^2} \right) \exp \left( jk_0 \frac{\Delta x_1 \sigma_1'^2}{2\sigma_1'^2} x_s \right), \tag{8.19}
\]
where we have introduced the auxiliary variable  
\[
\frac{1}{\sigma_1'^2} = \frac{1}{\sigma_1^2} + \frac{1}{2\rho_x^2}. \tag{8.20}
\]

As the FDI complex correlation coefficient does (see Sec. 7), also the SI complex correlation coefficient provides information about the width and center of the refractive-index irregularity distribution within the observation volume: the magnitude of \( c_{12}^{\text{SI}}(0) \) provides \( \sigma_1'^2 \), which is a measure of the width of the distribution in the \( \sigma_1 \)-direction, and the phase of \( c_{12}^{\text{SI}}(0) \) provides \( x_s \), i.e., the \( \sigma_1 \)-coordinate of the center of gravity of the distribution.

Now, we consider the two special cases \( \rho_x \ll \sigma_1 \) (a scattering-center that is well-localized with respect to the \( \sigma_1 \)-coordinate) and \( \rho_x \gg \sigma_1 \) (refractive-index irregularity distribution within the observation volume quasi-homogeneous with respect to the \( \sigma_1 \)-coordinate).
In the case \( \rho_x \ll \sigma_t \), we have \( \sigma_t^2 = 2\rho^2 \), that is

\[
c_S^{\text{SI}}(0) = \exp \left( -\frac{k_0^2 \Delta x_{12}^2 \rho^2}{4 r_0^2} \right) \exp \left( j k_0 \frac{\Delta x_{12} \sigma_t^2}{2 \rho^2 r_0} \right). \tag{8.21}
\]

For very small \( \rho_x \), the magnitude of \( c_S^{\text{SI}}(0) \) approaches 1 (ideal coherence), and the phase of \( c_S^{\text{SI}}(0) \) is identical with the result that we obtained in the case of the point-scatterer (see Sec. 8.2).

In the opposite case, \( \rho_x \gg \sigma_t \), we have \( \sigma_t^2 = \sigma_t^2 \), and we find

\[
c_S^{\text{SI}}(0) = \exp \left( -\frac{k_0^2 \Delta x_{12}^2 \sigma_t^2}{4 r_0^2} \right) \exp \left( j k_0 \frac{\Delta x_{12} \sigma_t^2}{2 \rho^2 r_0} \right). \tag{8.22}
\]

Inserting

\[
\Delta x_{12} = \frac{\lambda}{b \Theta_1} \tag{8.23}
\]

and

\[
\sigma_t^2 = \frac{\Theta_1 r_0^2}{8 \ln 2} \tag{8.24}
\]

leads to

\[
|c_S^{\text{SI}}(0)| = \exp \left( -\frac{\pi^2}{8(\ln 2)b^2} \right). \tag{8.25}
\]

This equation defines the lower limit for the coherence. If the dimensionless angular interval of unambiguity, \( b = \Delta \Theta / \Theta_1 \), is large, then the coherence \( |c_S^{\text{SI}}(0)| \) is practically equal to 1 and does not contain any information about \( \rho_x \), and then it is also not possible to gain any information about \( x_s \) since for the \( x_s \)-retrieval \( \rho_x \) must be known. The requirement that \( |c_S^{\text{SI}}(0)| \) should be considerably less than 1 in the case of \( \rho_x \gg \sigma_t \) implies a requirement for the dimensionless parameter \( b \). The value \( \exp(-1) = 0.368 \) is certainly a reasonable value for the lower limit of the coherence. Assuming that value, we obtain

\[
b = \frac{\pi}{\sqrt{8 \ln 2}} = 1.33. \tag{8.26}
\]

This results in the following antenna-spacing:

\[
\Delta x_{12} = \frac{\lambda}{b \Theta_1} = \frac{\sqrt{8 \ln 2}}{\pi} \frac{\lambda}{\Theta_1} = 0.750 \frac{\lambda}{\Theta_1}. \tag{8.27}
\]
For a typical boundary-layer SA wind-profiler with the parameters $\Theta_1 = 10^\circ$ and $\lambda = 32.8$ cm ($\nu = 915$ MHz), that specification for the coherence minimum leads to an antenna-spacing of $\Delta x_{12} = 4.3 \lambda = 1.4$ m.

### 8.3 The first moment

As we have done when deriving relationships for the interpretation of FDI cross-spectra (see Sec. 7.3), we now define a generalized (complex) Doppler shift also for SI cross-spectra:

$$
\omega_{D}^{SI} = \frac{M_{12}^{(1)}}{\sqrt{M_{11}^{(0)} M_{22}^{(0)}}} \approx \frac{M_{12}^{(1)}}{M_{11}^{(0)}}. \tag{8.28}
$$

After inserting the simplified spectral SI cross-sampling-function $H_{12}^{SI}(\vec{\sigma}, \vec{k})$, see Eq. (3.83), we obtain

$$
\omega_{D}^{SI} = -\omega_{D}^{SI} = \frac{\iint B(\vec{\sigma}) \Phi_n(\vec{\sigma}, \vec{k}_B(\vec{\sigma})) \vec{v}(\vec{\sigma}) \cdot \vec{k}_B(\vec{\sigma}) \exp \left( jk_0 \frac{\Delta x_{12} \cdot \vec{n}}{r_0} \right) d^3 \sigma}{\iint B(\vec{\sigma}) \Phi_n(\vec{\sigma}, \vec{k}_B(\vec{\sigma})) d^3 \sigma}, \tag{8.29}
$$

where

$$
\vec{v}(\vec{\sigma}) \cdot \vec{k}_B(\vec{\sigma}) = \begin{pmatrix} \bar{v}_1(\vec{\sigma}) \\ \bar{v}_2(\vec{\sigma}) \\ \bar{v}_3(\vec{\sigma}) \end{pmatrix} \cdot 2k_0 \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ 1 \end{pmatrix} = 2k_0 \left( \bar{v}_1(\vec{\sigma}) \frac{\sigma_1}{r_0} + \bar{v}_2(\vec{\sigma}) \frac{\sigma_2}{r_0} + \bar{v}_3(\vec{\sigma}) \right). \tag{8.30}
$$

#### 8.3.1 The generalized Doppler shift in the case of backscattering from a point-scatterer

It is to be expected that $\omega_{D}^{SI}$ contains information about the transversal components of the wind velocity vector. First, let us consider a point-scatterer

$$
\Phi_n(\vec{\sigma}, \vec{k}_B(\vec{\sigma})) = \Phi_n(2k_0) \delta_D(\vec{\sigma} - \vec{\sigma}_s), \tag{8.31}
$$

which at the measurement time is at the position

$$
\vec{\sigma}_s = \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix}. \tag{8.32}
$$
and which moves through the observation volume with the velocity vector

\[ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{8.33} \]

For a three-receiver configuration we obtain the following three Doppler shifts:

\[ \omega_D^{SI(x)} = -2k_0 \left( \frac{x_s}{r_0} + \frac{v_2 y_s}{r_0} + v_3 \right) \exp \left( jk_0 \frac{\Delta x x_s}{r_0} \right) \tag{8.34} \]

for the receiving-antenna pair specified by \( \Delta x_{12} = \Delta x, \Delta y_{12} = 0, \Delta z_{12} = 0 \); and

\[ \omega_D^{SI(y)} = -2k_0 \left( \frac{x_s}{r_0} + \frac{v_2 y_s}{r_0} + v_3 \right) \exp \left( jk_0 \frac{\Delta y y_s}{r_0} \right) \tag{8.35} \]

for the receiving-antenna pair specified by \( \Delta x_{12} = 0, \Delta y_{12} = \Delta y, \Delta z_{12} = 0 \); and

\[ \omega_D^{SI(z)} = -2k_0 \left( \frac{x_s}{r_0} + \frac{v_2 y_s}{r_0} + v_3 \right) \tag{8.36} \]

for the receiving-antenna pair specified by \( \Delta x_{12} = \Delta y_{12} = \Delta z_{12} = 0 \). Thus, again the formalism provides the classical relation \( \omega_D = -2k_0 v_r \) since \( \omega_D^{SI(z)} \) is by definition identical to the standard Doppler shift and since

\[ v_r = v_1 \frac{x_s}{r_0} + v_2 \frac{y_s}{r_0} + v_3 \tag{8.37} \]

is the radial velocity of the point scatterer.

Obviously, the three equations for \( \omega_D^{SI(x)}, \omega_D^{SI(y)} \) and \( \omega_D^{SI(z)} \) as stated above define a system of equations for the five unknowns \( v_1, v_2, v_3, x_s, y_s \). If one writes the real and the imaginary parts of the equations for \( \omega_D^{SI(x)} \) and \( \omega_D^{SI(y)} \) separately, one obtains altogether five equations for five unknowns. We notice that \( v_1 \) can be retrieved only if \( x_s \neq 0 \), and \( v_2 \) can be retrieved only if \( y_s \neq 0 \). Thus, an SI measurement of the transversal velocity components is not possible if the scatterer is a point scatterer and if the scatterer is on the beam axis at the measurement time.

The above-mentioned system of equations is non-linear, and it is valid only if the refractive-index irregularities are concentrated in one point within the observation volume. The question of whether this assumption is fulfilled or not can be answered if one evaluates the magnitudes of the coherences.

### 8.3.2 The generalized Doppler shift in the case of locally homogeneous and isotropic refractive-index irregularities

It is to be expected that in practical cases the refractive-index irregularities are not homogeneously distributed over the observation volume. In particular, it is to be expected that for a
short measurement time $T$ the autocovariance function

$$
R_n(\bar{\sigma}, \bar{\delta}) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} n'(\bar{\sigma})n'(\bar{\sigma} + \bar{\delta})d\tau
$$

(8.38)

is fairly inhomogeneous. If $T$, however, is sufficiently long, such that the $n'$-distribution within
the observation volume is at least once completely renewed, then the inhomogeneities within
the observation volume are "smeared out", at least along the direction of the mean wind. If
the velocity field is turbulent, as it is the case in the convective boundary layer, it is to be
expected that after a certain time, $R_n(\bar{\sigma}, \bar{\delta})$ is not longer a function of $\bar{\sigma}$. That is, homogeneity
of $R_n(\bar{\sigma}, \bar{\delta})$ in all three directions is to be expected if the measurement time amounts to one or
a few "renewal times"

$$
T_e = \frac{\sigma_t}{\nu},
$$

(8.39)

In the case $\Theta_1 = 10^\circ$ and $r_0 = 1$ km, we have $\sigma_t = 74$ m, and we obtain $T_e \approx 7$ s if we assume $\nu = 10$ m s$^{-1}$. Because $\sigma_t$ is proportional to $r_0$, we expect even smaller values for $T_e$ at lower altitudes. In summary, we expect $T_e$ to be on the order of several seconds for measurements in
the planetary boundary layer as long as the wind velocities are not too small, and the assumption
that $R_n(\bar{\sigma}, \bar{\delta})$ does not depend sensitively on $\bar{\sigma}$ seems reasonable as long as the measurement
time is not shorter than several seconds. At UHF Bragg wavenumbers, the assumption that
the refractive-index irregularities are locally isotropic is widely used as a standard assumption.
Thus, we assume

$$
\Phi_n(\bar{\sigma}, \bar{k}_B(\bar{\sigma})) = \Phi_n(2k_0) = \text{const.}
$$

(8.40)

Moreover, we assume that the velocity field, averaged over a measurement time which is suffi-
ciently long, is homogeneous within the observation volume:

$$
\bar{\nu}(\bar{\sigma}) = \begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 
\end{pmatrix} = \text{const.},
$$

(8.41)

and we obtain the following equations for the generalized SI Doppler shifts:

$$
\omega_D^{\text{SI}(\nu)} = -2k_0 \frac{\iint_{-\infty}^{\infty} B(\bar{\sigma}) \left( \frac{\nu_1 \sigma_1}{r_0} + \frac{\nu_2 \sigma_2}{r_0} + \nu_3 \right) \exp \left( jk_0 \frac{\Delta x_1}{r_0} \right) d^3 \sigma}{\iiint_{-\infty}^{\infty} B(\bar{\sigma}) d^3 \sigma},
$$

(8.42)
In a more compact notation, we have

\[
\omega_D^{S_I(x)} = \kappa_{x1} v_1 + \kappa_{x2} v_2 + \kappa_{x3} v_3, \quad (8.45)
\]

\[
\omega_D^{S_I(y)} = \kappa_{y1} v_1 + \kappa_{y2} v_2 + \kappa_{y3} v_3, \quad (8.46)
\]

\[
\omega_D^{S_I(z)} = \kappa_{z1} v_1 + \kappa_{z2} v_2 + \kappa_{z3} v_3. \quad (8.47)
\]

The set of the coefficients \(\kappa_{ij}\) is determined by the radar parameters and may be computed in closed form (see Appendix). A determination of the center of gravity of the refractive-index irregularity distribution is not necessary if \(R_n(\sigma, \delta)\) is homogeneous.

The method of measuring the transversal components of the wind vector by using SI as outlined here is similar, if not equivalent, to the method put forward by LATAITIS et al. (1995). As LATAITIS et al. (1995) have pointed out, the temporal decay of refractive-index irregularities needs not to be accounted for when applying "zero-lag" methods, in constrast to the so-called "full correlation analysis" (see, e.g., BRIGGS et al. 1950, LIU et al. 1990, DOVIAK et al. 1994, DOVIAK et al. 1996). It can be shown, however, that the correlation time is much more influenced by turbulence shuffling the relative positions of Bragg scatterers than by the scatterer's lifetime, i.e., the temporal decay of the refractive-index irregularities (DOVIAK ET AL. 1996, p. 160., col. 1). Recently, this has been assumed by MUSCHINSKI ET AL. (1998b) in their simulation of weather signals based on the Large-Eddy Simulation (LES) technique.
9 Concluding Remarks

In this work, a general theory of the moments of the variance- and cross-spectra of standard and interferometric clear-air Doppler-radar signals (frequency-domain interferometry as well as spaced-antenna interferometry) has been presented. The theory is complete in the sense that it has not been assumed that the spatial autocovariance- and cross-covariance functions of the refractive index and of the three wind-velocity components are homogeneous and/or isotropic within the observation volume. Only statistical stationarity during the measurement period has been assumed. (Of course, when discussing special cases, we did assume homogeneity and isotropy). Since everything has been described in space and not in time, questionable assumptions like Taylor’s hypothesis were not needed.

An important result is that the theory provides an algorithm that allows, in the sense of a forward calculation, in principle all moments of all variance- and cross-spectra of the standard and interferometric Doppler signals to be calculated in terms of the spatial variance spectra of the refractive-index irregularities and of the wind-irregularities, respectively, and of the spatial wind-/refractive-index cross-spectra. This has been explicitly carried out for the zeroth and the first moments but it has only been indicated for the second moments. It has come out that not only the spatial mean values and variance spectra of $n, v_1, v_2, v_3$ but also their spatial cross-spectra determine the first moments of the Doppler spectra. That is, in principle, a zero mean wind does not necessarily imply a zero mean Doppler shift and vice versa, not even in the case that the variance spectra of $n, v_1, v_2, v_3$ are homogeneous within the observation volume and isotropic at the Bragg wavenumber.

This result has been pointed out in Sec. 5, but the respective terms have not been evaluated quantitatively since, at heights well above the atmospheric surface layer, little is known about the spatial cross-spectra of $n, v_1, v_2, v_3$ at the length scales comparable to half the radar wavelength (D. Lenschow, Boulder, May 1997, priv. comm.). High-resolution in-situ observations using airborne sensors and high-resolution large-eddy simulations might help to fill this gap in our knowledge.
Summary

During the last 25 years, UHF/VHF clear-air Doppler radars have become standard instruments to observe the planetary boundary layer, the free troposphere, and the lower stratosphere. One of the most important application of atmospheric UHF/VHF radars is to measure vertical profiles of the three components of the wind vector ("wind-profiles"). It has been shown, however, that spectra from wind-profiler data contain also information about the refractive-index structure parameter $C_n^2$ (proportional to the zeroth moment of the Doppler spectrum) and about the turbulent kinetic energy of small-scale turbulence within the observation volume (proportional to the second central moment of the Doppler spectrum). More recently, multi-receiver and/or multi-frequency configurations have been realized which enable interferometry in the spatial (spatial interferometry, SI) and/or in the frequency domain (frequency-domain interferometry, FDI).

In this work, a first-principle theory for the first moments of the variance- and cross-spectra of standard and interferometric (SI and FDI) signals measured with clear-air Doppler radars is developed. The theory relies on a generalization of Doviak and Zrnić's (1984) concept of the radar sampling-functions, and on Tatarskii's (1961) theory on wave propagation in turbulent media, which combines Maxwell's equations with the inertial-subrange theory of fully developed turbulence. In particular, the present work provides a thorough theoretical description for the effects of non-zero correlations between wind- and refractive-index fluctuations on the first moments of the Doppler variance- and cross-spectra, respectively.

The work is organized as follows. Section 1 points out that there is a general need for a long-term strategy to develop a universal net of remote-sensing facilities, and it gives a short overview on possibilities and limitations for the utilization of the UHF/VHF Doppler-radar technique in meteorology. Section 2 provides a summary of the mathematical tools of Fourier analysis, as far as they are needed in the present context. Section 3 generalizes Doviak and Zrnić's (1984) concept of the radar sampling-functions, i.e., additionally to the lag-space and wavenumber-space sampling function in the standard (monostatic) case, $G_{11}(\vec{\sigma}, \vec{\delta})$ and $H_{11}(\vec{\sigma}, \vec{k})$, respectively, also the lag-space and wavenumber-space sampling-functions for FDI, $G_{12}^{FDI}(\vec{\sigma}, \vec{\delta})$ and $H_{12}^{FDI}(\vec{\sigma}, \vec{k})$, and for SI, $G_{11}^{SI}(\vec{\sigma}, \vec{\delta})$ and $H_{11}^{SI}(\vec{\sigma}, \vec{k})$, are worked out and discussed. Section 4 defines the "atmospheric functions", i.e., the space-time autocorrelation function of the refractive index at zero time-lag $\tau$, $R_n(\vec{\sigma}, \vec{\delta}, \tau = 0)$, and its $\tau$-derivatives at zero time-lag, $R_n(\vec{\sigma}, \vec{\delta}, \tau = 0)$, $R_n(\vec{\sigma}, \vec{\delta}, \tau = 0)$, etc. It is shown how the Fourier transforms of the atmospheric functions with respect to the lag-space coordinates $\vec{\delta}$ can be written in closed form in terms of the spatial variance- and cross-spectra of the wind- and refractive-index fluctuations. In Section 5, it is shown that each moment of each Doppler variance- or cross-spectrum can be written as a 6D (six-dimensional) integral of the product of a 6D sampling-function and a 6D atmospheric function. In the Sections 6, 7, and 8, the equations for the zeroth and first moments for the standard, FDI, and SI case, respectively, are worked out, and analytically tractable examples are discussed. In the concluding remarks, Section 9, it is pointed out that little is known on the spatial cross-spectra of wind- and refractive-index fluctuations at UHF/VHF Bragg-wavenumbers in the planetary boundary layer above the surface layer and in the free atmosphere, such that there is a need for high-resolution in-situ measurements and computer simulations in order to get quantitative estimates for the contributions of non-zero correlations between wind- and refractive-index fluctuations on the
first moments of clear-air Doppler-radar spectra.
A The generalized Doppler shift of spaced-antenna Doppler cross-spectra

Although the spaced-antenna technique has been used for wind-profiling for a long time, the discussion about how to interpret the Doppler cross-spectra has not settled.

The sampling-function formalism presented in this work allows the Doppler-shift concept to be generalized from the standard case to the spaced-antenna case in a straightforward manner. While the (standard) Doppler-shift $\omega_{11}$ is defined as the center of gravity of the (real) variance spectrum $S_{11}(\omega)$ of a single Doppler signal $I_1(t)$,

$$\omega_{11} = \frac{\int_{-\infty}^{\infty} S_{11}(\omega) \omega d\omega}{\int_{-\infty}^{\infty} S_{11}(\omega) d\omega} = \frac{M_{11}^{(1)}}{M_{11}^{(0)}},$$

we now define the generalized Doppler shift $\omega_{12}$ as the center of gravity of the (complex) cross-spectrum $S_{12}(\omega)$ of the two phase-coherent Doppler signals $I_1(t)$ and $I_2(t)$ received by two receiving antennas $R_1$ and $R_2$, respectively:

$$\omega_{12} = \frac{\int_{-\infty}^{\infty} S_{12}(\omega) \omega d\omega}{\int_{-\infty}^{\infty} S_{12}(\omega) d\omega} = \frac{M_{12}^{(1)}}{M_{12}^{(0)}},$$

In the following we will show how $\omega_{12}$ is related to the components of the wind-vector within the observation volume if the following assumptions are fulfilled: (i) $\Phi_n(\vec{\sigma}, \vec{k})$ is homogeneous within the observation volume; (ii) $\Phi_n(\vec{\sigma}, \vec{k})$ is locally isotropic at the Bragg wavenumber $|\vec{k}| = 2k_0$; (iii) the temporally averaged wind-vector $\vec{v}(\vec{\sigma})$ is homogeneous within the observation volume:

$$\vec{v}(\vec{\sigma}) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$ 

As shown in Sec. 8, the zeroth and first moments of $S_{12}(\omega)$ are given by

$$M_{12}^{(0)} = \iiint_{-\infty}^{\infty} \int H_{12}(\vec{\sigma}, \vec{k}) \Phi_n(\vec{\sigma}, \vec{k}) d^3k d^3\sigma$$

and

\[\text{Note that we have here applied a different normalization for } \omega_{12} \text{ as compared to Sec. 8.3.}\]
We assume that the observation volume is large in comparison with the radar wavelength, so that we may use the simplified spectral SI cross-sampling-function as given by (3.83):

\[
H_{12}(\vec{\sigma}, \vec{k}) = (2\pi)^3 A^2 B(\vec{\sigma}) \delta_D \left( \vec{k} - \vec{k}_B(\vec{\sigma}) \right) \times \exp \left[ jk_0 \left( \Delta z + \frac{\Delta x}{r_0} \sigma_1 + \frac{\Delta y}{r_0} \sigma_2 \right) \right].
\]

Here,

\[
\Delta \vec{a} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}
\]

is the three-dimensional spacing-vector which defines the position of R1 and R2 relative to each other. Note that we allow the spacing-vector to be three-dimensional, which will lead to a formula for \( \omega_{12} \) that is not restricted to vertically pointing spaced-antenna configurations.

Since we have assumed that \( \Phi(\vec{\sigma}, \vec{k}) \) is homogeneous within the observation volume and isotropic at the Bragg wave-vector \( \vec{k}_B(\vec{\sigma}) \), we have

\[
\Phi_n \left( \vec{\sigma}, \vec{k}_B(\vec{\sigma}) \right) = \Phi_n(2k_0).
\]

Inserting (A.3), (A.6) and (A.8) into (A.4) and (A.5) leads to

\[
M_{12}^{(0)} = (2\pi)^3 A^2 \Phi_n(2k_0) \exp(jk_0 \Delta z) \times \int_{-\infty}^{\infty} B(\vec{\sigma}) \exp \left[ jk_0 \left( \frac{\Delta x}{r_0} \sigma_1 + \frac{\Delta y}{r_0} \sigma_2 \right) \right] d^3 \sigma
\]

and

\[
M_{12}^{(1)} = (2\pi)^3 A^2 \Phi_n(2k_0) \exp(jk_0 \Delta z) \times (-2k_0) \int_{-\infty}^{\infty} B(\vec{\sigma}) \left( \frac{v_1}{r_0} \sigma_1 + \frac{v_2}{r_0} \sigma_2 + \frac{v_3}{r_0} \right) \exp \left[ jk_0 \left( \frac{\Delta x}{r_0} \sigma_1 + \frac{\Delta y}{r_0} \sigma_2 \right) \right] d^3 \sigma.
\]
Eventually, we obtain

\[
\omega_{12} = -2k_0 \frac{\iint B(\bar{\sigma}) \left( \frac{v_1}{r_0} \sigma_1 + \frac{v_2}{r_0} \sigma_2 + v_3 \right) \exp \left[ jk_0 \left( \frac{\Delta z}{r_0} \sigma_1 + \frac{\Delta y}{r_0} \sigma_2 \right) \right] d^3\sigma}{\iint B(\bar{\sigma}) \exp \left[ jk_0 \left( \frac{\Delta z}{r_0} \sigma_1 + \frac{\Delta y}{r_0} \sigma_2 \right) \right] d^3\sigma}. \tag{A.11}
\]

Note that the phase factor \( \exp(jk_0 \Delta z) \) cancels out. That is, the longitudinal spacing \( \Delta z \) has no effect on \( \omega_{12} \).

In the foregoing, \( B(\bar{\sigma}) \) is the composite two-way weighting-function defined by the antenna- and receiver characteristics:

\[
B(\bar{\sigma}) = \exp \left( -\frac{\sigma^2_1 + \sigma^2_2}{2\sigma^2_i} - \frac{\sigma^2_3}{2\sigma^2_i} \right), \tag{A.12}
\]

where \( \sigma_i \) is given by

\[
\frac{1}{\sigma^2_i} = \frac{1}{2\sigma^2_{iR}} + \frac{1}{2\sigma^2_{iT}}, \tag{A.13}
\]

\[
\sigma^2_{iR} = 2r_0^2 \sigma^2_{\Theta R}, \tag{A.14}
\]

\[
\sigma^2_{iT} = 2r_0^2 \sigma^2_{\Theta T}. \tag{A.15}
\]

The angles \( \sigma_{\Theta R} \) and \( \sigma_{\Theta T} \) are measures of the beamwidths of the receiving and transmitting antennas, respectively.

The integrations in (A.11) can be carried out in closed form. After lengthy but elementary manipulations, by making use of the identities

\[
\int_{-\infty}^{\infty} e^{-ax^2} x \sin(bx) dx = \frac{2\sqrt{\pi}b}{a^{\frac{3}{2}}} \exp \left( -\frac{b^2}{4a} \right), \tag{A.16}
\]

\[
\int_{-\infty}^{\infty} e^{-ax^2} \cos(bx) dx = \sqrt{\frac{\pi}{a}} \exp \left( -\frac{b^2}{4a} \right), \tag{A.17}
\]

\[
\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \tag{A.18}
\]

we obtain the result.
\[ \omega_{12} = -2k_0 \left[ -j (4k_0 \sigma_0^2 \Delta x) v_1 - j (4k_0 \sigma_0^2 \Delta y) v_2 + v_3 \right]. \] (A.19)

Note that for zero spacing, i.e. for the monostatic case, (A.19) reduces to the well-known relationship between the standard Doppler shift \( \omega_{11} \) the radial velocity \( v_3 \):

\[ \omega_{11} = -2k_0 v_3. \] (A.20)

For simplicity, we consider an antenna pair which is spaced in the \( \sigma \)-direction, i.e. for which \( \Delta y = 0 \):

\[ \omega_{12} = -2k_0 \left[ v_3 - j \left( 4k_0 \sigma_0^2 \Delta x \right) v_1 \right]. \] (A.21)

That is, the real part of \( \omega_{12} \) provides the radial wind component \( v_3 \), and the imaginary part of \( \omega_{12} \) provides the “baseline wind”, i.e. the \( \sigma_1 \)-component of the wind vector. Of course, a corresponding relation holds for an antenna pair that is spaced in the \( \sigma_2 \)-direction.

While the components of the spacing-vector are precisely known in practice, it is often not trivial to determine the value of \( \sigma_\Theta \). An additional difficulty arises from the fact that we have assumed Gaussian radar weighting-functions. So there is a need to define an effective value for \( \sigma_\Theta \). Here we show that the ratio between \( M_{12}^{(0)} \) and \( M_{11}^{(0)} \) may be used to determine an effective value for \( \sigma_\Theta \).

From (A.9) follows

\[ \frac{M_{12}^{(0)}}{M_{11}^{(0)}} = \frac{\int \int \int \int B(\bar{\sigma}) \exp \left[ j k_0 \left( \frac{\Delta x}{r_0} \sigma_1 + \frac{\Delta y}{r_0} \sigma_2 \right) \right] d^3 \sigma}{\int \int \int \int B(\bar{\sigma}) d^3 \sigma}, \] (A.22)

and by using (A.16), (A.17), (A.18) we find

\[ \frac{M_{12}^{(0)}}{M_{11}^{(0)}} = \exp \left[ -\frac{1}{2} k_0^2 \sigma_\Theta^2 \left( \Delta x^2 + \Delta y^2 \right) \right], \] (A.23)

which allows us to determine \( \sigma_\Theta \) from \( M_{12}^{(0)}/M_{11}^{(0)} \).
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